# The 71st William Lowell Putnam Mathematical Competition
# Saturday, December 4, 2010

A1 Given a positive integer $n$, what is the largest $k$ such that the numbers $1, 2, \ldots, n$ can be put into $k$ boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest $k$ is *at least* 3.]

A2 Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers $x$ and all positive integers $n$.

A3 Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x,y) = a\frac{\partial h}{\partial x}(x,y) + b\frac{\partial h}{\partial y}(x,y)$$

for some constants $a, b$. Prove that if there is a constant $M$ such that $|h(x,y)| \leq M$ for all $(x,y) \in \mathbb{R}^2$, then $h$ is identically zero.

A4 Prove that for each positive integer $n$, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

A5 Let $G$ be a group, with operation $*$. Suppose that

(i) $G$ is a subset of $\mathbb{R}^3$ (but $*$ need not be related to addition of vectors);

(ii) For each $\mathbf{a}, \mathbf{b} \in G$, either $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$ or $\mathbf{a} \times \mathbf{b} = 0$ (or both), where $\times$ is the usual cross product in $\mathbb{R}^3$.

Prove that $\mathbf{a} \times \mathbf{b} = 0$ for all $\mathbf{a}, \mathbf{b} \in G$.

A6 Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that $\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}\,dx$ diverges.

B1 Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer $m$?

B2 Given that $A$, $B$, and $C$ are noncollinear points in the plane with integer coordinates such that the distances $AB$, $AC$, and $BC$ are integers, what is the smallest possible value of $AB$?

B3 There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer $n$. You may redistribute the balls by a sequence of moves, each of which consists of choosing an $i$ and moving *exactly* $i$ balls from box $B_i$ into any one other box. For which values of $n$ is it possible to reach the distribution with exactly $n$ balls in each box, regardless of the initial distribution of balls?

B4 Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$

B5 Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all $x$?

B6 Let $A$ be an $n \times n$ matrix of real numbers for some $n \geq 1$. For each positive integer $k$, let $A^{[k]}$ be the matrix obtained by raising each entry to the $k^{\text{th}}$ power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \ldots, n+1$, then $A^k = A^{[k]}$ for all $k \geq 1$.