ALTERNATING SERIES

ALL OF OUR " TESTS " FOR CONVERGENCE SO FAR HAVE DEALT WITH SERIES OF NON-NEGATIVE TERMS. IF $\sum_{k=1}^{\infty} a_k$ CONTAINS BOTH POSITIVE AND NEGATIVE TERMS (E.G., $\sum_{k=1}^{\infty} \frac{\sin k}{k}$) IT CAN BE DIFFICULT TO DECIDE IF IT CONVERGES OR NOT.

HOWEVER, IF THE SIGNS OF THE TERMS ALTERNATE, E.G.,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

OR

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$$

THEN IT'S NOT TOO BAD.

AN ALTERNATING SERIES IS ONE OF THE FORM

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots$$

OR

$$\sum_{k=1}^{\infty} (-1)^{k} a_k = -a_1 + a_2 - a_3 + a_4 - \cdots$$

WHERE $a_1, a_2, a_3, a_4, \ldots$ ARE ALL POSITIVE.

(I'LL STATE ALL OF THE RESULTS FOR THE FIRST CASE, BUT EVERYTHING IS TRUE FOR THE SECOND AS WELL.)

TO SEE WHAT'S BEHIND THE THEOREM (" TEST ") TO FOLLOW, CONSIDER THE EXAMPLE

$$\sum_{n=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

LOOK AT THE PARTIAL SUMS :

$S_1 = 1$

$S_3 = 1 - \frac{1}{2} + \frac{1}{3}$

$\quad = 1 - (\frac{1}{2} - \frac{1}{3})$

$S_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$

$\quad = 1 - (\frac{1}{2} - \frac{1}{3}) - (\frac{1}{4} - \frac{1}{5})$

$\vdots$

DECREASING AND BOUNDED FROM BELOW BY 0 (BECAUSE $a_k = \frac{1}{k}$ DECREASES)

$\{ S_1, S_3, S_5, \ldots \}$ CONVERGES TO SOMETHING (CALL IT A)

$S_2 = 1 - \frac{1}{2}$

$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

$\quad = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4})$

$S_6 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$

$\quad = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6})$

$\vdots$

INCREASING AND BOUNDED FROM ABOVE BY $a_1 = 1$ (BECAUSE $a_k = \frac{1}{k}$ DECREASES)

$\{ S_2, S_4, S_6, \ldots \}$ CONVERGES TO SOMETHING (CALL IT B)

THEN

$$\begin{aligned} A - B &= \lim_{n\to\infty} S_{2n-1} - \lim_{n\to\infty} S_{2n} \\ &= \lim_{n\to\infty} \left((1 - \tfrac{1}{2} + \tfrac{1}{3} - \cdots + \tfrac{1}{2n-1}) - (1 - \tfrac{1}{2} + \tfrac{1}{3} - \cdots + \tfrac{1}{2n-1} - \tfrac{1}{2n}) \right) \\ &= \lim_{n\to\infty} \frac{1}{2n} \\ &= 0 \end{aligned}$$

THUS, $A = B$ SO

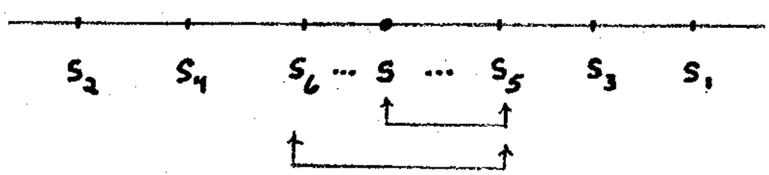
$$\{ S_1, S_2, S_3, S_4, S_5, S_6, \ldots \}$$

CONVERGES TO $A = B$, I.E.,

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$

CONVERGES.

ONE MORE THING TO NOTICE : IF $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = S$, THEN



$$|S - S_n| < |S_{n+1} - S_n| = \left| (-1)^{(n+1)+1} \frac{1}{n+1} \right| = \frac{1}{n+1}$$

SO THE ERROR MADE IN APPROXIMATING S BY S_n IS LESS THAN $\frac{1}{n+1}$.

<u>THEOREM</u> (<u>ALTERNATING SERIES TEST</u>): LET $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ (OR $\sum_{k=1}^{\infty} (-1)^k a_k$) BE AN ALTERNATING SERIES. IF

(a) $\{a_k\}_{k=1}^{\infty}$ IS DECREASING, AND

(b) $\lim_{k \to \infty} a_k = 0$

THEN THE SERIES CONVERGES. IN THIS CASE, THE ERROR MADE IN APPROXIMATING THE SUM S OF THE SERIES BY ITS n^{TH} PARTIAL SUM $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ (OR $\sum_{k=1}^{n} (-1)^k a_k$) IS LESS THAN a_{n+1}:

$$|S - S_n| < a_{n+1}$$

(MOREOVER, THE SIGN OF THE ERROR $S - S_n$ IS THE SAME AS THAT OF THE COEFFICIENT OF a_{n+1}).

<u>EXAMPLES</u>:

1. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ CONVERGES BECAUSE $\{a_k\} = \{\frac{1}{k}\}$ CLEARLY DECREASES TO 0.

LATER WE WILL SHOW THAT THIS SERIES ACTUALLY CONVERGES TO $\ln 2$:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1} \frac{1}{n} + \cdots = \ln 2$$

IF WE APPROXIMATE $\ln 2$ BY, SAY,

$$S_8 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8} = \frac{533}{840} = 0.6345238\overline{1}$$

THEN THE ERROR IS LESS THAN $\frac{1}{9} = 0.\overline{11}$ (AND $\ln 2$ IS LARGER BECAUSE $S - S_8 > 0$).

<u>CALCULATOR</u>: $\ln 2 \approx 0.69314718\overline{1}$

2. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k^2+k}$

$a_k = \frac{k+3}{k^2+k}$ IS DECREASING BECAUSE

$$a_k' = \frac{(k^2+k)(1)-(k+3)(2k+1)}{(k^2+k)^2}$$

$$= \frac{-k^2-6k-3}{(k^2+k)^2} = -\frac{k^2+6k+3}{(k^2+k)^2} < 0$$

MOREOVER,

$$\lim_{k\to\infty} \frac{k+3}{k^2+k} = \lim_{k\to\infty} \frac{\frac{1}{k}+\frac{3}{k^2}}{1+\frac{1}{k}} = \frac{0+0}{1+0} = 0$$

SO THE SERIES CONVERGES.

<u>NOTE</u> : $\sum_{k=1}^{\infty} \frac{1}{k}$ DOES NOT CONVERGE, BUT $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ DOES

(SUBTRACTING EVERY OTHER TERM KEEPS THE PARTIAL SUMS FROM BLOWING UP).

$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2}$ CONVERGES AND WILL STILL CONVERGE EVEN IF YOU MAKE ALL THE TERMS POSITIVE : $\sum_{k=1}^{\infty} \left| (-1)^{k+1} \frac{1}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$.

<u>DEFINITIONS</u> : $\sum_{k=1}^{\infty} a_k$ <u>CONVERGES</u> <u>ABSOLUTELY</u> IF $\sum_{k=1}^{\infty} |a_k|$ CONVERGES (E.G., $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2}$). $\sum_{k=1}^{\infty} a_k$ <u>CONVERGES</u> <u>CONDITIONALLY</u> IF IT CONVERGES, BUT $\sum_{k=1}^{\infty} |a_k|$ DOES NOT CONVERGE (E.G., $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$)

$$\boxed{\text{ABSOLUTE CONVERGENCE} \Rightarrow \text{CONVERGENCE}}$$

I.E., IF $\sum_{k=1}^{\infty} |a_k|$ CONVEGES, THEN SO DOES $\sum_{k=1}^{\infty} a_k$.

<u>EXAMPLES</u> :

1. $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots$

   SERIES DOES NOT ALTERNATE SO WE HAVE NO TESTS THAT APPLY TO IT DIRECTLY. HOWEVER, ITS SERIES OF ABSOLUTE VALUES IS

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \cdots = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

WHICH IS A CONVERGENT GEOMETRIC SERIES. THUS, THE ORIGINAL SERIES CONVERGES ABSOLUTELY (AND THEREFORE CONVERGES).

2. $\displaystyle\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$

<u>NOTE</u>: $\cos 1 > 0$

$\cos 2 < 0$

$\cos 3 < 0$

$\cos 4 < 0$

$\cos 5 > 0$

$\vdots$

TEST FOR ABSOLUTE CONVERGENCE :

$$\sum_{k=1}^{\infty} \left| \frac{\cos k}{k^2} \right| = \sum_{k=1}^{\infty} \frac{|\cos k|}{k^2} \ll \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

SO THE SERIES CONVERGES ABSOLUTELY (AND THEREFORE CONVERGES).

THE FOLLOWING VERSION OF THE RATIO TEST FOR ABSOLUTE CONVERGENCE WILL BE VERY IMPORTANT FOR MACLAURIN AND TAYLOR SERIES (NEXT SECTION) :

<u>THEOREM</u> : LET $\sum_{k=1}^{\infty} a_k$ BE A SERIES OF NONZERO TERMS AND LET

$$\rho = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| .$$

THEN

1. $\rho < 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ CONVERGES ABSOLUTELY (AND THEREFORE CONVERGES)

2. $\rho > 1$ (INCLUDING $\rho = \infty$) $\Rightarrow \sum_{k=1}^{\infty} a_k$ DIVERGES

3. $\rho = 1 \Rightarrow$ NOTHING (THE TEST FAILS)

<u>EXAMPLES</u> :

1. $\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k!}$: $\rho = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} 2^{k+1}}{(k+1)!} \cdot \frac{k!}{(-1)^k 2^k} \right|$

$$= \lim_{k\to\infty} \frac{2}{k+1} = 0 < 1 \Rightarrow \text{SERIES}$$

CONVERGES ABSOLUTELY

2. $\sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{3^k}$: $\rho = \lim_{k\to\infty} \left| \frac{(-1)^{k+1}(2k+1)!}{3^{k+1}} \cdot \frac{3^k}{(-1)^k (2k-1)!} \right|$

$$= \lim_{k\to\infty} \frac{(2k)(2k+1)}{3} = \infty$$

SO THE SERIES DIVERGES.