

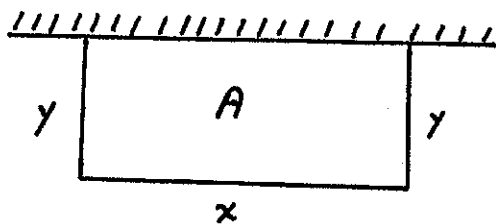
APPLIED MAXIMUM-MINIMUM PROBLEMS :

I'LL TRY TO ILLUSTRATE THE GENERAL FEATURES OF THESE PROBLEMS AND SOME RULES OF THUMB FOR SOLVING THEM WITH A SIMPLE EXAMPLE. THEN WE'LL GO THROUGH A NUMBER OF MORE INVOLVED EXAMPLES.

100 FT OF FENCING IS AVAILABLE WITH WHICH TO ENCLOSE A RECTANGULAR GARDEN. ONE SIDE OF THE GARDEN IS TO BE ALONG THE SIDE OF A HOUSE AND THEREFORE REQUIRES NO FENCING. HOW SHOULD THE DIMENSIONS OF THE GARDEN BE CHOSEN IN ORDER TO MAXIMIZE THE ENCLOSED AREA ?

GENERAL FEATURES : CHOOSE SOMETHING (E.G., DIMENSIONS) IN ORDER TO MAXIMIZE OR MINIMIZE SOMETHING ELSE (E.G., AREA).

PROCEDURE : DRAW A PICTURE (IF POSSIBLE) AND INTRODUCE NAMES FOR THE THINGS YOU ARE TO CHOOSE AND THE THING TO BE MAXIMIZED OR MINIMIZED.



$x$  = LENGTH OF GARDEN

$y$  = WIDTH OF GARDEN

$A$  = AREA OF GARDEN

WRITE THE THING TO BE MAXIMIZED/MINIMIZED IN TERMS OF THE THINGS TO BE CHOSEN :

$$A = xy$$

USE ANY CONSTRAINT IMPOSED ON THE WAY YOU CAN MAKE YOUR CHOICE (E.G., ONLY 100 FT OF FENCING AVAILABLE) TO ELIMINATE ALL BUT ONE OF THE "CHOICE" VARIABLES.

$$x + y + y = 100$$

$$x + 2y = 100$$

$$y = 50 - \frac{1}{2}x$$

$$A = x(50 - \frac{1}{2}x)$$

$$A(x) = 50x - \frac{1}{2}x^2$$

THE RESULT IS A FUNCTION OF ONE VARIABLE.

IF THE PROBLEM IMPOSES A RESTRICTION ON VALUES OF THE CHOICE VARIABLE, WRITE IT AS AN INTERVAL.

$$0 \leq x \leq 100$$

NOTE : IT MAY SEEM SILLY TO INCLUDE  $x=0$  AND  $x=100$  SINCE BOTH GIVE AN AREA OF 0, BUT A CLOSED INTERVAL IS ALWAYS PREFERABLE SINCE ABSOLUTE MAXIMA AND MINIMA ALWAYS EXIST.

NOW FIND THE MAXIMUM/MINIMUM VALUE OF THIS FUNCTION ON THIS INTERVAL.

MAXIMIZE  $A(x) = 50x - \frac{1}{2}x^2$  ON  $[0, 100]$ .

CRITICAL POINTS :  $A'(x) = 50 - x$

$$x = 50$$

$$A(50) = 50(50) - \frac{1}{2}(50)^2$$

ENDPOINTS :

$A(50) = 1250$
$A(0) = 0$
$A(100) = 0$

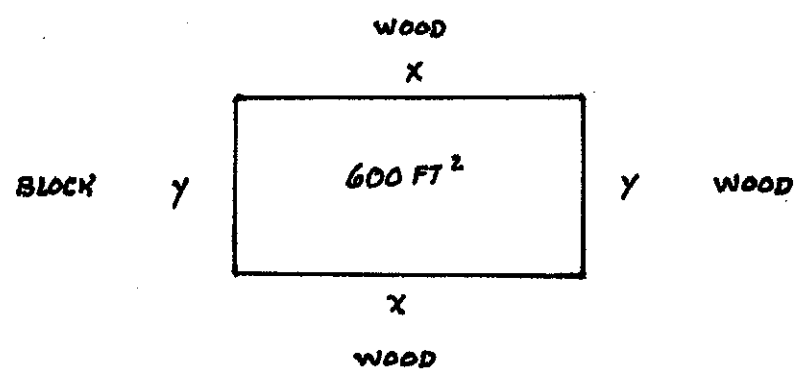
MAXIMUM OF  $1250 \text{ FT}^2$  WHEN

$$x = 50 \text{ FT}$$

$$y = 50 - \frac{1}{2}(50) = 25 \text{ FT}$$

MORE EXAMPLES :

1. A RECTANGULAR ENCLOSURE IS TO BE CONSTRUCTED WITH AREA  $600 \text{ FT}^2$ . THREE SIDES ARE TO BE MADE OF WOOD (COST :  $\$7.00$  PER FT ) AND THE FOURTH SIDE IS TO BE MADE OF CEMENT BLOCK (COST :  $\$14.00$  PER FT ). WHAT DIMENSIONS WILL MINIMIZE THE COST ?



$C = \text{COST}$

$C = 7x + 7x + 7y + 14y = 14x + 21y$

CONSTRAINT :  $xy = 600$   
 $y = \frac{600}{x}$

THUS,

$C = 14x + 21 \left( \frac{600}{x} \right)$

$C(x) = 14x + \frac{12,600}{x}$

RESTRICTION :  $x > 0$

MINIMIZE  $C(x) = 14x + \frac{12,600}{x}$  ON  $(0, \infty)$

NOTE THAT  $\lim_{x \rightarrow \infty} (14x + \frac{12,600}{x}) = \infty$  AND  $\lim_{x \rightarrow 0^+} (14x + \frac{12,600}{x}) = \infty$

SO THERE IS AN ABSOLUTE MINIMUM ON  $(0, \infty)$  THAT MUST OCCUR AT A CRITICAL POINT.

$C'(x) = 14 - \frac{12,600}{x^2}$

DEFINED EVERYWHERE ON  $(0, \infty)$ .  $C'(x) = 0$  WHEN

$14 - \frac{12,600}{x^2} = 0$

$$x^2 = \frac{12,600}{14} = 900$$

$$x = 30 \quad (\text{SINCE } x > 0)$$

MINIMUM COST :

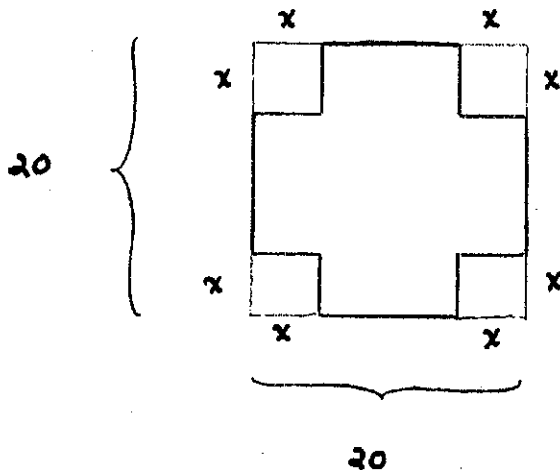
$$C(30) = 14(30) + \frac{12,600}{30} = \$840$$

OPTIMAL DIMENSIONS :

$$x = 30 \text{ FT}$$

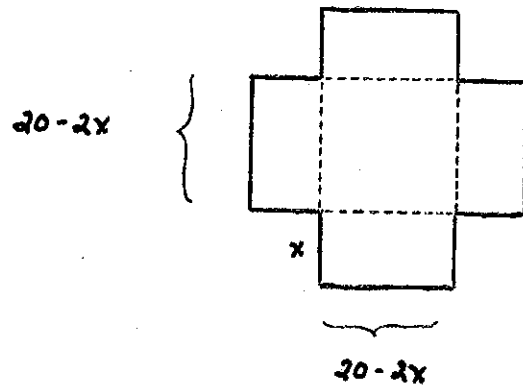
$$y = \frac{600}{30} = 20 \text{ FT}$$

2. A SQUARE SHEET OF CARDBOARD 20 IN ON A SIDE IS TO BE USED TO MAKE AN OPEN TOP BOX BY CUTTING A SMALL SQUARE FROM EACH CORNER AND THEN FOLDING THE SIDES UP. HOW LARGE A SQUARE SHOULD BE CUT FROM EACH CORNER TO MAXIMIZE THE VOLUME OF THE RESULTING BOX ?



$x$  = SIDE LENGTH OF  
THE SQUARE CUT OUT

$V$  = VOLUME OF THE  
RESULTING BOX



FOLD FLAPS UP ALONG  
THE DOTTED LINES

$$V = (\text{LENGTH})(\text{WIDTH})(\text{HEIGHT})$$

$$= (20-2x)(20-2x)(x)$$

$$V(x) = 4x^3 - 80x^2 + 400x$$

$$0 \leq x \leq 10$$

$$V'(x) = 12x^2 - 160x + 400 = 4(3x^2 - 40x + 100)$$

$$= 4(3x-10)(x-10)$$

CRITICAL POINTS IN  $(0 < x < 10)$  :  $x = \frac{10}{3}$

$$V\left(\frac{10}{3}\right) = \left(20 - \frac{20}{3}\right)\left(20 - \frac{20}{3}\right)\left(\frac{10}{3}\right)$$

$$V\left(\frac{10}{3}\right) = \frac{16,000}{9}$$

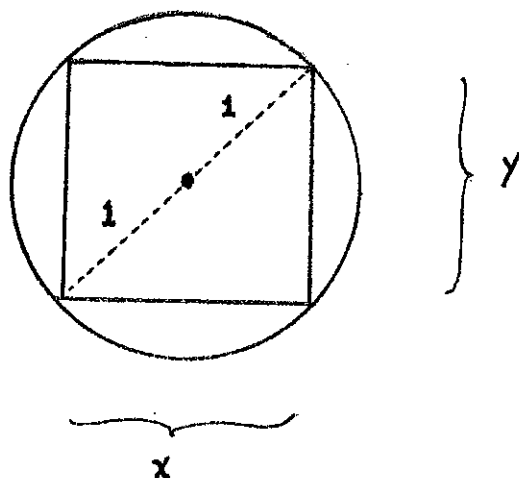
ENDPOINTS :

$$V(0) = 0$$

$$V(10) = 0$$

MAXIMUM VOLUME OF  $\frac{16,000}{9}$  IN<sup>3</sup> WHEN  $x = \frac{10}{3}$  IN

3. (SAWILL PROBLEM) FIND THE RECTANGLE OF MAXIMAL AREA THAT CAN BE INSCRIBED IN A CIRCLE OF RADIUS 1.



$x$  = LENGTH OF RECTANGLE

$y$  = WIDTH OF RECTANGLE

$A$  = AREA OF RECTANGLE

$$A = xy$$

PYTHAGOREAN THEOREM GIVES

$$x^2 + y^2 = 2^2$$

$$y = \sqrt{4 - x^2}$$

$$A(x) = x\sqrt{4 - x^2}$$

$$0 \leq x \leq 2$$

CRITICAL POINTS IN  $(0, 2)$  :

$$\begin{aligned} A'(x) &= (x(4-x^2)^{\frac{1}{2}})' = x\left(\frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)\right) + (4-x^2)^{\frac{1}{2}} \\ &= -x^2(4-x^2)^{-\frac{1}{2}} + (4-x^2)^{\frac{1}{2}} \\ &= (4-x^2)^{-\frac{1}{2}}(-x^2 + (4-x^2)) \\ &= \frac{4-2x^2}{\sqrt{4-x^2}} \end{aligned}$$

$A'(x) = \frac{4-2x^2}{\sqrt{4-x^2}}$  IS DEFINED EVERYWHERE ON  $(0, 2)$  AND  
EQUALS 0 ONLY WHEN

$$4 - 2x^2 = 0$$

$$x^2 = 2$$

$$x = \sqrt{2} \quad (-\sqrt{2} \text{ IS NOT IN } (0, 2))$$

$$A(\sqrt{2}) = \sqrt{2} \sqrt{4 - (\sqrt{2})^2} = \sqrt{2} \sqrt{2}$$

$$A(\sqrt{2}) = 2$$

ENDPOINTS :

$$A(0) = 0$$

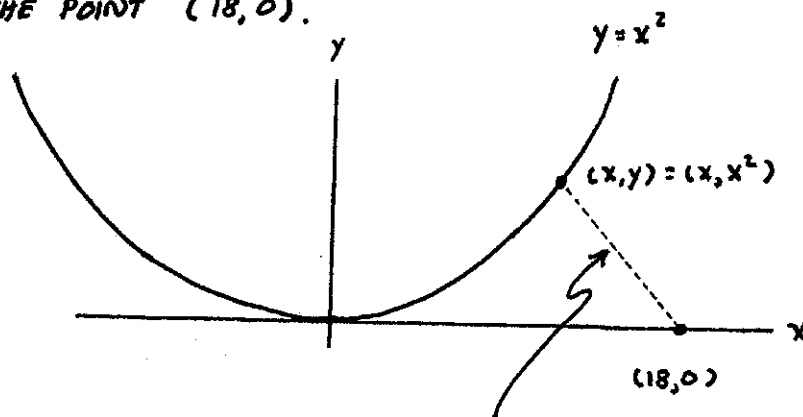
$$A(2) = 0$$

MAXIMUM AREA OF 2 WHEN

$$x = \sqrt{2}$$

$$y = \sqrt{4 - (\sqrt{2})^2} = \sqrt{2}$$

4. FIND THE POINT ON THE GRAPH OF  $y = x^2$  THAT IS CLOSEST TO THE POINT  $(18, 0)$ .



MINIMIZE THE LENGTH  $L$  OF THE LINE JOINING  $(18, 0)$  WITH A POINT ON THE GRAPH



FROM THE DISTANCE FORMULA, FOR ANY  $(x, y) = (x, x^2)$  ON THE GRAPH,

$$\begin{aligned} L &= \sqrt{(x-18)^2 + (y-0)^2} \\ &= \sqrt{(x-18)^2 + (x^2-0)^2} \\ &= \sqrt{x^4 + x^2 - 36x + 324} \end{aligned}$$

NOTE : IT WILL SIMPLIFY THE ARITHMETIC TO NOTICE THAT MINIMIZING THIS IS THE SAME AS MINIMIZING

$$f(x) = x^4 + x^2 - 36x + 324$$

SINCE  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = \infty$  THERE IS AN ABSOLUTE MINIMUM

WHICH MUST OCCUR AT A CRITICAL POINT.

$$f'(x) = 4x^3 + 2x - 36$$

$$f'(x) = 0$$

$$4x^3 + 2x - 36 = 0$$

$$2x^3 + x - 18 = 0$$

ONE SOLUTION IS EASY TO FIND BY INSPECTION :  $x = 2$

THUS,  $2x^3 + x - 18$  HAS A FACTOR OF  $x - 2$ . TO FIND THE OTHER FACTOR WE DO A DIVISION :

$$\begin{array}{r}
 2x^2 + 4x + 9 \\
 x-2 \overline{) 2x^3 + 0x^2 + x - 18} \\
 \underline{2x^3 - 4x^2} \phantom{+ x - 18} \\
 4x^2 + x \phantom{- 18} \\
 \underline{4x^2 - 8x} \phantom{- 18} \\
 9x - 18 \\
 \underline{9x - 18} \\
 0
 \end{array}$$

THUS,

$$2x^3 + x - 18 = (x-2)(2x^2 + 4x + 9)$$

SO

$$2x^3 + x - 18 = 0$$

$$(x-2)(2x^2 + 4x + 9) = 0$$

$$x-2=0$$

$$x = 2$$

$$2x^2 + 4x + 9 = 0$$

HAS ONLY COMPLEX ROOTS SINCE

$$b^2 - 4ac = 4^2 - 4(2)(9) < 0$$

CONSEQUENTLY, THERE IS ONLY ONE CRITICAL POINT, WHICH MUST GIVE THE ABSOLUTE MINIMUM.

CLOSEST POINT :

$$(x, x^2) = (2, 4)$$