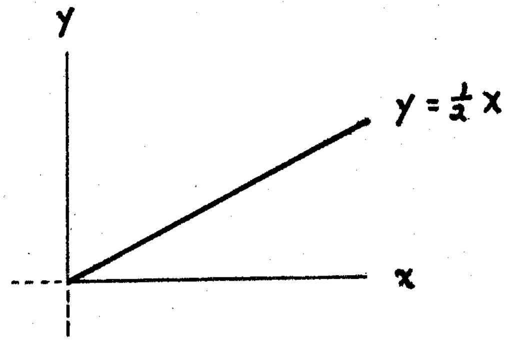
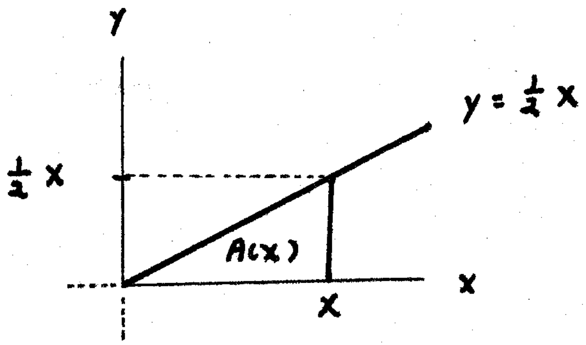
AREAS AND ANTIDERIVATIVES

THINK ABOUT THE FUNCTION $f(x) = \frac{1}{2}x$ ON THE INTERVAL $[0, \infty)$.



FOR ANY x IN $[0, \infty)$ LET

$A(x)$ = AREA UNDER THE GRAPH OF $f(x) = \frac{1}{2}x$ FROM 0 TO x



$$A(x) = \tfrac{1}{2}(\text{BASE})(\text{HEIGHT}) = \tfrac{1}{2}(x)(\tfrac{1}{2}x)$$

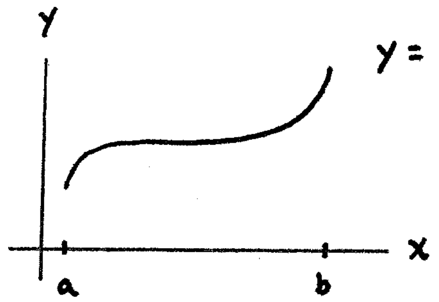
$$A(x) = \tfrac{1}{4}x^2$$

NOW, NOTICE SOMETHING PECULIAR :

$$A'(x) = (\tfrac{1}{4}x^2)' = \tfrac{1}{2}x = f(x)$$

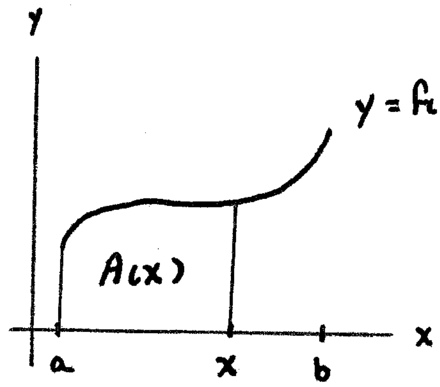
THE MOST "PECULIAR" THING ABOUT THIS IS THAT IT ISN'T REALLY PECULIAR AT ALL. IT HAPPENS ESSENTIALLY ALL THE TIME, AS I WILL NOW SHOW YOU :

LET $f(x)$ BE CONTINUOUS AND ≥ 0 ON THE INTERVAL $[a, b]$.

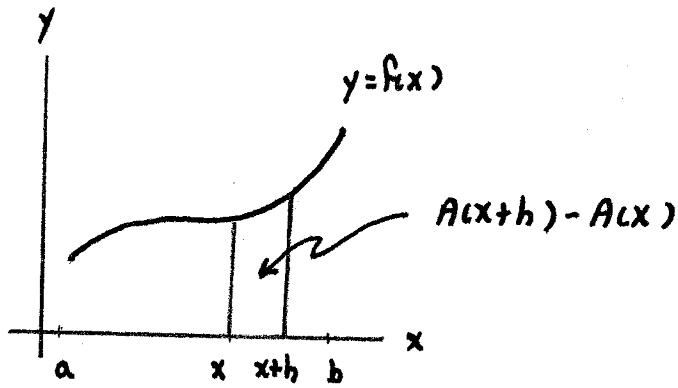


FOR ANY x IN $[a, b]$ LET

$A(x)$ = AREA UNDER THE GRAPH OF $f(x)$ FROM a TO x



NOW WE'LL "COMPUTE" THE DERIVATIVE $A'(x)$ GEOMETRICALLY.



$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

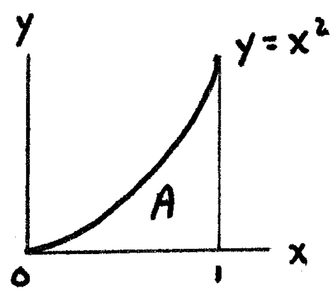
$A(x+h) - A(x) \approx f(x)h$ AND THE APPROXIMATION IMPROVES AS $h \to 0$

$$
\begin{aligned}
A'(x) &= \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x)h}{h} \\
&= \lim_{h \to 0} f(x) \\
&= f(x)
\end{aligned}
$$

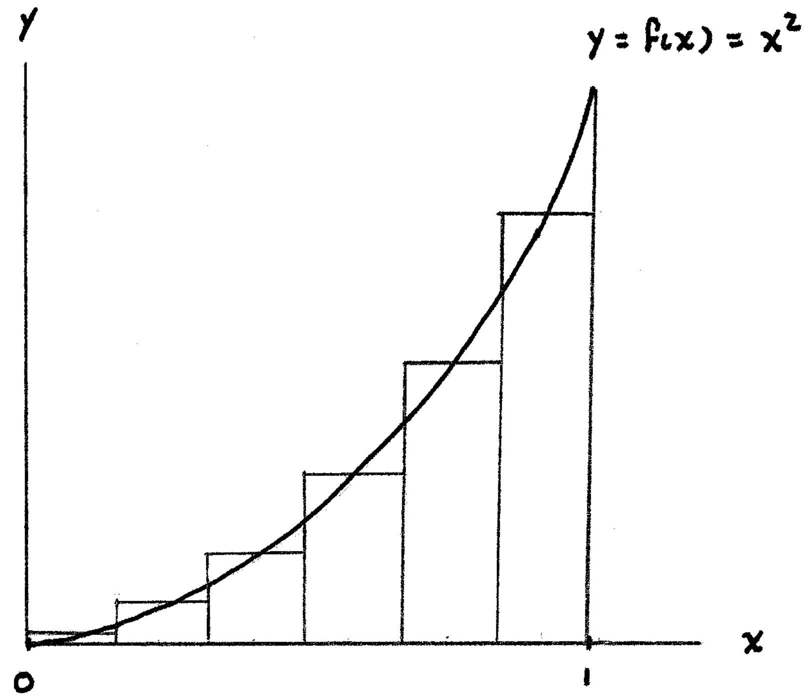
THE DERIVATIVE OF THE "AREA FUNCTION" OF $f(x)$ IS $f(x)$ ITSELF.

BECAUSE OF THIS CLOSE CONNECTION BETWEEN AREAS AND DERIVATIVES WE WILL SPEND SOME TIME DESCRIBING A DIRECT APPROACH TO THE PROBLEM OF COMPUTING SUCH AREAS.

<u>ILLUSTRATION</u>: COMPUTE THE AREA UNDER THE GRAPH OF $f(x) = x^2$ FROM $x = 0$ TO $x = 1$



THE IDEA IS VERY SIMPLE : BEGIN BY APPROXIMATING THE REGION WHOSE AREA WE WANT BY RECTANGULAR STRIPS.



THE SUM OF THE RECTANGULAR AREAS IS AN APPROXIMATION TO A.

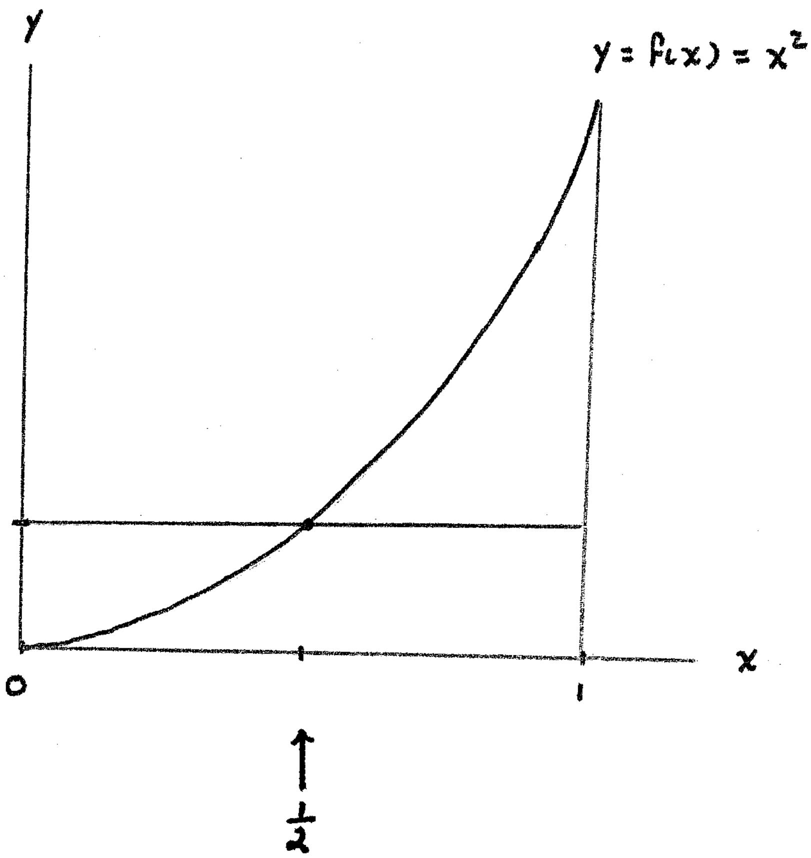
CHOOSING MORE AND MORE, SKINNIER AND SKINNIER RECTANGLES GIVES BETTER AND BETTER APPROXIMATIONS.

THE LIMIT OF THESE APPROXIMATIONS AS THE NUMBER OF RECTANGLES $\to \infty$ AND THEIR WIDTHS $\to 0$ GIVES PRECISELY THE AREA A WE WANT.

TO HAVE A SYSTEMATIC WAY OF BUILDING THESE RECTANGLES WE WILL (FOR THIS EXAMPLE) WE WILL

1. CHOOSE ALL OF THEIR BASES TO HAVE THE SAME LENGTH.
2. CHOOSE THE HEIGHT OF EACH TO BE THE VALUE OF THE FUNCTION AT THE MIDPOINT OF THE BASE (OTHER COMMON CHOICES ARE THE VALUES OF THE FUNCTION AT THE LEFT OR RIGHT HAND ENDPOINTS OF THE BASE).

1 RECTANGLE :

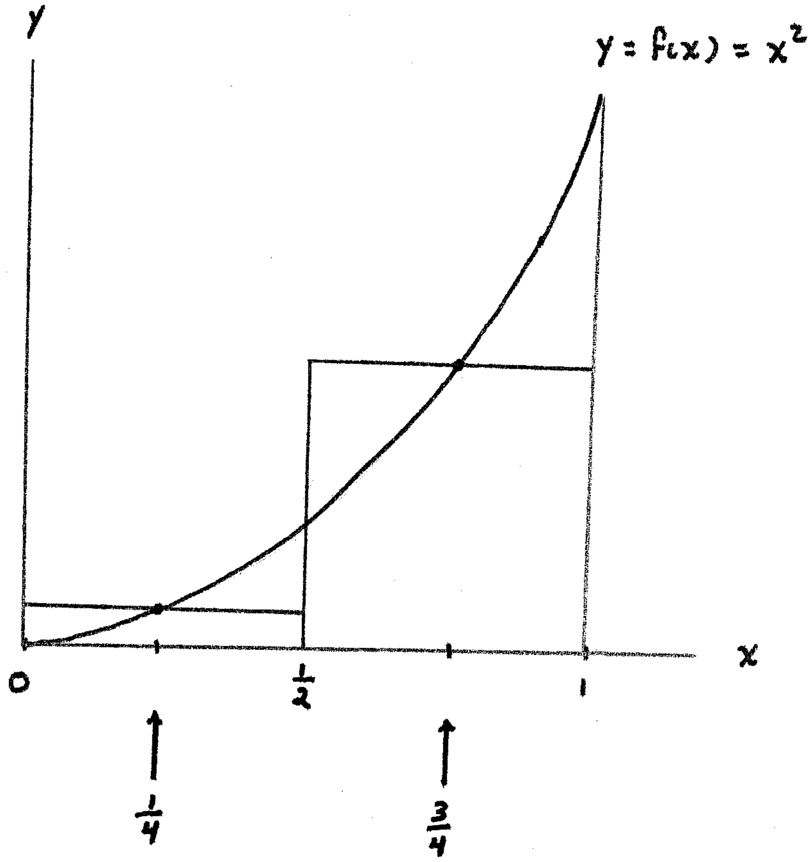


BASE : $[0,1]$

MIDPOINT : $\frac{1}{2}$

$$A \approx f(\tfrac{1}{2})\cdot 1 = (\tfrac{1}{2})^2 \cdot 1 = \tfrac{1}{4}$$

2 RECTANGLES :



BASES : $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$

MIDPOINTS : $\frac{1}{4}$, $\frac{3}{4}$

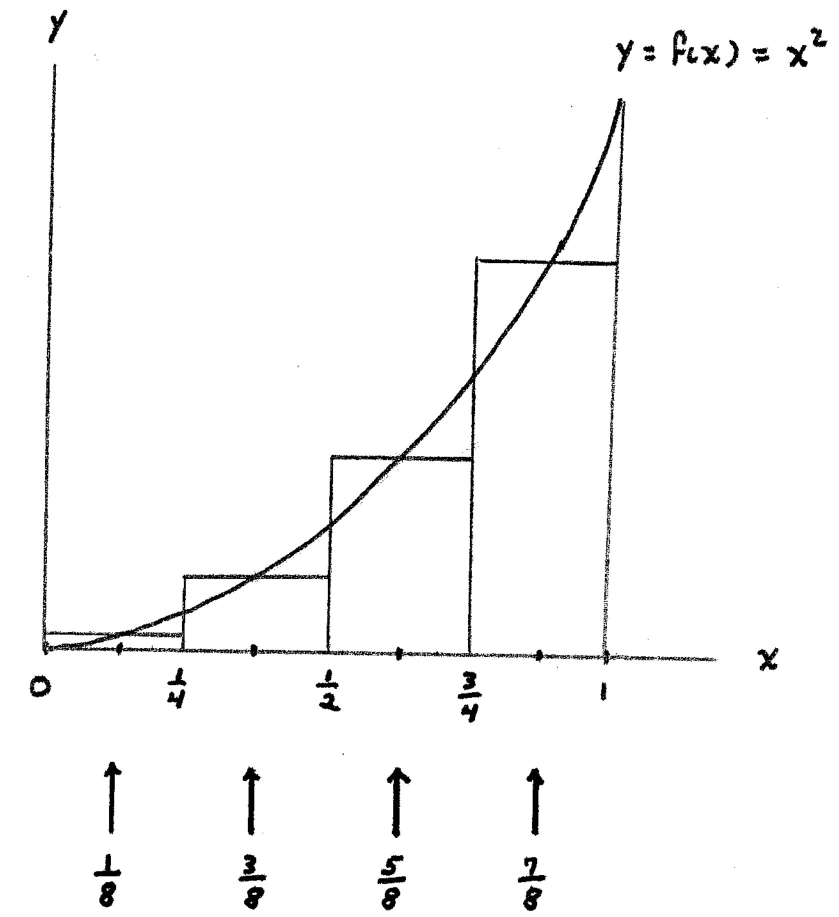
$$A \approx f(\tfrac{1}{4}) \cdot \tfrac{1}{2} + f(\tfrac{3}{4}) \cdot \tfrac{1}{2} = (\tfrac{1}{4})^2 \cdot \tfrac{1}{2} + (\tfrac{3}{4})^2 \cdot \tfrac{1}{2}$$

$$= (\tfrac{1}{16} + \tfrac{9}{16}) \cdot \tfrac{1}{2}$$

$$= \frac{5}{16}$$

$$= 0.3125$$

4 RECTANGLES :



BASES : $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$, $[\frac{3}{4}, 1]$

MIDPOINTS : $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$

$$A \approx f(\tfrac{1}{8})\cdot\tfrac{1}{4} + f(\tfrac{3}{8})\cdot\tfrac{1}{4} + f(\tfrac{5}{8})\cdot\tfrac{1}{4} + f(\tfrac{7}{8})\cdot\tfrac{1}{4} =$$

$$(\tfrac{1}{8})^2\cdot\tfrac{1}{4} + (\tfrac{3}{8})^2\cdot\tfrac{1}{4} + (\tfrac{5}{8})^2\cdot\tfrac{1}{4} + (\tfrac{7}{8})^2\cdot\tfrac{1}{4} =$$

$$(\tfrac{1}{64} + \tfrac{9}{64} + \tfrac{25}{64} + \tfrac{49}{64})\cdot\tfrac{1}{4} = \tfrac{21}{64}$$

$$= 0.3281$$

<u>NEXT</u> <u>TIME</u> : n RECTANGLES AND THE LIMIT AS $n \to \infty$