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Topology, Geometry, and Gauge Fields. Interactions. *By Gregory L. Naber.* Springer-Verlag, New York, 2000. $69.95. xiii+443 pp., hardcover. ISBN 0-387-98947-1.

An Overview.

The Subject Matter. The book under review covers a wide terrain of topics in mainstream differential geometry. These consist of Lie groups, principal and fiber bundles, connections and their curvatures, spin structures, and characteristic classes. The medium in which these topics happily coexist, sometimes synergistically, is gauge theory.

The Market. There are similar books on the market, some covering more ground by including, say, the rudiments of quantum field theory, or general relativity, or the Atiyah–Singer and Atiyah–Singer–Patodi index theorems. Well-received examples include the influential work by Eguchi, Gilkey, and Hanson [EGH], the comprehensive monograph by Nakahara [Nak], and a less ambitious but no less useful volume [NS] by Charles Nash and Siddhartha Sen. The reviewer has personally used these and finds them to be excellent desk references.

What Sets Naber's Book Apart? Naber's book, together with its predecessor [N4] subtitled *Foundations*, occupies a less populated niche in the market. This is the sector of *teachable texts* on differential geometry and its use in physics. Teachability does not refer to a definition-theorem-proof format. Nor does it imply anything about the depth of the treatment. Rather, it has to do with the organization of the topics, the selection of examples, the amount of instructive details provided, the ability to anticipate questions from the reader, and knowing when to stop.

The Raison D'Être of the Book.

Is the Subject Matter of Practical Importance? Curvatures are analogous to the electromagnetic field that permeates our lives. Their behavior is governed by generalizations of the Maxwell equations. Spinors describe particles of half-integer spin and represent what we (or physical matter in general) are made of. They obey Dirac-type equations.

Lie groups provide concise descriptions of the symmetries that are often present in systems of differential equations. Bundles afford us the settings in which local solutions can be patched together systematically to produce meaningful global objects. Characteristic classes give us the key tools to decide whether two bundles carry equivalent information.

Why Is Expert Help Needed? The range of topics cited is extensive, even for people trained in differential geometry. Those who specialize in other branches of mathematics often have the curiosity to venture into this vast terrain, only to find that they have no clue where to start. The few who somehow manage to get off the ground soon encounter discouraging barriers.

For instance, one could learn all about Lie groups and bundles, but would still have no idea why $U(1)$ bundles over S^2 and $SU(2)$

bundles over S^4 are classified by the first and second Chern classes, respectively. Or, one could learn all about spinors and gauge fields and still be mystified by the terms in the Seiberg–Witten equations.

In short, the problem is one of synthesis: the components learned need to be incorporated into one framework, and the interactions among the components need to be identified. Naber's book uses gauge theory and global differential geometry as the framework, and highlights the "interactions" at every opportunity.

Why Should Applied Mathematicians Care? Consider this hypothetical scenario. One solves a system of differential equations subject to boundary conditions. Say the (family of) solutions corresponding to any given set of boundary conditions are related to each other via a well-defined collection of gauge/symmetry transformations.

Each set of boundary conditions can be thought of as a point p on some manifold M. The collection of transformations relating the solutions in the said family may be viewed as a fiber over the point p. The picture that emerges is that of a fiber bundle whose geometry can be probed with the tools of connections, curvatures, and characteristic classes. Hopefully, the geometrical information obtained will then allow us to make conclusions about the solutions of the said differential equations, thereby shedding some light on the underlying physics or engineering problem.

The study of the Integer Quantum Hall Effect provides a concrete realization of this scenario. There, the time-independent Schrödinger equation is involved; it concerns a state with a fixed quantum number. Each choice of *periodic* boundary conditions corresponds to a specific point p on the 2-torus M and gives rise to a family of solutions that we schematically list along a fiber over that point p. Each such fiber is isomorphic to the Lie group $U(1)$. The resulting $U(1)$ bundle has a first Chern class which, upon integration over M, gives a first Chern number. According to Arovas et al., whether this Chern number is nonzero dictates whether the quantum state in question can carry current. Varying the quantum number over a discrete spectrum below the Fermi energy generates a family of $U(1)$ bundles, hence a series of first Chern numbers. Thouless et al. found that the sum of these first Chern numbers, when normalized by a physical constant, is equal to the boundary-condition-averaged Hall conductance. See [SW] for references to Arovas and Thouless.

Implicit in the above story is the assumption (by the reviewer) that applied mathematicians are more inclined to be strong in differential equations than in differential geometry. An anecdote involving Chern now comes to mind. Once, before a technical talk, Professor Chern asked a young person sitting next to him whether she was working in differential geometry. "No, PDEs," came the reply. Without even hesitating, Chern said: "Oh, [but they are the] same thing."

Information Proper about the Book.

What It Is, and What It Is Not. Naber writes in a most unpretentious style. His prose is not terse like Rudin's, but not verbose either. He gives full details to all difficult calculations and shows good judgment in deciding what is difficult versus what is not. This is one way in which a writer demonstrates rapport with his/her readers. Never once has Naber omitted anything out of laziness, under the pretense that it is routine.

The book is a carefully thought out and lecture-tested account of the subject matter listed earlier. It is rigorous, with an emphasis on the details in the examples. Naber favors examples that deal with concrete spaces (spheres, $U(1)$, $SU(2)$, etc.), and revisits them whenever appropriate.

Naber's book is not an axiomatic treatment of differential geometry or gauge theory. It does not adhere rigidly to a definition-theorem-proof format. It is not the book's goal to give a heuristic historical account or an account meant for spectators. If one is not already familiar with the subject, this book is unsuitable for bedtime reading. Some active participation is expected of the reader, in order to understand the examples and to follow the general progress of the lessons.

Gems. That active participation helps uncover many gems.

On the physical side, one finds an unusually lucid treatment of the minimal coupling of a gauge field to the Schrödinger equation. There is a detailed motivation for the gamma matrices and the Dirac equation; the how and why of the latter's relativistic invariance is then given a bare-hands treatment by quoting some representation theory from [N3]. We also find an accessible account of the Higgs field, as well as the t'Hooft–Polyakov monopole and its relation to the Dirac monopole. Throughout this excursion into theoretical particle physics, the author has deftly given precise mathematical meaning to much of the standard jargon.

On the mathematical side, the reviewer is delighted by Naber's frequent visits with the Hopf fibration of S^3 and S^7, elucidating the roles they play in the Dirac magnetic monopole and the instantons, respectively. Related to this is a thorough discussion of how $U(1)$ bundles over S^2 are classified by the first Chern number, and how $SU(2)$ bundles over S^4 are classified by the second Chern number. One also finds a careful treatment of the first and second Stieffel–Whitney classes, together with their use in ascertaining orientability and the existence of spin structures.

The appendix provides a rare treat. One finds there an excellent exposition of the Seiberg–Witten equations on flat $\mathbb{R}^4$.

Is This Book a Buy? In terms of its ability to teach a subject to the novice, this book ranks right up there with many classics. The reviewer's personal favorites include *Differential Topology* by Guillemin and Pollack [GP] and *Introduction to Quantum Field Theory* by Mandl [M]. People who collect classics should consider buying this one, whether or not they plan to study it chapter by chapter.

For someone who plans to compute right along with the examples, this book is a must-buy. Naber's goal is not to teach a sterile course on geometry and topology, but rather to enable us to see the subject in action, through gauge theory. The book is capable of fulfilling this goal because of Naber's efforts. He has undertaken the arduous task of researching the broad field with its extensive literature, learning the material himself, class testing it in lectures, and agonizing over the best ways to present it. Amazingly, the fruits of his labor (and his wife's typesetting) can be had for less than $70, thanks to Springer's consumer-friendly pricing.

The reviewer strongly recommends that the purchase be made together with the *Foundations* volume [N4] to which Naber refers copiously (but accurately), for the sake of efficiency and completeness. He also hopes that Naber will continue the scholarly program of bringing exciting mathematics and physics to a level of clarity that is within our reach. Perhaps there is a third volume in the works, somewhat along the lines of [Nas]?

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MATHEMATICAL SOCIETY (INC.)

Book Reviews

Topology, Geometry and Gauge Fields
Gregory L Naber

Texts in Applied Mathematics, 25 Springer-Verlag New Yourk, 1997, 396pp, DM 78.00. ISBN 0-387-94946-1.g

Mathematics and Physics have enjoyed a close relationship over the centuries. The study of physical and geometrical problems has often been a strong motivating factor in the development of new mathematics which in turn has contributed to the understanding of physical phenomena. In the last century the two disciplines have proceeded more independently with mathematics pursuing its fascination with abstraction. Nevertheless from time to time mathematicians and physicists discover that they are following similar lines of thought and the resulting interaction enriches both disciplines. A well-known example is the development of differential geometry and general relativity. More recently the work of physicists on the problem of quantizing classical field theory using gauge fields has provided an application of the theory of fibre bundles. The two groups, working independently, used different nomenclature, but once the links were realised the resulting activity and interactions produced mathematics of great depth and beauty along with profound insights into the structure of fundamental physical theories.

This book provides the mathematics needed to begin to appreciate this amazing parallel development of ideas. Its goal is to weave together notions from the classical gauge theory of physics with the topological and geometric concepts which become the mathematical models of these notions. Essentially it presents certain aspects of topology, algebra and differential geometry which form the foundation and vocabulary for describing gauge theories.

The author begins with a Chapter 0, designed to provide the physical and geometric motivation for the abstract mathematics which follows - ``an initial aerial view of the terrain" as he says. He traces the notion of a gauge field using, as an example, Dirac's magnetic monopole and the classical quantum mechanical description of the motion of a charged particle in its field. This points to the need for a fibre bundle and a path lifting procedure to keep track of the particle's phase. We then get informal descriptions of the Hopf bundle, connections on principal bundles and non-Abelian gauge fields and the moduli space of the bundle. All of this is punctuated with references forward to the main text.

The following chapters deal with the mathematics - topological spaces, homotopy groups, principal bundles, differentiable manifolds and matrix Lie groups. Here Naber assumes only a solid background in analysis, linear algebra and some of the terminology of modern algebra. Exercises are liberally scattered through the text, gently leading the reader to assist in proofs or to explore particular examples pertinent to the understanding of the concept. When the going gets tough there are references back to the motivational chapter, or forward to assure the reader that this further leap into abstraction is exactly what will be needed at a later stage.

In some sections of the final chapter on gauge fields and instantons Naber again reverts to a less formal survey, providing more ``excursions into the murky waters of physical motivation", and outlining results which require deeper mathematics.

Inevitably the book selects from topology and geometry only those topics (and they are substantial) which are needed for its purposes though there are often signposts and references to the broader fields. It is unusual to find a book so carefully tailored to the needs of this interdisciplinary area of mathematical physics. It is very self contained, all the definitions are here along with references for any results which are not proved. It is also very readable. Naber combines a deep knowledge of his subject with an excellent informal writing style. I recommend this book for graduate students and others with interests in mathematical physics, topology or differential geometry.

Gillian Thornley Massey University

Items Authored by Naber, Gregory L.

MR1444352 (99b:58001)
Naber, Gregory L.(1-CASCH)
Topology, geometry, and gauge fields. (English. English summary)
Foundations. Texts in Applied Mathematics, 25.
Springer-Verlag, New York, 1997. xviii+396 pp. *ISBN* 0-387-94946-1
58-01 (22E15 53-01 53C07 81T13)

During the past two decades gauge theory has been the focus of intense mathematical scrutiny due mainly to its topological relevance so brilliantly exposed by S. Donaldson.

For a beginning graduate student this may look like a forbidding subject because of its amazing ramifications and reliance on knowledge in many branches of mathematics: differential and algebraic topology/geometry, partial differential equations and functional analysis, K-theory and Lie groups, to name a few. The book under review is intended to be a gentle introduction to the very basics of this subject.

To be precise, the author has no intention of discussing Donaldson theory and he is content with a more modest, yet important goal, of presenting to an interested beginner some of the language and the atmosphere involved. Gauge theory is about certain connections on principal bundles over smooth manifolds and most of the book is about presenting these objects as detailed and concretely as possible.

The reader is given a brief taste of the subject in Chapter 0, which sketches the origins and the direction of this theory. The next two chapters are topological in content, and contain the fundamentals of homotopy theory from scratch: covering spaces, the fundamental group and higher homotopy groups. The manner in which they are organized suggests that the author has in mind a reader not very familiar with point set topology. There are many concrete examples worked out in great detail which in the reviewer's view are important for the understanding of this quite intricate theory. The next chapter is devoted to topological principal bundles and, as an application of the homotopy theory developed so far, the author classifies the principal bundles over a sphere.

Chapter 4 is an introduction to differential geometry and Lie groups. To spare the reader the formalism shock, the Lie groups in this book are mostly viewed as closed subgroups of a general linear group. This not only suffices for most applications but also many abstract arguments are easier to assimilate in this light.

Many of the examples contained in this chapter are built on situations studied in the previous chapters and serve a double purpose: an academic one, of familiarizing the reader with the new concepts, and a concrete one, of building the formalism leading to Chapter 5, the stated goal of the book. This last chapter contains the explicit construction of anti-selfdual connections on a principal ${\rm SU}(2)$-bundle over the round 4-sphere. The results are described in a quaternionic language. The last part of this chapter is expository and gives the reader a sense of why such a computation might be important. The author provides the rough outlines of the beginning of Donaldson theory.

The book is well written and the examples do a great service to the reader. It will be a helpful companion to anyone teaching or studying gauge theory and its first four chapters can serve as a one-semester course in the basics of differential geometry/topology.

Reviewed by *Liviu I. Nicolaescu*

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Naber, Gregory L. **The geometry of Minkowski spacetime: an introduction to the mathematics of the special theory of relativity**. Springer-Verlag, 1992. 257p bibl afp (Applied mathematical sciences, 92) ISBN 0-387-97848-8, $49.95. **OAB Winner!** Reviewed in 1993mar CHOICE.

Special relativity is less a subject for its own sake than a comparatively elementary and stable stepping-stone to the mathematically rich and intensely active disciplines of general relativity and quantum field theory. The emphasis Naber places here on the theory of spinors makes this exposition of special relativity especially suitable for readers who would later study the modern formulations of these more advanced subjects, especially future readers of R. Penrose and W. Rindler's Spinors and Space-time (2 v., 1984-86). By contrast, Rindler's own Introduction to Special Relativity (2nd ed., 1991; 1st ed., CH, Jan'83) employs exclusively the older language of tensors. Where many physics texts explain physical phenomena by means of mathematical models, here a rigorous and detailed mathematical development is accompanied by precise physical interpretations. As such, mathematicians should find this book to their particular taste. Other noteworthy features include Zeeman's theorem on causal automorphisms; Hawking, King, and McCarthy's <"path topology>"; and the clearest explanation of <"Dirac's Scissors Problem>" this reviewer has seen in print. For all its mathematical sophistication, this book can be read with just an undergraduate knowledge of linear algebra. Highly recommended. --*D. V. Feldman, University of New Hampshire*

Items Authored by Naber, Gregory L.

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Topology, geometry, and gauge fields. (English. English summary)
Interactions. Applied Mathematical Sciences, 141.
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53C07 (53C05 57R20 57R22 57R25 58A12 70S15 81T13)

References: 0	**Reference Citations: 4**	**Review Citations: 1**

This monograph is a continuation of the author's first part on the foundations of gauge theory \ref[*Topology, geometry, and gauge fields*, Springer, New York, 1997; MR1444352 (99b:58001)]. The book covers a collection of basic facts and techniques needed to penetrate this rather demanding subject. I could say that the main character in this book is the notion of connection on a principal bundle. This notion is introduced early on in the first chapter covering the basics of connections.

The presentation in the remaining five chapters is enriched by detailed discussions about the physical interpretations of connections, their curvatures and characteristic classes. I particularly enjoyed Chapter 2 where many fundamental physical examples are discussed at great length in a reader friendly fashion. No detail is left to the reader's imagination or interpretation. I am not aware of another source where these very important examples and ideas are presented at a level accessible to beginners. These examples serve an additional academic goal, to provide the reader with a convincing motivation to go through the more technical following four chapters covering integration, de Rham theory, characteristic classes (Chern-Weil construction) and a little bit of \v Cech cohomology. The book concludes with an appendix on the Seiberg-Witten equations.

The topics covered in this book can be found in many other sources, but the present volume discusses with great care those aspects and notions which are particularly important in gauge theory. Moreover, the presentation is backed by many useful and relevant examples and I am convinced that any beginner in gauge theory will find them very useful.

Reviewed by *Liviu I. Nicolaescu*

Items Authored by Naber, Gregory L.

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The geometry of Minkowski spacetime.
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Springer-Verlag, New York, 1992. xvi+257 pp. *ISBN* 0-387-97848-8
83A05 (83-01)

References: 0	**Reference Citations: 4**	**Review Citations: 3**

From the preface: "It is the intention of this monograph to provide an introduction to the special theory of relativity that is mathematically rigorous and yet spells out in considerable detail the physical significance of the mathematics."

The author fulfills this promise admirably. In Chapter One, he presents a detailed account of the geometrical structure of Minkowski space-time M, covering not only the customary topics but also causality relations along with Zeeman's characterisation of causal automorphisms and the correspondence between ${\rm SL}(2,**C**)$ and ${\rm SO}(1,3)$ with R. Penrose's \ref[Proc. Cambridge Philos. Soc. **55** (1959), 137--139; MR0099869 (20 \#6305)] result concerning the apparent shape of a relativistically moving sphere as an application. Chapter Two contains a thorough treatment of the tensorial formulation of electromagnetism, including the relation to the familiar three-vector formulation and the relevant linear algebra of skew rank-two tensors.

In the final chapter, the author begins with a very explicit description of the representations of the Lorentz group and then develops the algebraic theory of two-component spinors for M. It is particularly pleasing to find a clear exposition of this material at the introductory level which also demonstrates the utility of spinors. The chapter is welded neatly into the book through its connection with the discussion of spin transformations in Chapter One and its account of the spinor formulation of electromagnetism.

Finally, in two excellent appendices, the author first discusses the path topology of S. W. Hawking, A. R. King and P. J. McCarthy \ref[J. Math. Phys. **17** (1976), no. 2, 174--181; MR0395736 (52 \#16528)] for M and shows that its homeomorphism group is generated by translations, dilations and Lorentz transformations and, second, discusses the nature of "spinorial objects" by means of the Dirac "scissors problem", the fundamental group of SO(3), and the latter's covering space SU(2).

Apart from the appendices (where some simple topology is required), the text requires only familiarity with linear algebra and is a váluable contribution to the pedagogical literature which will be enjoyed by all those who delight in precise mathematics and physics.

Reviewed by *Peter R. Law*

Book review

Author(s) Naber, Gregory L.

Title The Geometry of Minkowski Spacetime

Publisher Springer

Year of publication 1992

Reviewed by Paul Blaga

Since two or three decades, we assist at a process of geometrization of physics. The geometry is more and more involved in most of the hot problems of contemporary theoretical physics. The book under review represents another step in this process. The author's confessed aim is to provide an introduction to special relativity that is mathematically rigorous and, on the other hand, to emphasize the physical significance of the mathematics involved.

The first chapter of the book presents the basic informations about the geometrical and the causal structure of the Minkowski spacetime. I have to quote, between other important results of this chapter the Zeeman characterization of the causal automorphisms of Minkowski spacetime and the Penrose theorem on the apparent shape of a relativistically moving sphere. The second chapter is devoted to the construction of a geometric theory of the electromagnetic field, represented as a skew-symmetric linear transformation. It is proved that the energy-momentum transformation associated to such a skew-symmetric linear map verifies the Dominant Energy Condition. There are written, also, the Maxwell equations by using skew-symmetric bilinear forms. The last chapter of the book is an introduction to the theory of spinors in Minkoviski spacetime. Several applications of spinor formalism are given here. Among others, there is given a classification of electromagnetic fields, similar to the Petrov classification of spacetime manifolds. The book contains, also, two appendices. The first introduces the so-called Zeeman topology for Minkowski spacetime, a topology that is not equivalent to Euclidian topology, but has more physical significance.The second appendix is concerned with Dirac's "Scissors Problem" and its relation with the representations of the Lorentz group. The preresquisites include only a knowledge of linear algebra and some point-set topology (for the two appendices).

This monograph represents, without any doubt, a valuable addition to the literature. It can be of a real help for anyone interested in special relativity, especially for mathematicians searching for rigor. In particular, the careful discussion of the geometry and causal structure of Minkowski spacetime could contribute to a better understanding of the structure of a general spacetime manifold. On the other hand, I should emphasize that this book contains more informations than the classical books on this topic. Beside the traditional menu (treated in a modern, geometrical language), it is presented much material hard to be find in other books, or even published for the first time in monograph literature. The reading of the book not presuming too many knowledges, the range of readers is very large, from undergraduate students to experts. I don't know if the reader is an expert active in special relativity, but his talent in choosing the most significant results and ordering them within the book can't be denied. The reading of the book is, really, a pleasure. Gregory Naber is, also, the author of "Spacetime and singularities", Cambridge University Press, 1990.

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Spacetime and Singularities—An Introduction.
By Gregory L. Naber.

1989, £22.50 (hbk), £7.95 (pbk), pp. 178, Cambridge University Press, ISBN 0 521 33612 0.
Scope: Text. Level: Postgraduate.

One of the intellectual summits of what P. Hajicek has called 'the golden era' of relativity theory was marked by the singularity theorems of Hawking and Penrose. They are often interpreted as implying that a non-quantum relativity theory is not self consistent, because a broad class of space-times exhibited the break-down of space-time itself. We relativists have traditionally regarded these famous but inaccessible theorems as our own private nature reserves, and so it is with surprise that we find Professor Naber organising day trips for all comers—or all who are equipped with a little basic mathematical expertise. Starting from sea-level, the reader is efficiently whisked through special relativity, relativistic mechanics, manifold theory and causal relations to Hawking's theorem (the one about space-times with a Cauchy surface having a mean curvature bounded below). The style is pedagogically good, with the right number of exercises to help the reader on his way, though with few rests to enjoy the scenery.

It is probably a book for mathematicians, which could leave many physicists feeling uneasy at the very axiomatic approach. There is no time to give much of a feeling for how relativistic mechanics differs from Newtonian mechanics, or what the energy–momentum tensor signifies. These are places where Professor Naber transgresses current norms of propriety in mathematics—but gets away with it; for example, in treating manifolds as subsets of $\mathbb{R}^n$, where one realises that in the hands of a good mathematician nothing is lost by so doing.

Impressive though the achievement is, of reaching such an advanced goal in so short a time, I am left with a few doubts as to the size of the potential audience. For the mathematician wanting to get a whiff of real modern Lorentzian geometry, or for the graduate student unsure whether to move into this area, it is ideal. But the student wanting a solid foundation in differential geometry and relativity theory would still be advised to follow more traditional routes, such as Beem and Ehrlich's book (now sadly out of print).

Professor C. J. S. Clarke (University of Southampton)

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SPACETIME AND SINGULARITIES: AN INTRODUCTION
(London Mathematical Society Student Texts 11)

By G. L. NABER: pp. 178. £22.50/£7.95. LMS members' price £16.88/£5.96. (Cambridge University Press, 1989)

Just over 20 years ago the famous singularity theorems of Penrose, Hawking and Geroch transformed our understanding of global questions in general relativity. These early advances were summarised in the early seventies in the SIAM lecture notes of Penrose and the rather better-known monograph of Hawking and Ellis. Since that time advances have accumulated at a rather slower rate and the use of the 'Techniques of differential topology in relativity', to quote the title of Penrose's lectures, has rather fallen off among physically-minded relativists and the subject has not retained the same central position it formerly held.

This is not the occasion to analyse in detail why that is, but part of the reason lies undoubtedly in the unfamiliarity of the ideas involved, not only for students whose original training was in physics but also those better prepared in pure mathematics and having familiarity with modern differential geometry. The point is that the subject has a flavour all of its own which needs to be carefully cultivated before one can feel at all confident with it. From this point of view existing textbooks require a rather abrupt transition which can be off-putting to all but the most persistent student. For that reason the present set of lecture notes seems admirably suited to fill a definite gap in the literature. It provides a gentle and pedagogically sound introduction to the salient ideas and culminates in the fourth and final chapter with a full proof of one of the significant theorems in the subject. The level is certainly within the capacity of a talented third year undergraduate and well within that of the average (albeit hardworking) graduate student. The first three chapters would make an excellent introductory course in general relativity for those wishing to emphasise global questions rather than the more conventional approach based on local differential geometry. I was especially impressed by the first chapter which is a beautiful summary for the mathematically minded undergraduate of the essential content of 'special relativity'.

G. W. GIBBONS

Topological methods in Euclidean spaces, by Gregory L. Naber. Pp 230. £20 hardcover, £6·95 paperback. 1980. ISBN 0 521 22746 1 and 29632 3 (Cambridge University Press)

Of the many books that now exist on Topology the early ones tried to give an overall view, but the rapid development of the subject caused a change. The majority of books began to concentrate on one aspect—usually 'General topology' or 'algebraic topology' and then pursued it ferociously with little or no explanation of purpose or cross-relations within Topology—let alone mathematics as a whole. A few popularising attempts have also been made, to give the attractive intuitive ideas: but it is difficult not to be superficial, if one is restricted to non-technical language. Indeed, an important part of the early development of topology was the very invention of a language precise enough to pin down elusive intuitions.

Gradually, however, topological ideas have been absorbed into other parts of mathematics, in particular into Calculus where there is strong contact with geometry through line and surface integrals, and tangent planes. But the natural habitat of Calculus is finite-dimensional euclidean space $\mathbb{R}^n$ (even though infinite-dimensional spaces are used in advanced work), and the attractive combination of ideas from Analysis, linear algebra and geometry, with relevant notions of topology, has grown into the subject 'Differential Topology'. It is a subject of increasing usefulness in Mathematical Physics, where problems can now be treated that were beyond the technique of older Applied Mathematicians. (For that reason, Poincaré was forced to leave problems of mechanics in order to make his decisive contributions in the early development of topology.)

Thus, many mathematicians other than specialist topologists need a topological education, but one that encourages them to look outwards. By taking for granted some knowledge of Calculus, the author of this book is able to get rapidly to interesting work. He is not the first to take the short cut of working in Euclidean space, but he goes, I think, further (and certainly more interestingly) than any other recent author I have read.

He begins with the usual undergraduate point-set topology of Euclidean space, but eschews defining 'metric space' or 'a topology' in favour of going as far as the Hahn-Mazurkiewicz theorem (every compact connected, locally connected subset of $\mathbb{R}^n$ is a continuous image of the unit interval $0 \leqslant x \leqslant 1$). By including there the (easy) 1-dimensional version of the Brouwer fixed-point theorem, he is able to set up the basic project of the book, which is to prove the n-dimensional case by three quite different methods—combinatorial, algebraic topology, and differential topology. The programme begins in Chapter 2 with barycentric simplices and Sperner's Lemma, with which the author proves both the Brouwer Theorem and the topological invariance of (Lebesgue) dimension. Line integrals in the plane prepare the stage for introducing homotopy and the fundamental group in Chapter 3, which also touches on covering spaces in order to compute the fundamental group of the circle. At this point we meet categories and functors, with a very light touch, ready for Chapter 4 on simplicial homology theory. Here the touch is still light, very sparing with notation, with a digression on the essential homological algebra that is needed. This chapter concludes with the Lefschetz fixed-point theorem, and some corollaries on the degree of spherical maps, and hence Brouwer's Theorem again.

In Chapter 5, we meet differential techniques which begin by showing how to approximate continuous maps by smooth ones, via the Stone-Weierstrass theorem. Since the ambient Euclidean space allows economies of exposition we can then move easily to smooth manifolds, tangent spaces and critical points of smooth maps (i.e. points where the gradient is zero). Sard's theorem, on the zero measure of critical values is proved, and the normal form of a Morse function is derived. This work is used to show that the only 1-dimensional manifolds in $\mathbb{R}^n$ are the circle and unit interval, from which Brouwer's theorem is once more derived. The final project is to give a characterisation of the n-sphere as a compact connected submanifold possessing a smooth function with only two non-degenerate critical points. One lemma within this Morse theory requires the apparatus of trajectories of a gradient field, so the author concludes with a few pages on smooth vector fields, taking up some of his remarks that introduce this chapter, on non-linear dynamical systems such as the pendulum.

Curiously, in view of his general good taste and maintenance of interest, the author does not even mention the famous (and historically first) theorem of Morse theory, that on the surface of the earth, 'pits plus peaks minus passes' equals 2, and its generalisations to other dimensions.

How can we cover so much ground in 240 pages? Partly by assuming such standard undergraduate knowledge as uniform convergence, integration, elementary Fourier series, and existence theorems of differential equations. Partly also because although no proof is avoided, many proofs are broken into short steps which are often intended as exercises; in that sense, the work is complete. There are some 260 exercises, of which (the author asserts) 162 establish results used elsewhere in the book, but the latter are sufficiently structured as to be genuine exercises. Whether or not a student would relish so much omission, I cannot say; but it certainly makes the text-book less forbidding in appearance.

I enjoyed this book—it's the kind I would like to have written myself.

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Gregory L. Naber, Spacetime and Singularities (Introduction). IX+178 pages, Cambridge University Press. Cambridge—New York—New Rochelle—Melbourne—Sydenye, 1988.

Starting from the most elementary facts of geometry, this book leads the reader through only 174 pages to one of the peaks of general relativity, to the singularity theorem of Hawking. The titles of the chapters are: The Geometry of Minkowski Spacetime; Some Concepts from Relativistic Mechanics; More General Spacetimes: Gravity; The Proof of Hawking's Theorem. This list shows the milestones of a straight road along which one has to pass to reach the summit. The book is written so well, that stopping anywhere between will not cause a disappointment. The text is aimed at a student in mathematics, but it can be recommended to a physicist too, who wants to get a more precise mathematical description of relativity than it is usual in physics books. On the other hand, short but sufficient physical background is presented everywhere, helping the reader to grasp the physical significance of the theorems deduced. The mathematical definitions are also well motivated, and besides the exact methods everything is explained in simple terms as well. One should state: almost everything is explained. Here and there one finds abbreviations like HE etc. The well educated reader suspects that this must be a reference to the book of Hawking and Ellis, perhaps. In fact the list of the references is missing from the volume. This should be corrected in a second edition, to have a really enjoyable text on this highly interesting subject.

M. G. Benedict (Szeged)

Naber, Gregory L. **Topology, geometry, and gauge fields: foundations**. Springer, 1997. 396p bibl index afp (Texts in applied mathematics, 25) ISBN 0-387-94946-1, $49.95. Reviewed in 1998jun CHOICE.

Though mathematicians and physicists have many common concerns, the two disciplines speak different languages, and students of each field do not have an easy time picking up what they need to know from the other. Thus, any book that may function as a Rosetta stone attracts considerable interest. Chapters 1 through 4 of Naber's volume form a reasonably conventional exposition of basic differential topology and geometry through fibre bundles and Lie groups. But since the author has applications to nuclear and particle physics in mind, physics students will find just what they need to know; mathematics students will learn what it all means for physics in the preliminary chapter 0 and in much more detail in the final chapter 5, which treats Yang-Mills theory and matter fields. A very useful book, for upper-division undergraduates and beyond. *—D. V. Feldman, University of New Hampshire*