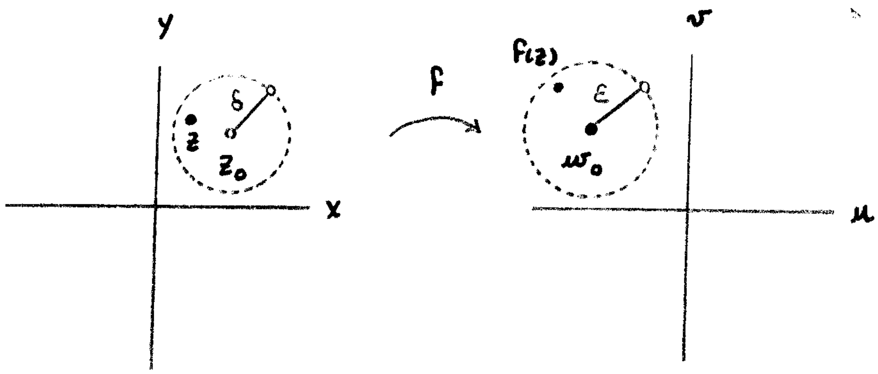
ANALYTIC FUNCTIONS AND CONFORMAL MAPPINGS :

NOW WE BEGIN TO DO CALCULUS. MANY OF THE DEFINITIONS (LIMITS, CONTINUITY, DERIVATIVE, ...) WILL SEEM VERY MUCH LIKE THOSE YOU ARE FAMILIAR WITH FROM REAL CALCULUS, BUT THE END RESULT ("COMPLEX ANALYSIS") WILL HAVE AN ENTIRELY DIFFERENT FLAVOR.

$w = f(z)$ A COMPLEX-VALUED FUNCTION OF A COMPLEX VARIABLE DEFINED NEAR, BUT NOT NECESSARILY AT z_0 (E.G., $w = f(z) = \dfrac{z^2+1}{z+i}$, $z_0 = -i$). THEN

$$\lim_{z \to z_0} f(z) = w_0$$

MEANS INTUITIVELY THAT $f(z)$ CAN BE MADE ARBITRARILY CLOSE TO w_0 BY CHOOSING z SUFFICIENTLY CLOSE, BUT NOT EQUAL TO z_0.



FOR ANY POSITIVE NUMBER ε (HOWEVER SMALL) ONE CAN FIND A SUFFICIENTLY SMALL POSITIVE NUMBER δ FOR WHICH

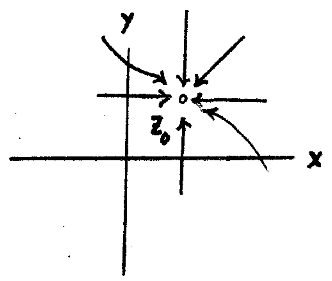
$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon$$

MANY OF THE RESULTS FAMILIAR FROM CALCULUS (E.G., THE LIMIT OF A SUM IS THE SUM OF THE LIMITS, ETC.) ARE TRUE IN THIS NEW CONTEXT ALSO AND PROVED IN EXACTLY THE SAME WAY. MANY OF THE CALCULATIONS ARE ALSO EXACTLY THE SAME, E.G.,

$$\lim_{z \to -i} \frac{z^2+1}{z+i} = \lim_{z \to -i} \frac{(z+i)(z-i)}{z+i} = \lim_{z \to -i} (z-i) = -i-i = -2i .$$

THE CRUCIAL DIFFERENCE IS THAT

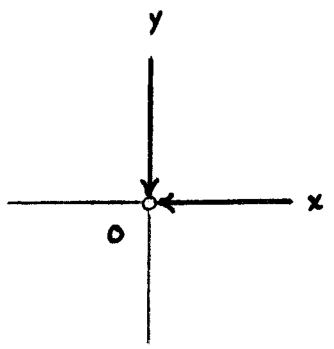
IN ORDER FOR $\lim_{z \to z_0} f(z)$ TO EXIST, $f(z)$ MUST APPROACH THE SAME THING HOWEVER ONE CHOOSES z APPROACHING z_0.



BECAUSE OF THIS SOME VERY SIMPLE FUNCTIONS CAN FAIL TO HAVE LIMITS, E.G.,

$$f(z) = \frac{\bar{z}}{z} \quad , \quad z_0 = 0$$

$$= \frac{x-yi}{x+yi}$$

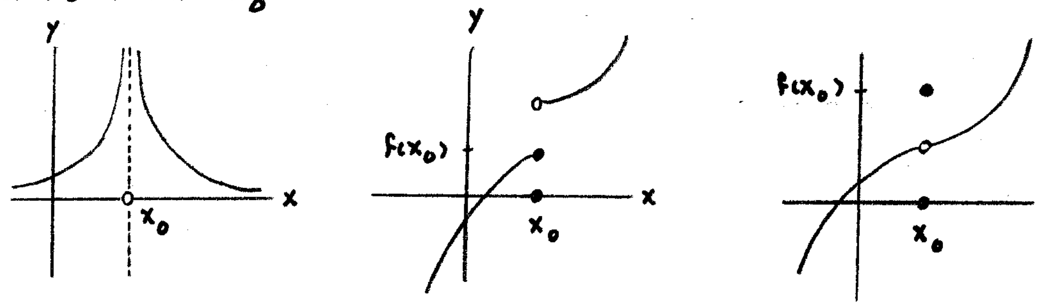


ALONG X-AXIS $(y=0)$: $f(z) = \frac{x-0i}{x+0i} = 1 \to 1$ AS $x \to 0$

ALONG Y-AXIS $(x=0)$: $f(z) = \frac{0-yi}{0+yi} = -1 \to -1$ AS $y \to 0$

$\lim_{z \to 0} \frac{\bar{z}}{z}$ DOES NOT EXIST

RECALL FROM CALCULUS: SOME FUNCTIONS $f(x)$ THAT ARE NOT "CONTINUOUS" AT x_0:



$f(x)$ NOT DEFINED AT x_0	$f(x)$ DEFINED AT x_0, BUT $\lim_{x\to x_0} f(x)$ DOES NOT EXIST	$f(x)$ DEFINED AT x_0, $\lim_{x\to x_0} f(x)$ EXISTS, BUT $\lim_{x\to x_0} f(x) \neq f(x_0)$

f DOES NOT MAP POINTS NEAR x_0 TO POINTS NEAR $f(x_0)$

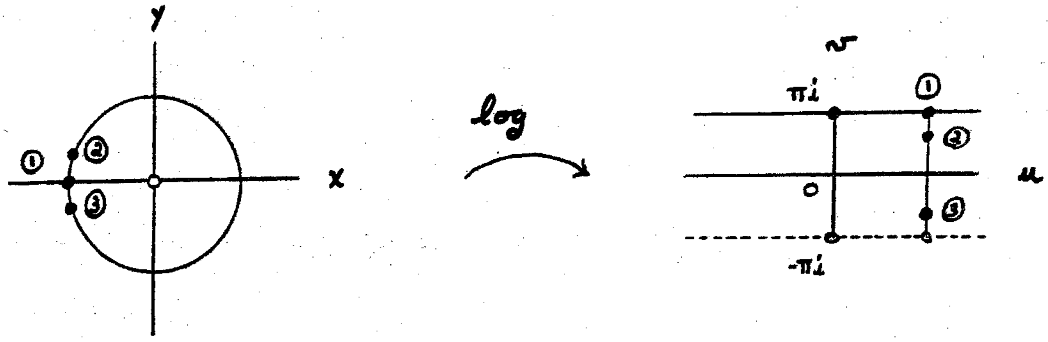
A COMPLEX-VALUED FUNCTION $f(z)$ OF A COMPLEX VARIABLE IS <u>CONTINUOUS AT</u> z_0 IF $f(z_0)$ IS DEFINED, $\lim_{z\to z_0} f(z)$ EXISTS AND

$$\lim_{z\to z_0} f(z) = f(z_0)$$

(POINTS NEAR z_0 ARE MAPPED TO POINTS NEAR $f(z_0)$). IF THIS IS TRUE FOR EACH z_0 IN SOME REGION R, THEN $f(z)$ IS SAID TO BE <u>CONTINUOUS ON</u> R.

EXACTLY AS IN CALCULUS, SUMS, PRODUCTS AND COMPOSITIONS OF CONTINUOUS FUNCTIONS ARE CONTINUOUS, QUOTIENTS OF CONTINUOUS FUNCTIONS ARE CONTINUOUS WHEREVER THE DENOMINATOR IS NONZERO, POLYNOMIALS ARE CONTINUOUS EVERYWHERE, AS ARE e^z, $\sin z$, $\cos z$, $\sinh z$, AND $\cosh z$.

THINGS ARE NOT SO SIMPLE FOR THE REST, HOWEVER. CONSIDER, FOR EXAMPLE, THE PRINCIPAL BRANCH OF THE LOGARITHM:

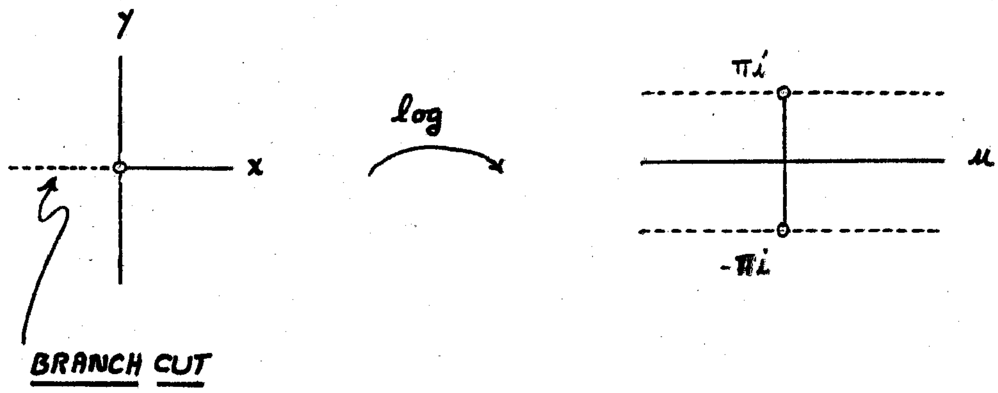


$\mathbb{C} - \{0\}$ $\qquad\qquad$ $-\pi < \mathrm{Im}(w) \le \pi$

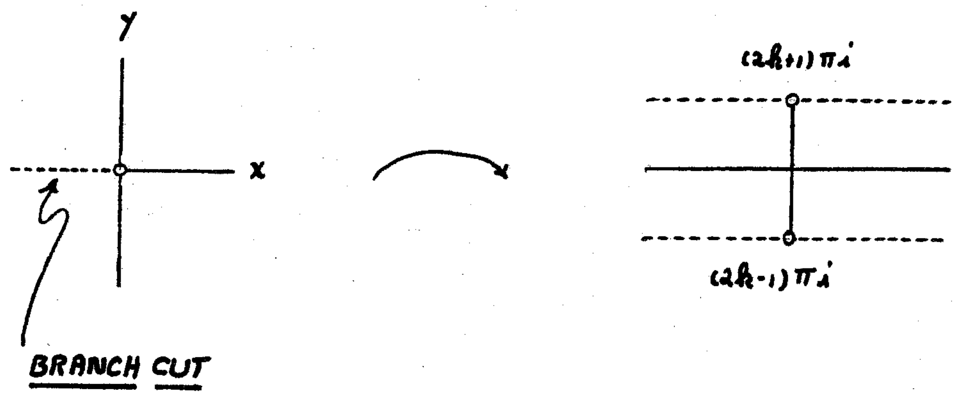
POINTS ON THE NEGATIVE X-AXIS ARE MAPPED TO POINTS WITH $\mathrm{Im}(w) = \pi$ (E.G., ① ABOVE), BUT "NEARBY" POINTS ARE SENT TO OPPOSITE SIDES OF THE HORIZONTAL STRIP (E.G., ② AND ③ ABOVE)

$$\lim_{z \to ①} \log z \text{ DOES NOT EXIST}$$

TO GET A CONTINUOUS BRANCH OF THE LOGARITHM WE MUST "CUT" POINTS OF TYPE ① OUT OF THE DOMAIN.



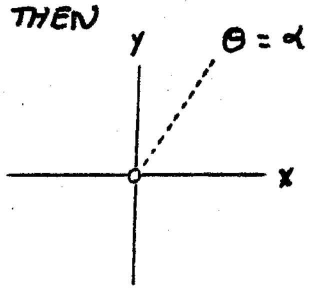
THE SAME BRANCH CUT WORKS FOR ANY OF THE OTHER BRANCHES OF $\log$ THAT WE HAVE DEFINED.



OFTEN IT IS CONVENIENT TO USE STILL OTHER CONTINUOUS BRANCHES OF $\log$, E.G., BY RESTRICTING θ IN $z = re^{i\theta}$ TO SOME INTERVAL $\alpha - \pi < \theta < \alpha + \pi$ AND DEFINING THE MAPPING

$$z = re^{i\theta} \longrightarrow \ln r + \theta i .$$

THE BRANCH CUT IS THEN



ALL OF THIS APPLIES EQUALLY WELL TO ANY FUNCTIONS DEFINED FROM THE LOGARITHM, E.G.,

<u>EXERCISE</u> 39 : DRAW A (MAPPING) PICTURE SHOWING WHY THE PRINCIPAL BRANCH OF THE SQUARE ROOT FUNCTION, DEFINED ON $\mathbb{C} - \{0\}$, IS DISCONTINUOUS AT POINTS OF THE NEGATIVE REAL AXIS.

THE DEFINITION OF THE DERIVATIVE IS FORMALLY IDENTICAL TO THAT FAMILIAR FROM CALCULUS.

$f(z)$ IS <u>DIFFERENTIABLE AT</u> z_0 IF THE LIMIT

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

EXISTS, IN WHICH CASE IT IS DENOTED $f'(z_0)$ AND CALLED THE <u>DERIVATIVE OF</u> $f(z)$ <u>AT</u> z_0.

BECAUSE OF THIS, MANY OF THE "USUAL" RESULTS FROM CALCULUS (PRODUCT RULE, QUOTIENT RULE, CHAIN RULE, DERIVATIVES OF POLYNOMIALS, DIFFERENTIABLE $\Rightarrow$ CONTINUOUS, ETC.) ARE STILL TRUE AND PROVED IN EXACTLY THE SAME WAY.

THERE ARE, HOWEVER, ENORMOUS DIFFERENCES AS WELL, MOST OF WHICH CAN BE TRACED TO THE FACT THAT LIMITS IN THE COMPLEX PLANE HAVE A PRONOUNCED TENDENCY TO FAIL TO EXIST (I.E., TO DEPEND ON HOW z APPROACHES z_0) SO ASSUMING THAT THEY DO EXIST IS A VERY STRONG ASSUMPTION. THIS IS SEEN MOST CLEARLY IN THE FOLLOWING STRINGENT CONSTRAINT IMPOSED ON THE REAL AND IMAGINARY PARTS

$$f(z) = u(x,y) + i\,v(x,y)$$

OF A DIFFERENTIABLE FUNCTION $f(z)$.

SUPPOSE $f(z) = u(x,y) + i\,v(x,y)$ IS DIFFERENTIABLE AT $z_0 = x_0 + i y_0$.

THEN

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{(x,y)\to(x_0,y_0)} \frac{[u(x,y) + i\,v(x,y)] - [u(x_0,y_0) + i\,v(x_0,y_0)]}{[x + i y] - [x_0 + i y_0]}$$

$$= \lim_{(x,y)\to(x_0,y_0)} \frac{[u(x,y) - u(x_0,y_0)] + i\,[v(x,y) - v(x_0,y_0)]}{(x - x_0) + i(y - y_0)}$$

<u>INDEPENDENT</u> <u>OF</u> <u>HOW</u> $(x,y) \to (x_0,y_0)$.

<u>TWO</u> <u>POSSIBLE</u> <u>APPROACHES</u> :

1. HORIZONTALLY (ALONG $y = y_0$)

$$f'(z_0) = \lim_{x \to x_0} \frac{[u(x,y_0) - u(x_0,y_0)] + i\,[v(x,y_0) - v(x_0,y_0)]}{x - x_0}$$

$$= \left(\lim_{x\to x_0} \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0}\right) + i\left(\lim_{x\to x_0} \frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}\right)$$

$$= u_x(x_0,y_0) + i\,v_x(x_0,y_0)$$

2. VERTICALLY (ALONG $x = x_0$)

$$f'(z_0) = \lim_{y \to y_0} \frac{[u(x_0,y) - u(x_0,y_0)] + i\,[v(x_0,y) - v(x_0,y_0)]}{i(y - y_0)}$$

$$= \left(\lim_{y\to y_0} \frac{v(x_0,y) - v(x_0,y_0)}{y - y_0}\right) - i\left(\lim_{y\to y_0} \frac{u(x_0,y) - u(x_0,y_0)}{y - y_0}\right)$$

$$= v_y(x_0,y_0) - i\,u_y(x_0,y_0)$$

THUS ,

$$f'(z_0) = u_x(x_0,y_0) + i\,v_x(x_0,y_0) = v_y(x_0,y_0) - i\,u_y(x_0,y_0)$$

SO

$$u_x(x_0,y_0) = v_y(x_0,y_0) \text{ AND } u_y(x_0,y_0) = -v_x(x_0,y_0) .$$

DROPPING THE REFERENCE TO (x_0,y_0) :

$$f \text{ DIFFERENTIABLE} \quad \Rightarrow \quad u_x = v_y \text{ AND } u_y = -v_x$$

(THE CAUCHY - RIEMANN EQUATIONS)

SO, LOOK HOW EASY IT IS TO WRITE DOWN PERFECTLY SIMPLE FUNCTIONS THAT CANNOT BE DIFFERENTIABLE ANYWHERE :

$f(z) = \bar{z} = x - y\,i$: $u(x,y) = x$ AND $v(x,y) = -y$

$u_x = 1$ $\qquad v_y = -1$

SO $u_x = v_y$ IS NEVER SATISFIED

AND $f(z) = \bar{z}$ IS NOWHERE DIFFERENTIABLE

WITH ONE SMALL ADDITIONAL ASSUMPTION, THE ARROW $\Rightarrow$ ABOVE GOES THE OTHER WAY ALSO :

IF $f(z) = u(x,y) + i\,v(x,y)$ IS DEFINED ON SOME OPEN SET CONTAINING $z_0 = x_0 + i y_0$ ON WHICH THE PARTIAL DERIVATIVES u_x, u_y, v_x AND v_y EXIST AND IF THESE PARTIALS ARE CONTINUOUS AT (x_0,y_0), THEN, AT z_0,

$$u_x = v_y \text{ AND } u_y = -v_x \quad \Rightarrow \quad f \text{ DIFFERENTIABLE}$$

BOTTOM LINE : ASSUMING CONTINUITY OF THE 1ST ORDER PARTIALS,

$$f(z) = u(x,y) + i\,v(x,y) \text{ DIFFERENTIABLE } \iff \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

E.G.,

$$e^z = \underbrace{e^x \cos y}_{u(x,y)} + i\,\underbrace{e^x \sin y}_{v(x,y)}$$

$$u_x = e^x \cos y \qquad v_x = e^x \sin y$$

$$u_y = -e^x \sin y \qquad v_y = e^x \cos y$$

$u_x = v_y$ AND $u_y = -v_x$ EVERYWHERE ON $\mathbb{C}$ SO

$$\boxed{e^z \text{ IS DIFFERENTIABLE EVERYWHERE ON } \mathbb{C}}$$

MOREOVER, IN PROVING THE CAUCHY-RIEMANN EQUATIONS WE SHOWED THAT

$$f'(z) = u_x + i v_x = v_y - i u_y$$

SO

$$(e^z)' = e^x \cos y + i\,e^x \sin y = e^z$$

$$\boxed{\frac{d}{dz}(e^z) = e^z}$$

A FUNCTION $f(z)$ THAT IS DIFFERENTIABLE EVERYWHERE ON SOME DOMAIN D IS SAID TO BE <u>ANALYTIC ON</u> D. IF $f(z)$ IS ANALYTIC ON ALL OF $\mathbb{C}$ IT IS SAID TO BE <u>ENTIRE</u>.

SOME EXAMPLES OF ANALYTIC FUNCTIONS :

1. POLYNOMIALS ARE ENTIRE FUNCTIONS

$$P(z) = a_n z^n + \cdots + a_1 z + a_0 \quad \Rightarrow P'(z) = n a_n z^{n-1} + \cdots + a_1$$

2. RATIONAL FUNCTIONS ARE ANALYTIC ON THEIR DOMAINS (QUOTIENT RULE)

3. $\sin z$, $\cos z$, $\sinh z$ AND $\cosh z$ ARE ENTIRE WITH DERIVATIVES

$$\frac{d}{dz}(\sin z) = \cos z$$

$$\frac{d}{dz}(\cos z) = -\sin z$$

$$\frac{d}{dz}(\sinh z) = \cosh z$$

$$\frac{d}{dz}(\cosh z) = \sinh z$$

E.G., $$\sin z = \underbrace{\sin x \cosh y}_{u(x,y)} + i \underbrace{\cos x \sinh y}_{v(x,y)}$$

$$u_x = \cos x \cosh y \qquad v_x = -\sin x \sinh y$$

$$u_y = \sin x \sinh y \qquad v_y = \cos x \cosh y$$

SO $u_x = v_y$ AND $u_y = -v_x$ EVERYWHERE AND

$$\begin{aligned} \frac{d}{dz}(\sin z) &= u_x + i v_x \\ &= \cos x \cosh y - i \sin x \sinh y \\ &= \cos z \end{aligned}$$

4. REMAINING TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS ARE ANALYTIC ON THEIR DOMAINS (QUOTIENT RULE) AND HAVE THE EXPECTED DERIVATIVES, E.G., $(\tan z)' = \sec^2 z$, ETC.

THE FOLLOWING THEOREM (WHICH WE WILL NOT PROVE HERE) IS OUR FIRST INDICATION OF JUST HOW STRONG THE ASSUMPTION OF ANALYTICITY IS FOR COMPLEX FUNCTIONS AND IS THE ULTIMATE JUSTIFICATION FOR OUR PECULIAR LOOKING DEFINITIONS OF e^z, $\sin z$, ... BASICALLY, THEY ARE THE ONLY POSSIBLE DEFINITIONS.

<u>THEOREM</u>: SUPPOSE f AND g ARE ANALYTIC ON SOME DOMAIN D AND $f(z) = g(z)$ FOR ALL z ON SOME LINE SEGMENT IN D. THEN $f(z) = g(z)$ FOR ALL z IN D.

IN PARTICULAR, IF f AND g AGREE ON THE REAL LINE, THEY MUST AGREE EVERYWHERE.

<u>EXERCISES</u>:

40. SHOW THAT $\cosh z$ IS ENTIRE AND $(\cosh z)' = \sinh z$

41. SHOW THAT $f(z) = z \, \mathrm{Im}(z)$ IS DIFFERENTIABLE ONLY AT $z = 0$ AND $f'(0) = 0$.

42. SHOW THAT $e^{\bar{z}}$ IS NOWHERE DIFFERENTIABLE.

ACCORDING TO THE CHAIN RULE A COMPOSITION OF ANALYTIC FUNCTIONS IS ANALYTIC ON ITS DOMAIN,

E.G., $\cosh(e^z)$ IS ENTIRE.

IN ORDER TO EXPAND OUR INVENTORY OF ANALYTIC FUNCTIONS WE NEED THE POLAR FORM OF THE CAUCHY-RIEMANN EQUATIONS.

$$f(z) = u(x,y) + i\,v(x,y) = u(r,\theta) + i\,v(r,\theta)$$

WE NOW SHOW THAT

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \iff \begin{cases} u_r = \frac{1}{r} v_\theta \\ u_\theta = -r v_r \end{cases}$$

$x = r\cos\theta$ AND $y = r\sin\theta$ SO THE CHAIN RULE GIVES

$$u_r = \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_x \cos\theta + u_y \sin\theta$$

$$v_r = v_x \cos\theta + v_y \sin\theta$$

$$u_\theta = \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = u_x(-r\sin\theta) + u_y(r\cos\theta)$$

$$v_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta)$$

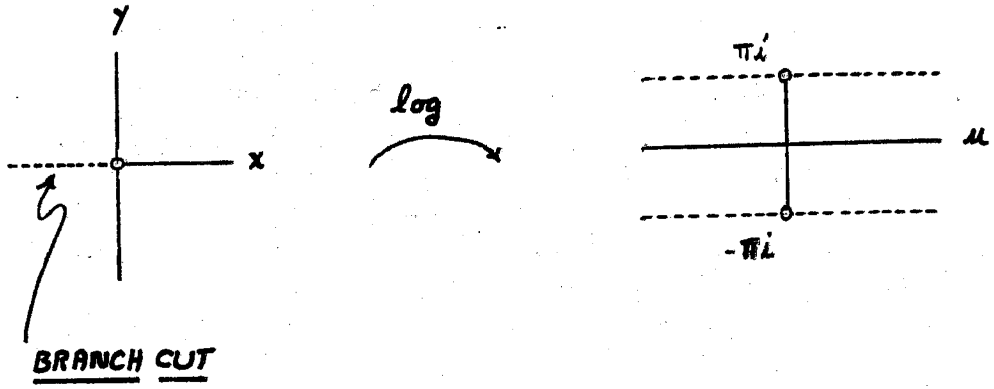
NOW ASSUME $u_x = v_y$ AND $u_y = -v_x$. THEN

$$u_r = v_y \cos\theta - v_x \sin\theta = \frac{1}{r} v_\theta$$

$$u_\theta = v_y(-r\sin\theta) - v_x(r\cos\theta) = -r v_r$$

THE REVERSE IMPLICATION IS PROVED IN THE SAME WAY AFTER SOLVING THE EQUATIONS ABOVE FOR u_x, u_y, v_x, v_y IN TERMS OF $u_r, u_\theta, v_r, v_\theta$.

WITH THE POLAR FORM OF THE CAUCHY-RIEMANN EQUATIONS WE CAN SHOW THAT ANY CONTINUOUS BRANCH OF log IS ANALYTIC ON ITS OPEN DOMAIN, E.G.,



$$\log(z) = \ln r + \theta i \qquad (r>0,\ -\pi<\theta<\pi)$$

$$u(r,\theta) = \ln r$$

$$v(r,\theta) = \theta$$

$$u_r = \frac{1}{r} \qquad v_r = 0$$

$$u_\theta = 0 \qquad v_\theta = 1$$

SO

$$u_r = \frac{1}{r} v_\theta \quad \text{AND} \quad u_\theta = -r v_r$$

EXERCISE 43 : SHOW THAT IF $f(z) = u(x,y) + i v(x,y) = u(r,\theta) + i v(r,\theta)$ IS ANALYTIC ON SOME DOMAIN D, THEN OUR EXPRESSION $f'(z) = u_x + i v_x$ FOR THE DERIVATIVE CAN BE WRITTEN

$$f'(z) = e^{-\theta i}(u_r + i v_r)$$

APPLYING THIS TO $\log z$ GIVES

$$\frac{d}{dz}(\log z) = e^{-\theta i}\left(\frac{1}{r} + 0i\right) = (re^{\theta i})^{-1} = \frac{1}{z}$$

AS EXPECTED.

THE SAME CALCULATIONS GIVE THE SAME RESULT FOR ANY CONTINUOUS BRANCH OF THE LOGARITHM.

EXERCISE 44 : SHOW THAT THE PRINCIPAL BRANCH OF THE SQUARE ROOT $\sqrt{z}$

$$\sqrt{r}\, e^{i\theta/2} \qquad (r > 0, -\pi < \theta < \pi)$$

IS ANALYTIC ON THE GIVEN DOMAIN AND THAT ITS DERIVATIVE IS THE PRINCIPAL BRANCH OF $\frac{1}{2\sqrt{z}}$, AS EXPECTED.

SIMILAR CONSIDERATION APPLY TO ALL OF THE OTHER "MULTI-VALUED FUNCTIONS" WE HAVE DEFINED FROM $\log z$:

CHOOSE AN ANALYTIC BRANCH OF $\log z$, GET AN ANALYTIC BRANCH OF z^c, ARCSIN z, ...

EXERCISE 45 :

(a) SHOW THAT $\log(z-i)$ IS ANALYTIC EVERYWHERE EXCEPT ON THE HALF-LINE $y = 1$ $(x \leq 0)$.

(b) DETERMINE THE LARGEST DOMAIN ON WHICH $\frac{\log(z+4)}{z^2+i}$ IS ANALYTIC.

ONLY VERY SPECIAL FUNCTIONS CAN BE THE REAL AND IMAGINARY PARTS OF AN ANALYTIC FUNCTION :

$$f(z) = u(x,y) + i\,v(x,y)$$

$$u_x = v_y \qquad\qquad u_y = -v_x$$

$$u_{xx} = v_{yx} \qquad\qquad u_{yy} = -v_{xy}$$

$$v_{yx} = v_{xy}$$

$$\Rightarrow \qquad \boxed{u_{xx} + u_{yy} = 0} \qquad \text{(LAPLACE EQUATION)}$$

SIMILARLY,

$$\boxed{v_{xx} + v_{yy} = 0}$$

THE REAL AND IMAGINARY PARTS OF AN ANALYTIC FUNCTION MUST BE SOLUTIONS TO THE LAPLACE EQUATION (<u>HARMONIC FUNCTIONS</u>).

E.G., $u(x,y) = x^2(y^2+1)$ CANNOT BE THE REAL (OR IMAGINARY) PART OF ANY ANALYTIC FUNCTION BECAUSE

$$u = x^2y^2 + x^2 \quad \Rightarrow \quad \begin{aligned} u_x &= 2xy^2 + 2x \\ u_{xx} &= 2y^2 + 2 \\ u_y &= 2x^2y \\ u_{yy} &= 2x^2 \end{aligned}$$

SO

$$u_{xx} + u_{yy} = 2(x^2+y^2+1)$$

WHICH IS NEVER ZERO.

MOREOVER, YOU CAN'T JUST PICK YOUR TWO FAVORITE HARMONIC FUNCTIONS AND MAKE THEM THE REAL AND IMAGINARY PARTS ; THEY HAVE TO BE RELATED BY THE CAUCHY - RIEMANN EQUATIONS.

E.G., x^2-y^2 IS HARMONIC $(\mathrm{Re}(z^2))$ AND SO IS $e^x \cos y$ $(\mathrm{Re}(e^z))$, BUT $(x^2-y^2) + i\, e^x \cos y$ IS NOT ANALYTIC $(2x \neq -e^x \sin y)$.

HOWEVER, GIVEN ONE HARMONIC FUNCTION $u(x,y)$ IT IS GENERALLY POSSIBLE TO FIND ANOTHER HARMONIC FUNCTION $v(x,y)$ FOR WHICH $u(x,y) + i\, v(x,y)$ IS ANALYTIC ($v(x,y)$ IS CALLED A <u>HARMONIC CONJUGATE</u> FOR $u(x,y)$).

E.G., $u(x,y) = y^3 - 3x^2 y$ IS HARMONIC EVERYWHERE BECAUSE

$$u_{xx} + u_{yy} = -6y + 6y = 0.$$

TO FIND A HARMONIC CONJUGATE $v(x,y)$ WE MUST SOLVE

$$u_x = v_y \qquad\qquad u_y = -v_x$$
$$-6xy = v_y \qquad\qquad 3y^2 - 3x^2 = -v_x$$

FOR v. INTEGRATE $v_y = -6xy$ WITH RESPECT TO y TO GET

$$v(x,y) = -3xy^2 + c(x)$$

WHERE $c(x)$ IS AN ARBITRARY FUNCTION OF x. THEN $v_x = -3y^2 + c'(x)$ SO WE SET

$$\begin{aligned} -3y^2 + c'(x) &= -3y^2 + 3x^2 \\ c'(x) &= 3x^2 \\ c(x) &= x^3 + C \end{aligned}$$

WHERE C IS AN ARBITRARY CONSTANT. THUS,

$$v(x,y) = -3xy^2 + x^3 + C$$

IS A HARMONIC CONJUGATE FOR ANY CHOICE OF C, E.G., TAKING $C=0$,

$$(y^3 - 3x^2y) + i(x^3 - 3xy^2)$$

IS ANALYTIC EVERYWHERE.

EXERCISES :

46. SHOW THAT IF $f(z) = u(r,\theta) + i\,v(r,\theta)$ IS ANALYTIC, THEN $u(r,\theta)$ SATISFIES

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$$

(POLAR FORM OF THE LAPLACE EQUATION). SIMILARLY FOR $v(r,\theta)$.

47. SHOW THAT $u(x,y) = 2x(1-y)$ IS HARMONIC EVERYWHERE AND FIND AN ENTIRE FUNCTION $f(z)$ WHOSE REAL PART IS $u(x,y)$.

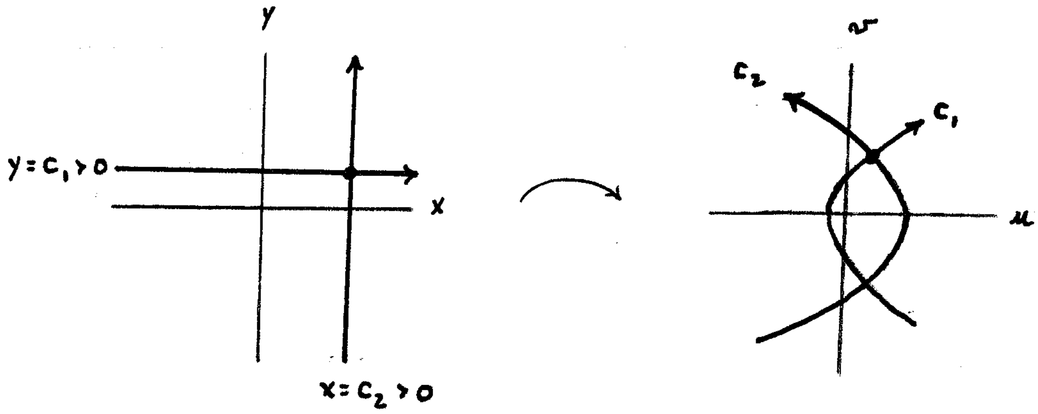
48. SHOW THAT $u(x,y) = e^{x^2-y^2}\cos(2xy)$ IS HARMONIC EVERYWHERE AND FIND A HARMONIC CONJUGATE FOR $u(x,y)$.

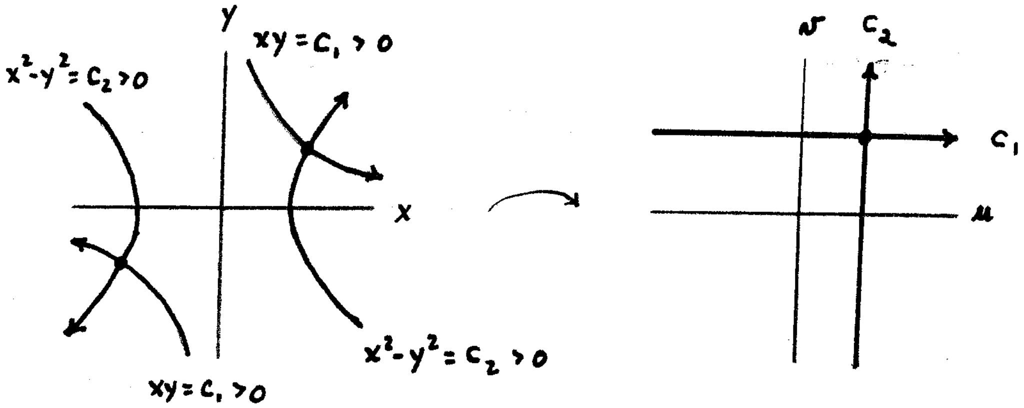
HINT : THINK BEFORE YOU COMPUTE.

49. SHOW THAT IF v IS A HARMONIC CONJUGATE FOR u, THEN $u^2 - v^2$ IS ALSO HARMONIC.

WE CONCLUDE WITH ONE LAST OBSERVATION ABOUT ANALYTIC FUNCTIONS THAT IS CRUCIAL FOR MANY OF ITS APPLICATIONS.

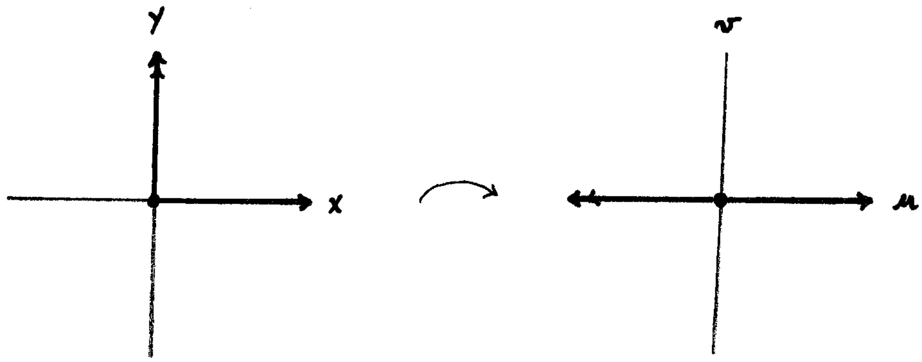
CONSIDER AGAIN SOME MAPPING PROPERTIES OF $w = f(z) = z^2$





CURVES INTERSECT ORTHOGONALLY

IMAGES INTERSECT ORTHOGONALLY

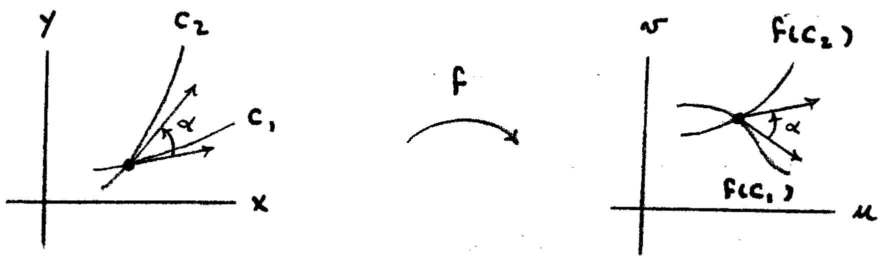


ANGLES DOUBLED

NOTE : $f(z) = z^2$ IS ANALYTIC EVERYWHERE AND

$f'(z) = 0 \iff z = 0$

A MAPPING $f(z)$ ON A DOMAIN D IS SAID TO BE CONFORMAL ON D IF IT PRESERVES THE ANGLE (BOTH MAGNITUDE AND ORIENTATION) BETWEEN ANY TWO INTERSECTING CURVES IN D.



THEOREM : $f(z)$ IS CONFORMAL ON $D \iff f(z)$ IS ANALYTIC ON D AND $f'(z)$ IS NONZERO ON D.

NOTE : WE WILL NOT PROVE THIS HERE.

E.G., $f(z) = z^2$ PRESERVES ANGLES EVERYWHERE EXCEPT AT $z = 0$ (WHERE IT DOUBLES THEM).

CONFORMAL MAPPINGS HAVE MANY APPLICATIONS (E.G., TO ELECTROSTATICS, STEADY STATE TEMPERATURE DISTRIBUTIONS, FLUID FLOW, ETC.) ALL OF WHICH REVOLVE AROUND BOUNDARY VALUE PROBLEMS FOR THE LAPLACE EQUATION (FINDING HARMONIC FUNCTIONS SATISFYING VARIOUS SPECIFIED CONDITIONS ON THE BOUNDARY OF SOME REGION). IN ORDER TO EXHIBIT

ONE SIMPLE EXAMPLE WE WILL STATE ONE MORE THEOREM WITHOUT PROOF.

THEOREM : SUPPOSE THAT THE ANALYTIC FUNCTION

$$f(z) = u(x,y) + i\,v(x,y)$$

MAPS A DOMAIN D_z OF THE z-PLANE ONTO A DOMAIN D_w OF THE w-PLANE. IF $h(u,v)$ IS HARMONIC ON D_w , THEN

$$H(x,y) = h(u(x,y), v(x,y))$$

IS HARMONIC ON D_z. IF, IN ADDITIONAL, $f(z)$ IS CONFORMAL ON D_z AND C IS SOME SMOOTH CURVE IN D_z WITH IMAGE $f(C)$ IN D_w AND $h(u,v)$ IS CONSTANT ON $f(C)$

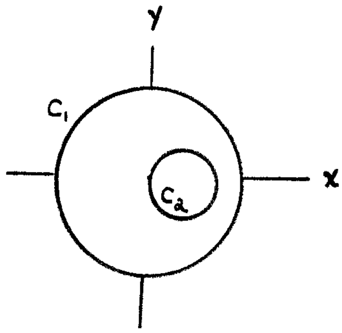
$$h = h_0 \quad \text{ON } f(C)$$

THEN H TAKES THE SAME CONSTANT VALUE ON C

$$H = h_0 \quad \text{ON } C.$$

SAMPLE APPLICATION :

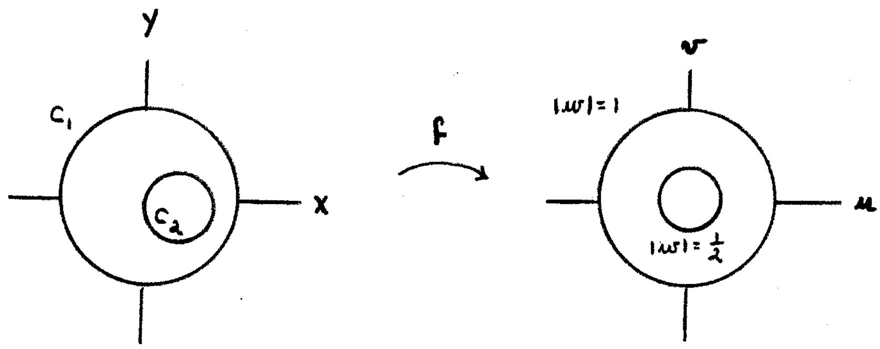
WE CONSIDER TWO CHARGED CYLINDERS C_1 ($|z| = 1$) AND C_2 ($|z - \frac{2}{5}| = \frac{2}{5}$), THE FIRST OF WHICH IS GROUNDED (HELD AT POTENTIAL $U_1 = 0$) AND THE SECOND HELD AT POTENTIAL U_2 = 110 VOLTS



THE PROBLEM IS TO FIND THE ELECTROSTATIC POTENTIAL $H(x,y)$ BETWEEN THE TWO CYLINDERS, I.E., TO SOLVE THE BOUNDARY VALUE PROBLEM

$$\begin{cases} H_{xx} + H_{yy} = 0 & \text{BETWEEN } C_1 \text{ AND } C_2 \\ H = 0 \text{ ON } C_1 \\ H = 110 \text{ ON } C_2 \end{cases}$$

EXECISE 26 SHOWS THAT THE MAPPING $f(z) = \dfrac{2z-1}{z-2}$ IS ANALYTIC ON THE REGION BETWEEN C_1 AND C_2, MAPS C_1 ONTO $|w| = 1$ AND MAPS C_2 ONTO $|w| = \frac{1}{2}$. SINCE $f'(z) = \dfrac{-3}{(z-2)^2}$, $f(z)$ IS ALSO CONFORMAL.



OVER HERE THE PROBLEM

$$\begin{cases} h_{xx} + h_{yy} = 0 \text{ BETWEEN } |w|=1 \text{ AND } |w| = \frac{1}{2} \\ h = 0 \text{ ON } |w| = 1 \\ h = 110 \text{ ON } |w| = \frac{1}{2} \end{cases}$$

IS EASY TO SOLVE. THE RADIALLY SYMMETRIC HARMONIC FUNCTION

$$a \ln |w| + b$$

SATISFIES THE BOUNDARY CONDITIONS IF $b = 0$ AND $a = -110/\ln 2$ SO

$$h(u,v) = -\frac{110}{\ln 2} \ln \sqrt{u^2+v^2}$$

SINCE

$$f(z) = \frac{2z-1}{z-2} = u(x,y) + i\, v(x,y)$$

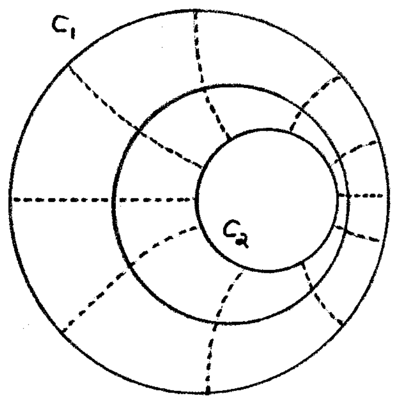
THE PREVIOUS THEOREM GIVES AS THE SOLUTION TO OUR ORIGINAL PROBLEM AS

$$H(x,y) = h(u(x,y), v(x,y))$$

$$= -\frac{110}{\ln 2} \ln \sqrt{(u(x,y))^2 + (v(x,y))^2}$$

$$= -\frac{110}{\ln 2} \ln \left| \frac{2z-1}{z-2} \right|$$

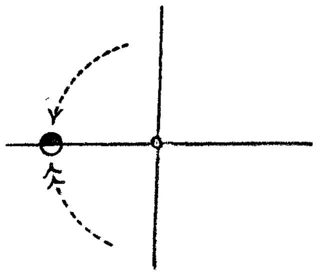
ONE CAN VISUALIZE THE POTENTIAL THROUGH ITS EQUIPOTENTIAL CURVES $H(x,y)$ = CONSTANT AND LINES OF FORCE (ORTHOGONAL TRAJECTORIES OF THE EQUIPOTENTIAL CURVES). THE FORMER ARE CIRCLES SURROUNDING C_2 AND THE LATTER ARE CIRCULAR ARCS (SHOWN DASHED BELOW).

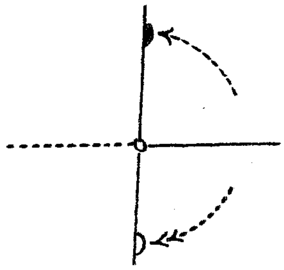


SOLUTIONS TO THE EXERCISES :

39. PRINCIPAL BRANCH OF THE SQUARE ROOT FUNCTION :

$$\sqrt{z} = z^{\frac{1}{2}} = (re^{i\theta})^{\frac{1}{2}} = \sqrt{r}\, e^{i\theta/2} \quad , \; r>0, \; -\pi < \theta \leq \pi$$





POINTS NEAR THOSE ON THE NEGATIVE REAL AXIS ARE NOT SENT TO "NEARBY" POINTS.

40. TWO POSSIBLE SOLUTIONS :

(a) $\cosh z = \frac{1}{2}(e^{z}+e^{-z})$ IS ENTIRE BECAUSE e^{z} IS ENTIRE AND $(\cosh z)' = \frac{1}{2}\left((e^{z})' + (e^{-z})'\right)$

$$= \frac{1}{2}(e^{z}-e^{-z}) = \sinh z$$

(b) $$\cosh z = \underbrace{\cosh x \cos y}_{u(x,y)} + i\, \underbrace{\sinh x \sin y}_{v(x,y)}$$

$u_x = \sinh x \cos y$ $\qquad$ $v_x = \cosh x \sin y$

$u_y = -\cosh x \sin y$ $\qquad$ $v_y = \sinh x \cos y$

(CONTINUOUS EVERYWHERE)

$u_x = v_y$ AND $u_y = -v_x$

EVERYWHERE

41. $f(z) = z\,\mathrm{Im}(z) = (x+yi)y = xy + y^2 i$

$u(x,y) = xy \qquad v(x,y) = y^2$

$u_x = y \qquad v_x = 0$

$u_y = x \qquad v_y = 2y$

(CONTINUOUS EVERYWHERE)

$u_x = v_y \Rightarrow y = 2y \Rightarrow y = 0$

$u_y = -v_x \Rightarrow x = 0$

CAUCHY-RIEMANN SATISFIED ONLY AT $z = 0+0i = 0$ AND

$f'(0) = u_x(0,0) + i v_x(0,0) = 0 + 0i = 0$

42. $f(z) = e^{\bar{z}} = e^{x-yi} = e^x \cos(-y) + i e^x \sin(-y)$

$= e^x \cos y - i e^x \sin y$

$u(x,y) = e^x \cos y \qquad v(x,y) = -e^x \sin y$

$u_x = e^x \cos y \qquad v_x = -e^x \sin y$

$u_y = -e^x \sin y \qquad v_y = -e^x \cos y$

$u_x = v_y \Rightarrow e^x \cos y = -e^x \cos y \Rightarrow \cos y = -\cos y$

$\Rightarrow \cos y = 0 \Rightarrow y = \frac{\pi}{2} + 2k\pi$

$u_y = -v_x \Rightarrow -e^x \sin y = e^x \sin y \Rightarrow -\sin y = \sin y$

$\Rightarrow \sin y = 0 \Rightarrow y = k\pi$

NO y CAN SATISFY BOTH OF THESE SO THE CAUCHY-RIEMANN EQUATIONS ARE NEVER SATISFIED.

43. $f(z) = u(x,y) + i v(x,y) = u(r,\theta) + i v(r,\theta)$

$$f'(z) = u_x + i v_x$$

$$u_r = \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_x \cos\theta + u_y \sin\theta$$

$$u_\theta = \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = u_x(-r\sin\theta) + u_y(r\cos\theta)$$

NOW SOLVE

$$\begin{cases} u_x \cos\theta + u_y \sin\theta = u_r \\ u_x(-r\sin\theta) + u_y(r\cos\theta) = u_\theta \end{cases}$$

FOR u_x AND u_y TO GET

$$u_x = u_r \cos\theta - \frac{1}{r} u_\theta \sin\theta$$

$$u_y = u_r \sin\theta + \frac{1}{r} u_\theta \cos\theta$$

CHANGING u TO v GIVES SIMILAR FORMULAS FOR v_x AND v_y. THUS,

$$f'(z) = u_x + i v_x$$

$$= \left(u_r \cos\theta - \frac{1}{r} u_\theta \sin\theta\right) + i\left(v_r \cos\theta - \frac{1}{r} v_\theta \sin\theta\right)$$

$$= (u_r \cos\theta + v_r \sin\theta) + i(v_r \cos\theta - u_r \sin\theta)$$

BY THE CAUCHY-RIEMANN EQUATIONS

$$= (\cos\theta - i\sin\theta)(u_r + i v_r)$$

$$= e^{-\theta i}(u_r + i v_r)$$

44. PRINCIPAL BRANCH OF THE SQUARE ROOT FUNCTION:

$$\sqrt{z} = \sqrt{r}\, e^{i\theta/2}$$

$$= \sqrt{r}\cos\frac{\theta}{2} + i\sqrt{r}\sin\frac{\theta}{2} \qquad , r>0, -\pi<\theta<\pi$$

$$u(r,\theta) = r^{\frac{1}{2}}\cos\frac{\theta}{2} \qquad v(r,\theta) = r^{\frac{1}{2}}\sin\frac{\theta}{2}$$

$$u_r = \frac{1}{2}r^{-\frac{1}{2}}\cos\frac{\theta}{2} \qquad v_r = \frac{1}{2}r^{-\frac{1}{2}}\sin\frac{\theta}{2}$$

$$u_\theta = -\frac{1}{2}r^{\frac{1}{2}}\sin\frac{\theta}{2} \qquad v_\theta = \frac{1}{2}r^{\frac{1}{2}}\cos\frac{\theta}{2}$$

(CONTINUOUS ON $r>0, -\pi<\theta<\pi$)

$$\frac{1}{r}v_\theta = \frac{1}{2}r^{-\frac{1}{2}}\cos\frac{\theta}{2} = u_r$$

$$-r\,v_r = -\frac{1}{2}r^{\frac{1}{2}}\sin\frac{\theta}{2} = u_\theta$$

SO THE FUNCTION IS ANALYTIC ON THE GIVEN DOMAIN. MOREOVER, THE DERIVATIVE IS GIVEN (BY EXERCISE 43) BY

$$e^{-\theta i}(u_r + i v_r) = e^{-\theta i}\left(\frac{1}{2}r^{-\frac{1}{2}}\cos\frac{\theta}{2} + i\,\frac{1}{2}r^{-\frac{1}{2}}\sin\frac{\theta}{2}\right)$$

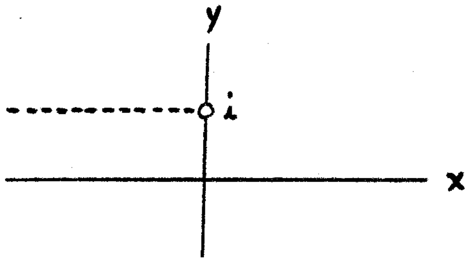
$$= \frac{1}{2}r^{-\frac{1}{2}}e^{-\theta i}\, e^{i\theta/2}$$

$$= \frac{1}{2}r^{-\frac{1}{2}}e^{-i\theta/2}$$

$$= \frac{1}{2(re^{i\theta})^{1/2}}$$

$$= \frac{1}{2\sqrt{z}} \qquad \text{(PRINCIPAL BRANCH)}$$

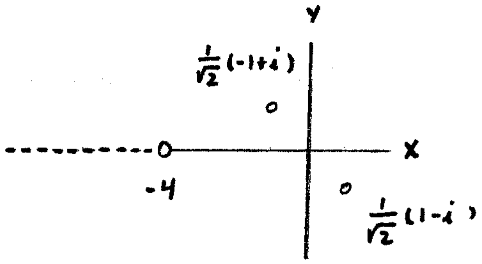
45. (a) $f(z) = \log(z - i) = \log(x + (y-1)i)$ IS ANALYTIC EXCEPT WHEN $x + (y-1)i$ IS ON THE NON-POSITIVE REAL AXIS, I.E., EXCEPT WHEN $y = 1$ AND $x \le 0$.



(b) $f(z) = \dfrac{\log(z+4)}{z^2 + i}$ IS ANALYTIC EXCEPT WHERE

(1) $z^2 + i = 0$, I.E., $z = \pm \frac{1}{\sqrt{2}}(1 - i)$

(2) $z + 4 = (x+4) + yi$ IS ON THE NON-POSITIVE REAL AXIS, I.E., EXCEPT WHEN $y = 0$ AND $x \le -4$



46. $f(z) = u(r, \theta) + i\,v(r, \theta)$

CAUCHY-RIEMANN : $u_r = \frac{1}{r} v_\theta \qquad u_\theta = -r v_r$

$$r u_r = v_\theta \qquad u_{\theta\theta} = \frac{\partial}{\partial \theta}(-r v_r)$$

$$= -r v_{r\theta}$$

$$u_{rr} = \frac{\partial}{\partial r}(r^{-1} v_\theta)$$

$$= r^{-1} v_{\theta r} - r^{-2} v_\theta$$

$$r^2 u_{rr} = r v_{\theta r} - v_\theta = r v_{r\theta} - v_\theta = -u_{\theta\theta} - r u_r$$

$$\Rightarrow \quad r^2 u_{rr} + r u_r + u_{\theta\theta} = 0 \quad \text{(SIMILARLY FOR } v\text{)}$$

47. $u(x,y) = 2x(1-y) = 2x - 2xy$

$u_x = 2-2y \qquad u_y = -2x$

$u_{xx} = 0 \qquad u_{yy} = 0 \qquad \Rightarrow u_{xx} + u_{yy} = 0$ EVERYWHERE

HARMONIC CONJUGATE : $v(x,y)$

$$v_y = u_x = 2-2y \Rightarrow v(x,y) = 2y - y^2 + C(x)$$
$$\Rightarrow v_x = C'(x)$$
$$v_x = -u_y = 2x \Rightarrow C'(x) = 2x \Rightarrow C(x) = x^2 + C$$
$$\Rightarrow v(x,y) = 2y - y^2 + x^2 + C$$

TAKE $C = 0$. THEN

$$f(z) = u(x,y) + i\,v(x,y)$$
$$= 2x(1-y) + i(2y - y^2 + x^2)$$

IS ENTIRE.

48. $u(x,y) = e^{x^2-y^2}\cos(2xy)$: $e^z = e^x\cos y + i e^x \sin y$

$$z^2 = (x^2-y^2) + 2xyi$$
$$\Rightarrow e^{z^2} = e^{x^2-y^2}\cos 2xy + i e^{x^2-y^2}\sin 2xy$$

IS ENTIRE SO ITS REAL PART

$$e^{x^2-y^2}\cos 2xy$$

IS HARMONIC EVERYWHERE AND

$$e^{x^2-y^2}\sin 2xy$$

IS A HARMONIC CONJUGATE.

49. u HARMONIC AND v A HARMONIC CONJUGATE $\Rightarrow u + iv$ ANALYTIC $\Rightarrow (u+iv)^2$ ANALYTIC. BUT $(u+iv)^2 = (u^2 - v^2) + 2uvi$ SO $u^2 - v^2$ IS HARMONIC.