M = SMOOTH n-MANIFOLD WITH RIEMANNIAN METRIC g.

(U, φ) = CHART WITH COORDINATE FUNCTIONS $x^1, \dots, x^n$

(g_{ij}) = METRIC COMPONENTS IN (U, φ), I.E., $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$

$(g^{ij}) = (g_{ij})^{-1}$

SUPPOSE THERE IS ANOTHER CHART (V, ψ), COORDINATE FUNCTIONS $y^1, \dots, y^n$, FOR WHICH

$$g\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right) = \delta_{ab} = \begin{cases} 1 , & a = b = 1, \dots, n \\ 0 , & a \neq b \end{cases}$$

ON V AND WITH $U \cap V \neq \emptyset$.

TRANSFORMATION EQUATIONS :

$$\psi \circ \varphi^{-1} : \begin{cases} y^1 = y^1(x^1, \dots, x^n) \\ \vdots \\ y^n = y^n(x^1, \dots, x^n) \end{cases}$$

$$\frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i}\frac{\partial}{\partial y^a} \qquad \frac{\partial}{\partial x^j} = \frac{\partial y^b}{\partial x^j}\frac{\partial}{\partial y^b}$$

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = g\left(\frac{\partial y^a}{\partial x^i}\frac{\partial}{\partial y^a}, \frac{\partial y^b}{\partial x^j}\frac{\partial}{\partial y^b}\right)$$

$$g_{ij} = \frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^j} g\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right)$$

$$g_{ij} = \frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^i} \delta_{ab}$$

(1) $$g_{ij} = \sum_{a=1}^{n} \frac{\partial y^a}{\partial x^i}\frac{\partial y^a}{\partial x^j} , \quad i, j = 1, \dots, n$$

EQUATIONS (1) ARE EQUIVALENT TO THE MATRIX EQUATION

$$(g_{ij}) = \left(\frac{\partial y^a}{\partial x^i}\right)^T \left(\frac{\partial y^a}{\partial x^i}\right)$$

$$\begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^1} \\ \vdots & & \vdots \\ \frac{\partial y^1}{\partial x^n} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix} \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix}$$

MOMENTARILY SIMPLIFY THE NOTATION TO

$$G = J^T J .$$

THEN

$$G^{-1} = (J^T J)^{-1} = J^{-1} (J^T)^{-1}$$

OR

$$J G^{-1} J^T = Id$$

NOW BACK TO ENTRIES

$$\begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix} \begin{pmatrix} g^{11} & \cdots & g^{1n} \\ \vdots & & \vdots \\ g^{n1} & \cdots & g^{nn} \end{pmatrix} \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^1} \\ \vdots & & \vdots \\ \frac{\partial y^1}{\partial x^n} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

(2) $$\frac{\partial y^a}{\partial x^i} g^{ij} \frac{\partial y^b}{\partial x^j} = \delta^{ab} \qquad , a, b = 1, \dots, n$$

NOW, DIFFERENTIATE (1) WITH RESPECT TO x^k TO GET

$$\frac{\partial g_{ij}}{\partial x^k} = \sum_{a=1}^{n} \frac{\partial}{\partial x^k}\left(\frac{\partial y^a}{\partial x^i}\frac{\partial y^a}{\partial x^j}\right) = \sum_{a=1}^{n}\left(\frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^k \partial x^j} + \frac{\partial y^a}{\partial x^j}\frac{\partial^2 y^a}{\partial x^k \partial x^i}\right)$$

$$= \sum_{a=1}^{n}\left(\frac{\partial y^a}{\partial x^j}\frac{\partial^2 y^a}{\partial x^i \partial x^k} + \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^j \partial x^k}\right)$$

<u>EXERCISE</u>: WRITE DOWN SIMILAR EXPRESSIONS FOR $\dfrac{\partial g_{ik}}{\partial x^j}$ AND $\dfrac{\partial g_{jk}}{\partial x^i}$ AND COMBINE THEM TO GET

$$\text{(3)} \qquad \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right) = \sum_{a=1}^{n} \frac{\partial^2 y^a}{\partial x^j \partial x^k}\frac{\partial y^a}{\partial x^i}$$

ALGEBRAIC TRICK:

FIX SOME INDEX $b = 1, \dots, n$, MULTIPLY ON BOTH SIDES OF (3) BY

$$g^{i\beta}\frac{\partial y^b}{\partial x^\beta} \qquad (\text{SUMMED OVER } \beta = 1, \dots, n)$$

AND SUM OVER i AS REQUIRED BY THE SUMMATION CONVENTION.

$$\frac{1}{2} g^{i\beta}\frac{\partial y^b}{\partial x^\beta}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right) = \sum_{a=1}^{n} g^{i\beta}\frac{\partial y^b}{\partial x^\beta}\frac{\partial^2 y^a}{\partial x^j \partial x^k}\frac{\partial y^a}{\partial x^i}$$

$$\frac{\partial y^a}{\partial x^i} g^{i\beta}\frac{\partial y^b}{\partial x^\beta} = \delta^{ab}$$

BY (2)

$$= \sum_{a=1}^{n} \delta^{ab}\frac{\partial^2 y^a}{\partial x^j \partial x^k}$$

$$\text{(4)} \qquad \left[\frac{1}{2} g^{\beta i}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)\right]\frac{\partial y^b}{\partial x^\beta} = \frac{\partial^2 y^b}{\partial x^j \partial x^k}, \quad b, j, k = 1, \dots, n$$

NOTATION AND TERMINOLOGY :

CHRISTOFFEL SYMBOLS : $\Gamma^{\beta}_{jk} = \frac{1}{2} g^{\beta i} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right)$

$\beta, j, k = 1, \dots, n$

SO (4) CAN BE WRITTEN

$$(5) \qquad \frac{\partial^2 y^b}{\partial x^j \partial x^k} = \Gamma^{\beta}_{jk} \frac{\partial y^b}{\partial x^{\beta}} \qquad b, j, k = 1, \dots, n$$

NEXT, FIX AN INDEX $b = 1, \dots, n$ AND DEFINE

$$z = (z_1, \dots, z_n) = \left(\frac{\partial y^b}{\partial x^1}, \dots, \frac{\partial y^b}{\partial x^n} \right)$$

SO THAT (5) CAN BE WRITTEN

$$\frac{\partial z_j}{\partial x^k} = \Gamma^{\beta}_{jk} z_{\beta} \qquad , \quad j, k = 1, \dots, n$$

FOR EACH j, k AND ℓ WE MUST HAVE

$$\frac{\partial^2 z_j}{\partial x^{\ell} \partial x^k} = \frac{\partial^2 z_j}{\partial x^k \partial x^{\ell}}$$

SO

$$\frac{\partial}{\partial x^{\ell}} \left(\frac{\partial z_j}{\partial x^k} \right) = \frac{\partial}{\partial x^k} \left(\frac{\partial z_j}{\partial x^{\ell}} \right)$$

$$\frac{\partial}{\partial x^{\ell}} \left(\Gamma^{\beta}_{jk} z_{\beta} \right) = \frac{\partial}{\partial x^k} \left(\Gamma^{\beta}_{j\ell} z_{\beta} \right)$$

$$\Gamma^{\beta}_{jk}\,\frac{\partial z_\beta}{\partial x^{\ell}} + \frac{\partial \Gamma^{\beta}_{jk}}{\partial x^{\ell}}\, z_\beta = \Gamma^{\beta}_{j\ell}\,\frac{\partial z_\beta}{\partial x^{k}} + \frac{\partial \Gamma^{\beta}_{j\ell}}{\partial x^{k}}\, z_\beta$$

$$\Gamma^{\beta}_{jk}\,(\Gamma^{\gamma}_{\beta\ell}\, z_\gamma) + \underbrace{\frac{\partial \Gamma^{\beta}_{jk}}{\partial x^{\ell}}\, z_\beta}_{\frac{\partial \Gamma^{\gamma}_{jk}}{\partial x^{\ell}}\, z_\gamma} = \Gamma^{\beta}_{j\ell}\,(\Gamma^{\gamma}_{\beta k}\, z_\gamma) + \underbrace{\frac{\partial \Gamma^{\beta}_{j\ell}}{\partial x^{k}}\, z_\beta}_{\frac{\partial \Gamma^{\gamma}_{j\ell}}{\partial x^{k}}\, z_\gamma}$$

$$\left(\frac{\partial \Gamma^{\gamma}_{jk}}{\partial x^{\ell}} + \Gamma^{\beta}_{jk}\,\Gamma^{\gamma}_{\beta\ell} - \frac{\partial \Gamma^{\gamma}_{j\ell}}{\partial x^{k}} - \Gamma^{\beta}_{j\ell}\,\Gamma^{\gamma}_{\beta k} \right) z_\gamma = 0$$

<u>NOTATION</u> : FOR EACH $\gamma, j, k, \ell = 1, \dots, n$ LET

$$R^{\gamma}_{jk\ell} = \frac{\partial \Gamma^{\gamma}_{j\ell}}{\partial x^{\ell}} + \Gamma^{\gamma}_{\beta\ell}\,\Gamma^{\beta}_{jk} - \frac{\partial \Gamma^{\gamma}_{j\ell}}{\partial x^{k}} - \Gamma^{\gamma}_{\beta k}\,\Gamma^{\beta}_{j\ell}$$

AND NOTE THAT THESE ARE ENTIRELY DETERMINED BY THE METRIC COMPONENTS (g_{ij}) IN THE CHART (U, φ).

$$R^{\gamma}_{ijk}\, z_\ell = 0 \qquad i, j, k = 1, \dots, n$$

$$R^{\gamma}_{ijk}\, \frac{\partial y^b}{\partial x^\gamma} = 0 \qquad b, i, j, k = 1, \dots, n$$

FOR ANY FIXED $i, j, k = 1, \dots, n$ THIS CAN BE REGARDED AS A HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS IN $(R^1_{ijk}, \dots, R^n_{ijk})$ AND, SINCE THE JACOBIAN $\left(\frac{\partial y^b}{\partial x^\gamma}\right)$ IS NONSINGULAR, WE CONCLUDE THAT

$$R^{\gamma}_{ijk} = 0 \qquad \gamma, i, j, k = 1, \dots, n$$

WE HAVE SHOWN THAT THE EXISTENCE OF A CHART (V, ψ) IN WHICH THE METRIC COMPONENTS ARE δ_{ab} EVERYWHERE ON V IMPLIES THAT $R^{r}_{\ ijk} = 0$ EVERYWHERE ON V.

INSTEAD OF REGARDING SUCH A LOCAL COORDINATE SYSTEM AS BEING GIVEN ONE CAN REGARD

$$(1) \qquad \sum_{a=1}^{n} \frac{\partial y^a}{\partial x^i} \frac{\partial y^a}{\partial x^j} = g_{ij}$$

AS A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS FOR SUCH A SET $y^1, \dots, y^n$ OF COORDINATES.

IT CAN BE SHOWN THAT $R^{r}_{\ ijk} = 0$ IS NOT ONLY NECESSARY, BUT ALSO SUFFICIENT FOR THE EXISTENCE OF A SOLUTION TO THESE EQUATIONS

CONSEQUENCE : THE CONDITION THAT $R^{r}_{\ ijk} = 0$ ON AN OPEN SET IN M IS INDEPENDENT OF THE CHOICE OF COORDINATES.

SO, WHAT IS THE POINT OF THIS LONG AND ANNOYING CALCULATION ? WHAT WE HAVE DONE IS ARRIVE AT n^4 RIDICULOUSLY COMPLICATED FUNCTIONS $R^{r}_{\ ijl}$ COMPUTABLE FROM THE METRIC COMPONENTS g_{ij} IN SOME CHART (U, φ), THE VANISHING OF WHICH ON U IS EQUIVALENT TO AN AFFIRMATIVE ANSWER TO THE QUESTION,

" IS THERE ANOTHER COORDINATE SYSTEM $y^1, \dots, y^n$ ON U IN WHICH THE METRIC IS 'FLAT' : $dy^1 \otimes dy^1 + \dots + dy^n \otimes dy^n$? "

THE FUNCTIONS R^{r}_{jkl} ARE WHAT WE WILL SOON COME TO CALL THE "COMPONENTS OF THE RIEMANN CURVATURE TENSOR FOR THE LEVI-CIVITA CONNECTION ASSOCIATED TO THE RIEMANNIAN METRIC g".

WHAT IS COMING NEXT IS A BIT ABSTRACT AND COMPUTATIONALLY INTENSIVE SO WE SHOULD TRY A LITTLE MOTIVATION :

LET X BE AN ARBITRARY MANIFOLD, V A VECTOR FIELD ON X AND f A SMOOTH, REAL-VALUED FUNCTION ON X.

WE ALREADY HAVE A NATURAL, BUILT-IN NOTION OF THE "DIRECTIONAL DERIVATIVE OF f WITH RESPECT TO V"

$df(V) = V(f) =$ REAL-VALUED FUNCTION ON X WHOSE VALUE AT p IS

$$V(f)(p) = V_p(f)$$
$$= \frac{d}{dt} f(\alpha(t))\Big|_{t=0}$$

WHERE α IS ANY SMOOTH CURVE IN X WITH

$$\alpha(0) = p$$
$$\alpha'(0) = V_p$$

IF $X = \mathbb{R}^n$, THEN (BECAUSE OF THE GLOBAL COORDINATE SYSTEM) WE CAN DO BETTER. WE CAN DEFINE THE "DIRECTIONAL DERIVATIVE OF ONE VECTOR FIELD W WITH RESPECT TO ANOTHER VECTOR FIELD V".

YOU JUST DO IT COMPONENTWISE: WRITE

$$W = w^i \frac{\partial}{\partial x^i}$$

AND DEFINE

$$\nabla_V W = \nabla_V \left(w^i \frac{\partial}{\partial x^i} \right) = V(w^i) \frac{\partial}{\partial x^i}$$

$$= dw^i(V) \frac{\partial}{\partial x^i}$$

(THIS FAILS MISERABLY IF THE COORDINATE SYSTEM IS NOT GLOBAL BECAUSE IT GIVES DIFFERENT RESULTS ON THE INTERSECTION OF TWO CHARTS)

HERE ARE SOME BASIC PROPERTIES OF THIS $\nabla_V W$ ON $\mathbb{R}^n$:

1. $\nabla_{a_1 V_1 + a_2 V_2} W = a_1 \nabla_{V_1} W + a_2 \nabla_{V_2} W \quad \forall\, a_1, a_2 \in \mathbb{R}$
2. $\nabla_V (a_1 W_1 + a_2 W_2) = a_1 \nabla_V W_1 + a_2 \nabla_V W_2 \quad \forall\, a_1, a_2 \in \mathbb{R}$
3. $\nabla_{fV} W = f \nabla_V W \quad \forall\, f \in C^\infty(\mathbb{R}^n)$
4. $\nabla_V (fW) = f \nabla_V W + (V(f)) W \quad \forall\, f \in C^\infty(\mathbb{R}^n)$

SAMPLE PROOF :

4. $$\nabla_V (fW) = \nabla_V \left(f W^i \frac{\partial}{\partial x^i} \right) = V(f W^i) \frac{\partial}{\partial x^i}$$

$$= \left(f V(W^i) + V(f) W^i \right) \frac{\partial}{\partial x^i}$$

$$= f V(W^i) \frac{\partial}{\partial x^i} + V(f) W^i \frac{\partial}{\partial x^i}$$

$$= f \nabla_V W + V(f) W .$$

<u>EXERCISE</u> : PROVE # 1-3.

THESE FOUR PROPERTIES WE WILL TAKE TO BE AXIOMS FOR A GENERAL DEFINITION.

A <u>CONNECTION</u> (OR <u>AFFINE</u> <u>CONNECTION</u>) ON A SMOOTH MANIFOLD M IS A MAPPING

$$\nabla : \mathcal{T}(TM) \times \mathcal{T}(TM) \longrightarrow \mathcal{T}(TM)$$

WHICH ASSIGNS TO EACH ORDERED PAIR (V, W) OF SMOOTH VECTOR FIELDS ON M A THIRD SMOOTH VECTOR FIELD

$$\nabla_V W$$

(CALLED THE <u>COVARIANT</u> <u>DERIVATIVE</u> OF W WITH RESPECT TO V FOR THE CONNECTION ∇) SUCH THAT

$$= (VW - WV)(x^i)\frac{\partial}{\partial x^i}$$

$$= [V,W](x^i)\frac{\partial}{\partial x^i}$$

$$= [V,W] \qquad \square$$

2. IF V, W AND X ARE SMOOTH VECTOR FIELDS ON $\mathbb{R}^n$ AND IF DENOTE THE POINTWISE $\mathbb{R}^n$- INNER PRODUCT OF VECTOR FIELDS WITH A DOT $\cdot$, THEN

$$X(V\cdot W) = V\cdot \nabla_X W + \nabla_X V \cdot W$$

PROOF : FIRST COMPUTE

$$V\cdot \nabla_X W = (V^i \frac{\partial}{\partial x^i})\cdot (X(W^j)\frac{\partial}{\partial x^j})$$

$$= V^i X(W^j)\frac{\partial}{\partial x^i}\cdot\frac{\partial}{\partial x^j} = V^i X(W^j)\,\delta_{ij}$$

$$= \sum_{i=1}^{n} V^i X(W^i)$$

SIMILARLY,

$$\nabla_X V\cdot W = \sum_{i=1}^{n} X(V^i)W^i$$

SO

$$V\cdot\nabla_X W + \nabla_X V\cdot W = \sum_{i=1}^{n}(V^i X(W^i) + X(V^i)W^i)$$

NOW COMPUTE

$$X(V\cdot W) = X((V^i\frac{\partial}{\partial x^i})\cdot(W^j\frac{\partial}{\partial x^j}))$$

$$= X(\sum_{i=1}^{n} V^i W^i)$$

$$= \sum_{i=1}^{n} X(v^i w^i)$$

$$= \sum_{i=1}^{n} (v^i X(w^i) + X(v^i) w^i)$$

$$= V \cdot \nabla_X W + \nabla_X V \cdot W \qquad \square$$

3. WITH THE SAME NOTATION AS IN #2,

$$2 \nabla_V W \cdot X = V(W \cdot X) - X(V \cdot W) + W(X \cdot V)$$
$$+ W \cdot [X, V] - V \cdot [W, X] + X \cdot [V, W]$$

<u>PROOF</u>: THIS FOLLOWS DIRECTLY FROM THE FIRST TWO PROPERTIES:

$$V(W \cdot X) - X(V \cdot W) + W(X \cdot V) = \boxed{\begin{array}{l} W \cdot \nabla_V X + \nabla_V W \cdot X \\ - V \cdot \nabla_X W - \nabla_X V \cdot W \\ + X \cdot \nabla_W V + \nabla_W X \cdot V \end{array}}$$

$$W \cdot [X, V] - V \cdot [W, X] + X \cdot [V, W] =$$

$$W \cdot (\nabla_X V - \nabla_V X)$$
$$- V \cdot (\nabla_W X - \nabla_X W)$$
$$+ X \cdot (\nabla_V W - \nabla_W V)$$

$$= W \cdot \nabla_X V - W \cdot \nabla_V X$$
$$- V \cdot \nabla_W X + V \cdot \nabla_X W$$
$$+ X \cdot \nabla_V W - X \cdot \nabla_W V$$

$$= \boxed{\begin{array}{l} - W \cdot \nabla_V X \;+\; \nabla_V W \cdot X \\ + V \cdot \nabla_X W \;+\; \nabla_X V \cdot W \\ - X \cdot \nabla_W V \;-\; \nabla_W X \cdot V \end{array}}$$

ADDING THE TWO BOXES GIVES THE RESULT. $\square$

THESE LAST TWO PROPERTIES RELY ON THE DOT PRODUCT (RIEMANNIAN METRIC) ON $\mathbb{R}^n$ AND WE WILL RETURN TO THEM LATER.

NOW WE NEED TO GO BACK TO THE GENERAL DEFINITION OF A CONNECTION AND BE SURE THAT THE AXIOMS CONTAIN ENOUGH INFORMATION TO ENSURE THAT EVERYTHING ONE WOULD WANT TO BE TRUE OF A "DERIVATIVE" IS, IN FACT, TRUE.

<u>LEMMA</u>: LET U BE AN OPEN SET IN M. IF EITHER V OR W VANISHES IDENTICALLY ON U, THEN SO DOES $\nabla_V W$.

<u>PROOF</u>: SUPPOSE W IS IDENTICALLY ZERO ON U (THE PROOF FOR V IS SIMILAR). MUST SHOW THAT $\forall\, g \in C^\infty(M)$

$$(\nabla_V W)(g) \equiv 0$$

ON U. LET $p \in U$. CHOOSE A SMOOTH ("BUMP") FUNCTION $f \in C^\infty(M)$ THAT IS 0 AT p AND 1 OUTSIDE OF U. THEN

$W = fW$ so

$$(\nabla_V W)(g) = (\nabla_V (fW))(g)$$
$$= (f\nabla_V W + (Vf)W)(g)$$
$$= f((\nabla_V W)(g)) + (Vf)(Wg)$$

AT p,

$$((\nabla_V W)(g))(p) = \underbrace{f(p)}_{0}\,((\nabla_V W)(g))(p) + \underbrace{((Vf)(p))}_{V_p(f)=0}((Wg)(p))$$
$$= 0$$

SINCE $p \in U$ WAS ARBITRARY THE PROOF IS COMPLETE. $\square$

<u>NOTE</u>: WE WILL STRENGTHEN THIS SHORTLY TO SHOW THAT IF V VANISHES AT A POINT p, THEN $\nabla_V W$ ALSO VANISHES AT p FOR ANY W.

BY LINEARITY WE CONCLUDE FROM THE LEMMA THAT IF V AND V' (OR W AND W') AGREE ON U, THEN $\nabla_V W$ AND $\nabla_{V'} W$ (OR $\nabla_V W$ AND $\nabla_V W'$) ALSO AGREE ON U.

WITH THIS WE CAN SHOW THAT AN AFFINE CONNECTION ON M INDUCES AN AFFINE CONNECTION ON ANY OPEN SUBMANIFOLD U OF M:

LET ∇ BE A CONNECTION ON M AND SUPPOSE U IS OPEN IN M. DEFINE ∇_U ON U AS FOLLOWS:

LET V AND W BE TWO VECTOR FIELDS ON U. FOR EACH $p \in U$ $\exists$ VECTOR FIELDS V' AND W' ON M THAT AGREE WITH V AND W ON A NEIGHBORHOOD U' OF p IN U. SET

$$((\nabla_U)_V(W))_q = (\nabla_{V'} W')_q$$

FOR EVERY $q \in U'$. BY THE LEMMA THE RIGHT HAND SIDE IS INDEPENDENT OF THE CHOICE OF V' AND W'. THUS, $(\nabla_U)_V(W)$ IS WELL-DEFINED ON U'. WE CAN DO THIS FOR EVERY p IN U AND, BY THE LEMMA AGAIN, THESE LOCAL DEFINITIONS OF $(\nabla_U)_V(W)$ AGREE ON THE INTERSECTIONS OF THEIR DOMAINS SO $(\nabla_U)_V(W)$ IS WELL-DEFINED ON U. ∇_U OBVIOUSLY HAS THE DEFINING PROPERTIES OF A CONNECTION BECAUSE ∇ DOES.

SUPPOSE NOW THAT U HAPPENS TO BE THE DOMAIN OF SOME CHART (U, φ). LET $x^1, \ldots, x^n$ BE THE CORRESPONDING COORDINATE FUNCTIONS. THEN

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$$

ARE VECTOR FIELDS ON U. THUS, $(\nabla_U)_{\frac{\partial}{\partial x^i}}$ OPERATES ON SMOOTH VECTOR FIELDS ON U TO GIVE SMOOTH VECTOR FIELDS ON U.

FOR SIMPLICITY LET'S WRITE

$$\nabla_i = (\nabla_U)_{\frac{\partial}{\partial x^i}}$$

THEN, FOR EACH $j = 1, \ldots, n$, $\nabla_i\left(\frac{\partial}{\partial x^j}\right)$ IS A SMOOTH VECTOR FIELD ON U AND SO IT CAN BE WRITTEN AS

$$\nabla_i\left(\frac{\partial}{\partial x^j}\right) = T^k_{ij}\,\frac{\partial}{\partial x^k}$$

FOR SOME SMOOTH FUNCTIONS T^k_{ij} ON U.

> <u>NOTE</u>: TO ECONOMIZE ON NOTATION I'LL USE THE SAME SYMBOLS T^k_{ij} FOR THE COORDINATE EXPRESSIONS $T^k_{ij} \circ \varphi^{-1}$.
> THUS, WE'LL THINK OF
> $$T^k_{ij} = T^k_{ij}(x^1, \ldots, x^n)$$

NOW, IF WE HAVE ANOTHER CHART $(\hat{U}, \hat{\varphi})$ FOR M WITH COORDINATE FUNCTIONS $\hat{x}^1, \ldots, \hat{x}^n$, THEN WE CAN DO THE SAME THING ON $\hat{U}$ TO GET ANOTHER SET OF FUNCTIONS $\hat{T}^k_{ij}$.

TO GET THE RELATIONSHIP BETWEEN THESE TWO SETS OF FUNCTIONS Γ^k_{ij} AND $\hat{\Gamma}^k_{ij}$ ONE SIMPLY APPLIES THE CONNECTION AXIOMS TO (DROPPING THE $U \cap \hat{U}$ IN $\nabla_{U \cap \hat{U}}$)

$$\nabla_{\frac{\partial}{\partial \hat{x}^i}} \left(\frac{\partial}{\partial \hat{x}^j} \right) = \hat{\Gamma}^k_{ij} \frac{\partial}{\partial \hat{x}^k}$$

$$\nabla_{\frac{\partial x^\alpha}{\partial \hat{x}^i} \frac{\partial}{\partial x^\alpha}} \left(\frac{\partial x^\beta}{\partial \hat{x}^j} \frac{\partial}{\partial x^\beta} \right) = \hat{\Gamma}^k_{ij} \frac{\partial}{\partial \hat{x}^k}$$

$$\frac{\partial x^\alpha}{\partial \hat{x}^i} \nabla_{\frac{\partial}{\partial x^\alpha}} \left(\frac{\partial x^\beta}{\partial \hat{x}^j} \frac{\partial}{\partial x^\beta} \right) = \hat{\Gamma}^k_{ij} \frac{\partial}{\partial \hat{x}^k}$$

$$\frac{\partial x^\alpha}{\partial \hat{x}^i} \left(\frac{\partial x^\beta}{\partial \hat{x}^j} \nabla_{\frac{\partial}{\partial x^\alpha}} \left(\frac{\partial}{\partial x^\beta} \right) + \frac{\partial}{\partial x^\alpha} \left(\frac{\partial x^\beta}{\partial \hat{x}^j} \right) \frac{\partial}{\partial x^\beta} \right) = \hat{\Gamma}^k_{ij} \frac{\partial}{\partial \hat{x}^k}$$

$$\frac{\partial x^\alpha}{\partial \hat{x}^i} \frac{\partial x^\beta}{\partial \hat{x}^j} \Gamma^\gamma_{\alpha\beta} \frac{\partial}{\partial x^\gamma} + \frac{\partial x^\alpha}{\partial \hat{x}^i} \frac{\partial}{\partial x^\alpha} \left(\frac{\partial x^\beta}{\partial \hat{x}^j} \right) \frac{\partial}{\partial x^\beta} = \hat{\Gamma}^k_{ij} \frac{\partial}{\partial \hat{x}^k}$$

$$\frac{\partial x^\alpha}{\partial \hat{x}^i} \frac{\partial x^\beta}{\partial \hat{x}^j} \frac{\partial \hat{x}^k}{\partial x^\gamma} \Gamma^\gamma_{\alpha\beta} \frac{\partial}{\partial \hat{x}^k} + \frac{\partial x^\alpha}{\partial \hat{x}^i} \frac{\partial}{\partial x^\alpha} \left(\frac{\partial x^\beta}{\partial \hat{x}^j} \right) \frac{\partial \hat{x}^k}{\partial x^\beta} \frac{\partial}{\partial \hat{x}^k} = \hat{\Gamma}^k_{ij} \frac{\partial}{\partial \hat{x}^k}$$

$$\left(\frac{\partial x^\alpha}{\partial \hat{x}^i} \frac{\partial x^\beta}{\partial \hat{x}^j} \frac{\partial \hat{x}^k}{\partial x^\gamma} \Gamma^\gamma_{\alpha\beta} + \frac{\partial^2 x^\beta}{\partial \hat{x}^i \partial \hat{x}^j} \frac{\partial \hat{x}^k}{\partial x^\beta} \right) \frac{\partial}{\partial \hat{x}^k} = \hat{\Gamma}^k_{ij} \frac{\partial}{\partial \hat{x}^k}$$

SO

$$\boxed{\hat{\Gamma}^k_{ij} = \frac{\partial x^\alpha}{\partial \hat{x}^i} \frac{\partial x^\beta}{\partial \hat{x}^j} \frac{\partial \hat{x}^k}{\partial x^\gamma} \Gamma^\gamma_{\alpha\beta} + \frac{\partial^2 x^\beta}{\partial \hat{x}^i \partial \hat{x}^j} \frac{\partial \hat{x}^k}{\partial x^\beta}}$$

THUS, GIVEN A CONNECTION ∇ ON M ONE OBTAINS A CONNECTION ON EACH COORDINATE NEIGHBORHOOD WHICH IS COMPLETELY DETERMINED BY THE n^3 FUNCTIONS Γ^{ℓ}_{ij} AND, ON THE INTERSECTION OF TWO SUCH COORDINATE NEIGHBORHOODS THE CORRESPONDING FUNCTIONS ARE RELATED BY THE RATHER NASTY TRANSFORMATION LAW ON THE PREVIOUS PAGE.

<u>NOTE</u> : THE WHOLE PROCESS CAN BE REVERSED. GIVEN A COVER OF M BY COORDINATE NEIGHBORHOODS AND, FOR EACH OF THESE, A COLLECTION OF n^3 FUNCTIONS Γ^{ℓ}_{ij} RELATED BY THIS TRANSFORMATION LAW ON ANY OVERLAPS, ONE DEFINES ∇_U ON EACH U BY $(\nabla_U)_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) = \Gamma^{\ell}_{ij}\frac{\partial}{\partial x^{\ell}}$ AND THESE PIECE TOGETHER INTO A CONNECTION ∇ ON M.

<u>LEMMA</u> : IF $V, W \in \Gamma(TM)$ AND $V(p) = 0$ FOR SOME $p \in M$, THEN

$$(\nabla_V W)(p) = 0$$

AS WELL.

<u>PROOF</u> : LET (U, φ) BE A CHART AT p WITH COORDINATE FUNCTIONS $x^1, \ldots, x^n$. ON U WRITE

$$V = v^i \frac{\partial}{\partial x^i}$$

THEN $v^i(p) = 0 \quad \forall\, i = 1, \dots, n.$

$$\begin{aligned}
(\nabla_V W)(p) &= \left((\nabla_U)_{V|U} (W|U) \right)(p) \\
&= \left((\nabla_U)_{v^i \frac{\partial}{\partial x^i}} (W|U) \right)(p) \\
&= v^i(p)\, \nabla_i (W|U)(p) \\
&= 0
\end{aligned}$$

□

BY LINEARITY, IF V AND V' ARE TWO VECTOR FIELDS WITH $V(p) = V'(p)$, THEN

$$(\nabla_V W)(p) = (\nabla_{V'} W)(p).$$

SINCE ANY $v \in T_p(M)$ CAN BE EXTENDED TO A VECTOR FIELD V ON M WE CAN THEREFORE DEFINE

$$\nabla_v W = (\nabla_V W)(p)$$

FOR ANY W.