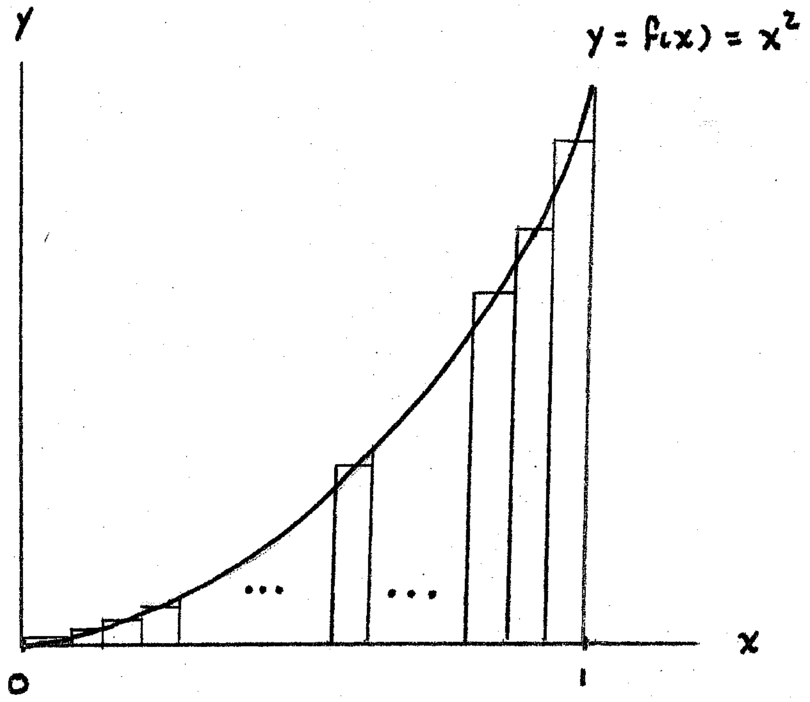
AREAS AND RIEMANN SUMS

n RECTANGLES



BASES : $[0, \frac{1}{n}]$, $[\frac{1}{n}, \frac{2}{n}]$, ... , $[\frac{n-1}{n}, 1]$

MIDPOINTS : $\frac{1}{2n}$, $\frac{3}{2n}$, ... , $\frac{2n-1}{2n}$

$$A \approx f\left(\frac{1}{2n}\right)\cdot\frac{1}{n} + f\left(\frac{3}{2n}\right)\cdot\frac{1}{n} + \cdots + f\left(\frac{2n-1}{2n}\right)\cdot\frac{1}{n} =$$

$$\left(\frac{1}{2n}\right)^2\cdot\frac{1}{n} + \left(\frac{3}{2n}\right)^2\cdot\frac{1}{n} + \cdots + \left(\frac{2n-1}{2n}\right)^2\cdot\frac{1}{n} =$$

$$\frac{1^2}{4n^2}\cdot\frac{1}{n} + \frac{3^2}{4n^2}\cdot\frac{1}{n} + \cdots + \frac{(2n-1)^2}{4n^2}\cdot\frac{1}{n} =$$

$$\left(1^2 + 3^2 + \cdots + (2n-1)^2\right)\cdot\frac{1}{4n^3}$$

E.G., IF $n = 5$,

$$A \approx (1^2+3^2+5^2+7^2+9^2)\cdot\frac{1}{4\cdot 5^3} = \frac{165}{500} = 0.3300$$

<u>FACT</u>: $1^2+3^2+\cdots+(2n-1)^2 = \dfrac{4n^3-n}{3}$

E.G., $n = 3$: $1^2+3^2+5^2 = 1+9+25 = 35$

$$\frac{4\cdot 3^3-3}{3} = \frac{108-3}{3} = \frac{105}{3} = 35$$

(THIS IS PROVED BY "MATHEMATICAL INDUCTION")

THUS,

$$A \approx (1^2+3^2+\cdots+(2n-1)^2)\frac{1}{4n^3} = \frac{4n^3-n}{3}\ \frac{1}{4n^3}$$

$$= \frac{4n^3-n}{12n^3}$$

AND THE APPROXIMATION IMPROVES AS $n \to \infty$, I.E.,

$$A = \lim_{n\to\infty} \frac{4n^3-n}{12n^3} = \lim_{n\to\infty} \frac{4-\frac{1}{n^2}}{12}$$

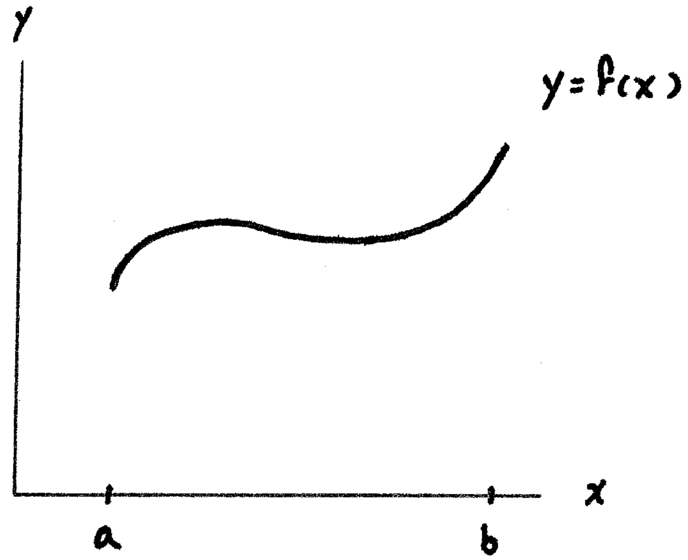
$$= \frac{4-0}{12}$$

$$= \frac{1}{3}$$

<u>CONCLUSION</u>: THE AREA UNDER THE GRAPH OF $f(x) = x^2$ FROM $x = 0$ TO $x = 1$ IS <u>PRECISELY</u> $\frac{1}{3}$.

THE GENERAL CONSTRUCTION:

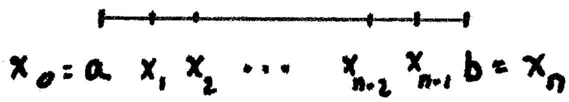
LET $f(x)$ BE CONTINUOUS AND NON-NEGATIVE ($f(x) \geq 0$) ON $[a, b]$.



TO COMPUTE THE AREA UNDER THE GRAPH OF $f(x)$ AND ABOVE THE INTERVAL $[a, b]$ WE PROCEED AS FOLLOWS:

1. SUBDIVIDE THE INTEVAL $[a, b]$ INTO n SUBINTERVALS WITH ENDPOINTS

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} < x_n = b$$



FOR EACH $i = 1, 2, \ldots, n-1, n$, LET

$$\Delta x_i = x_i - x_{i-1} = \text{LENGTH OF } [x_{i-1}, x_i]$$

NOTE : IF ALL OF THE SUBINTERVALS HAVE THE SAME LENGTH WE DENOTE ALL OF THE Δx_i SIMPLY Δx. OTHERWISE THE LARGEST OF THE Δx_i WILL BE DENOTED Δx_{max}.

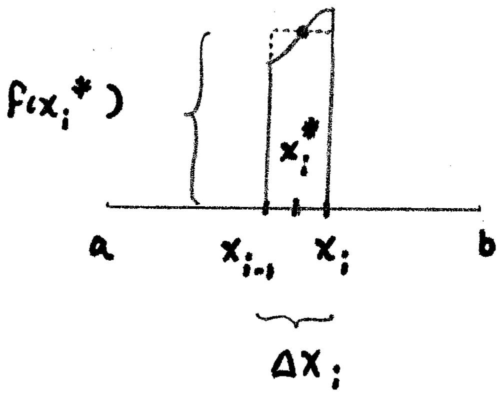
2. INSIDE EACH $[x_{i-1}, x_i]$ SELECT A POINT x_i^*

EVALUATE

$$f(x_1^*),\ f(x_2^*),\ \ldots,\ f(x_{n-1}^*),\ f(x_n^*)$$

AND COMPUTE

$$f(x_1^*)\Delta x_1,\ f(x_2^*)\Delta x_2,\ \ldots,\ f(x_{n-1}^*)\Delta x_{n-1},\ f(x_n^*)\Delta x_n$$



3. FORM THE <u>RIEMANN</u> <u>SUM</u> APPROXIMATION

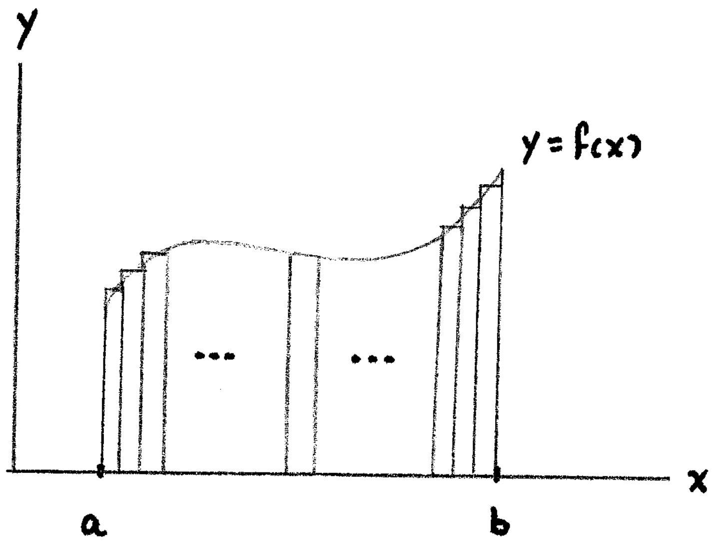
$$f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_{n-1}^*)\Delta x_{n-1} + f(x_n^*)\Delta x_n$$

$$= \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

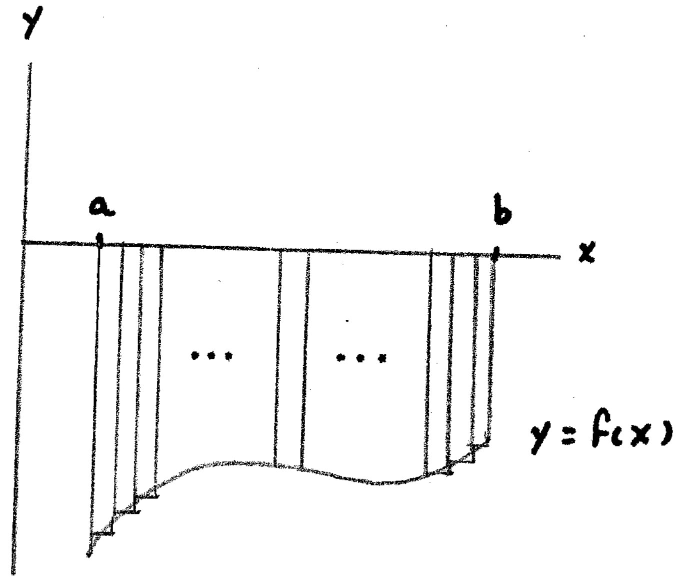
(SIGMA NOTATION : THE SUM OF ALL THE $f(x_i^*)\Delta x_i$ FOR i TAKING VALUES FROM 1 TO n)

4. REPEAT STEPS #1-3 OVER AND OVER WITH FINER AND FINER SUBDIVISIONS OF $[a,b]$ (I.E., SMALLER AND SMALLER Δx_{max}) AND TAKE THE LIMIT

$$\lim_{\Delta x_{max} \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$



NOTICE THAT IF $f(x) \leq 0$ (RATHER THAN $f(x) \geq 0$) ON $[a,b]$, THEN THE RESULT OF THIS PROCEDURE WILL BE <u>MINUS</u> THE AREA BETWEEN THE GRAPH OF $f(x)$ AND $[a,b]$.



IF $f(x)$ TAKES BOTH POSITIVE AND NEGATIVE VALUES ON $[a,b]$, THEN THE PROCEDURE YIELDS THE <u>NET SIGNED AREA</u> BETWEEN THE GRAPH OF $f(x)$ AND THE INTERVAL $[a,b]$.