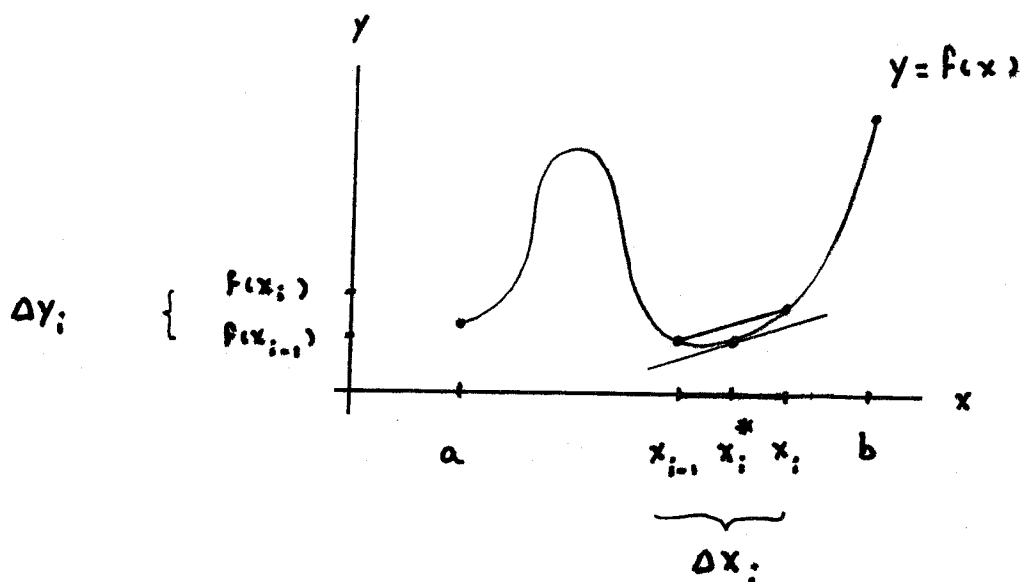


ARC LENGTH

LENGTHS OF CURVES IN THE PLANE



A SHORT ENOUGH SEGMENT IS NEARLY STRAIGHT SO ITS LENGTH CAN BE APPROXIMATED BY THE PYTHAGOREAN THEOREM.

$$\text{LENGTH OF GRAPH ABOVE } [x_{i-1}, x_i] \approx \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

WHICH WE REWRITE AS FOLLOWS :

$$\begin{aligned} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} &= \sqrt{(\Delta x_i)^2 \left(1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2\right)} \\ &= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i \\ &= \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2} \Delta x_i \\ &= \sqrt{1 + (f'(x_i^*))^2} \Delta x_i \end{aligned}$$

WHERE x_i^* IS THE POINT GUARANTEED BY THE MEAN VALUE THEOREM AT WHICH THE SLOPE OF THE TANGENT LINE ($f'(x_i^*)$) IS THE SAME AS THE SLOPE OF THE SECANT LINE JOINING THE ENDPONTS OF THE GRAPH ABOVE $[x_{i-1}, x_i]$ ($\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$).

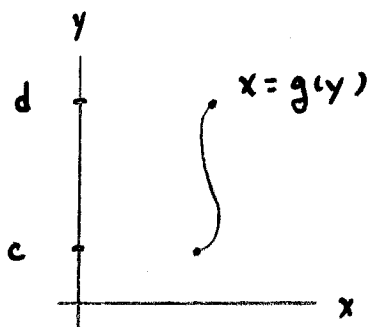
THUS,

$$\text{LENGTH OF GRAPH} \approx \sum_{i=1}^n \sqrt{1 + (f'(x_i^*))^2} \Delta x_i$$

SO

$$\begin{aligned} \text{LENGTH OF GRAPH} &= \lim_{\Delta x_{\max} \rightarrow 0} \sum_{i=1}^n \sqrt{1 + (f'(x_i^*))^2} \Delta x_i \\ &= \int_a^b \sqrt{1 + (f'(x))^2} dx \end{aligned}$$

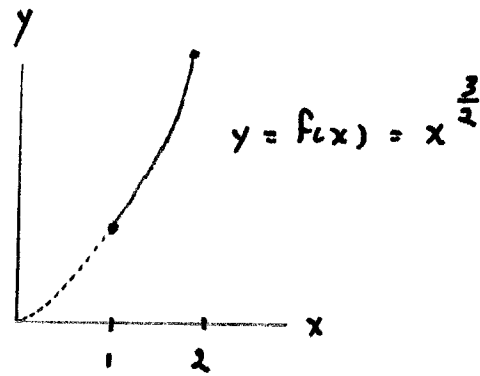
SIMILARLY FOR GRAPHS OF FUNCTIONS OF y :



$$\text{LENGTH} = \int_c^d \sqrt{1 + (g'(y))^2} dy$$

EXAMPLES :

1. COMPUTE THE LENGTH OF THE CURVE $y = x^{\frac{3}{2}}$ OVER THE INTERVAL $[1, 2]$.



$$\text{LENGTH} = \int_1^2 \sqrt{1 + (f'(x))^2} dx$$

$$f'(x) = \frac{3}{2} x^{\frac{1}{2}}$$

$$(f'(x))^2 = \frac{9}{4} x$$

$$= \int_1^2 \sqrt{1 + \frac{9}{4}x} dx = \frac{4}{9} \int_{\frac{13}{4}}^{\frac{22}{4}} u^{\frac{1}{2}} du = \frac{8}{27} u^{\frac{3}{2}} \Big|_{\frac{13}{4}}^{\frac{22}{4}}$$

$$= \frac{8}{27} \left[\left(\frac{22}{4}\right)^{\frac{3}{2}} - \left(\frac{13}{4}\right)^{\frac{3}{2}} \right]$$

$$= \frac{8}{27} \left(\frac{22\sqrt{22} - 13\sqrt{13}}{8} \right)$$

$$= \frac{22\sqrt{22} - 13\sqrt{13}}{27}$$

2. COMPUTE THE LENGTH OF $x = g(y) = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$ OVER THE INTERVAL $1 \leq y \leq 4$.

$$\text{LENGTH} = \int_1^4 \sqrt{1 + (g'(y))^2} dy$$

FIRST WE'LL COMPUTE

$$g'(y) = \left(\frac{1}{8} y^4 + \frac{1}{4} y^{-2} \right)' = \frac{1}{2} y^3 - \frac{1}{2} y^{-3}$$

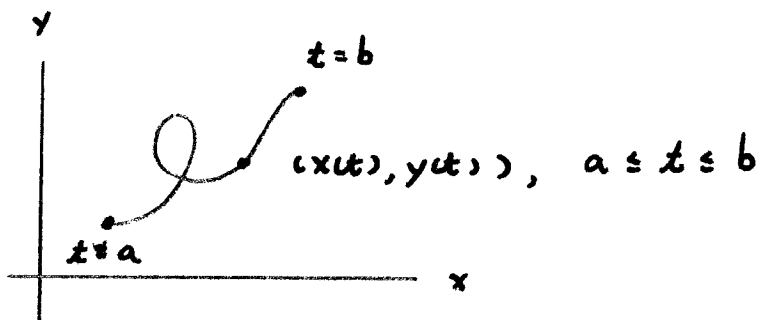
$$\begin{aligned} (g'(y))^2 &= \left(\frac{1}{2} y^3 - \frac{1}{2} y^{-3} \right)^2 = \frac{1}{4} y^6 - 2 \left(\frac{1}{2} y^3 \right) \left(\frac{1}{2} y^{-3} \right) + \frac{1}{4} y^{-6} \\ &= \frac{1}{4} y^6 - \frac{1}{2} + \frac{1}{4} y^{-6} \end{aligned}$$

$$\begin{aligned} 1 + (g'(y))^2 &= \frac{1}{4} y^6 - \frac{1}{2} + \frac{1}{4} y^{-6} + 1 \\ &= \frac{1}{4} y^6 + \frac{1}{2} + \frac{1}{4} y^{-6} \\ &= \left(\frac{1}{2} y^3 + \frac{1}{2} y^{-3} \right)^2 \end{aligned}$$

$$\sqrt{1 + (g'(y))^2} = \frac{1}{2} y^3 + \frac{1}{2} y^{-3}$$

$$\begin{aligned} \text{LENGTH} &= \int_1^4 \left(\frac{1}{2} y^3 + \frac{1}{2} y^{-3} \right) dy \\ &= \left. \frac{1}{8} y^4 \right|_1^4 - \left. \frac{1}{4} y^{-2} \right|_1^4 \\ &= \frac{1}{8} (256 - 1) - \frac{1}{4} \left(\frac{1}{16} - 1 \right) \\ &= \frac{255}{8} - \frac{1}{4} \left(-\frac{15}{16} \right) = \frac{255}{8} + \frac{15}{64} \\ &= \frac{2040}{64} + \frac{15}{64} = \frac{2055}{64} \end{aligned}$$

SOMETIMES A CURVE IN THE xy -PLANE IS DESCRIBED PARAMETRICALLY BY GIVING THE x AND y COORDINATES OF A POINT ON THE CURVE AS FUNCTIONS OF SOME PARAMETER t (THINK OF t AS TIME AND THE CURVE AS THE PATH OF AN OBJECT MOVING IN THE PLANE).



ARGUMENTS JUST LIKE THOSE GIVEN EARLIER SHOW

$$\text{LENGTH} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

EXAMPLE : $(x(t), y(t)) = (a \cos t, a \sin t)$, $0 \leq t \leq 2\pi$

(WITH $a > 0$ A CONSTANT) DESCRIBES A CIRCLE OF RADIUS a

BECAUSE

$$\begin{aligned} (x(t))^2 + (y(t))^2 &= (a \cos t)^2 + (a \sin t)^2 \\ &= a^2 \cos^2 t + a^2 \sin^2 t \\ &= a^2 (\cos^2 t + \sin^2 t) \\ &= a^2 \end{aligned}$$

IT'S LENGTH IS

$$\begin{aligned} &\int_0^{2\pi} \sqrt{((a \cos t)')^2 + ((a \sin t)')^2} dt = \int_0^{2\pi} \sqrt{(-a \sin t)^2 + (a \cos t)^2} dt \\ &= \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} dt = \int_0^{2\pi} \sqrt{a^2 (\sin^2 t + \cos^2 t)} dt \\ &= \int_0^{2\pi} a dt = a t \Big|_0^{2\pi} = 2\pi a \end{aligned}$$

AS EXPECTED.