COMPACTNESS

RECALL FROM CALCULUS:

IF $[a,b]$ IS A CLOSED, BOUNDED INTERVAL IN $\mathbb{R}$, THEN

- ANY INFINITE SEQUENCE IN $[a,b]$ HAS A CONVERGENT SUBSEQUENCE
- ANY CONTINUOUS REAL-VALUED FUNCTION ON $[a,b]$ TAKES ON A MAXIMUM AND A MINIMUM VALUE

CONSEQUENCES OF A PROPERTY OF $[a,b]$ CALLED "COMPACTNESS" WHICH WE NOW DEFINE IN GENERAL.

X A TOPOLOGICAL SPACE.

AN <u>OPEN</u> <u>COVER</u> OF X IS A COLLECTION $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ OF OPEN SETS IN X WITH $\bigcup_{\alpha\in\mathcal{A}} U_\alpha = X$.

E.G., FOR $X = S^n$

1. $\{U_N, U_S\}$
2. $\{U_k^{\pm} : k = 1,\dots,n+1\}$, WHERE
   $$U_k^+ = \{(x^1,\dots,x^{n+1}) \in S^n : x^k > 0\}$$
   $$U_k^- = \{(x^1,\dots,x^{n+1}) \in S^n : x^k < 0\}$$
3. $\{S^n \cap U_\varepsilon(p) : p \in S^n,\ \varepsilon > 0\}$

A <u>SUBCOVER</u> OF $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ IS A SUBCOLLECTION

$$\{U_\alpha\}_{\alpha\in\mathcal{A}'}\ ,\ \mathcal{A}'\subseteq\mathcal{A}$$

FOR WHICH $\bigcup_{\alpha\in\mathcal{A}'} U_\alpha = X$.

E.G., A SUBCOVER OF $\{ S^n \cap U_\varepsilon(p) : p\in S^n, \varepsilon>0\}$ IS $\{ S^n\cap U_1(p) : p\in S^n\}$

<u>THEOREM</u>: ANY OPEN COVER OF A SECOND COUNTABLE SPACE (E.G., ANY SUBSPACE OF SOME EUCLIDEAN SPACE $\mathbb{R}^n$) HAS A <u>COUNTABLE</u> SUBCOVER.

THIS IS EASY BECAUSE EVERY POINT IN THE SPACE IS IN AN ELEMENT OF THE COUNTABLE BASIS CONTAINED IN SOME ELEMENT OF THE COVER.

A HAUSDORFF SPACE X IS <u>COMPACT</u> IF EVERY OPEN COVER OF X HAS A <u>FINITE</u> SUBCOVER.

Once more, the concept of compact sets eludes Greg. The open cover seems totally irrelevant.

FROM REAL ANALYSIS :

HEINE - BOREL THEOREM : A SUBSPACE X OF $\mathbb{R}^n$ IS COMPACT IF AND ONLY IF IT IS CLOSED AND BOUNDED.

NOTE : "BOUNDED" MAKES NO SENSE IN A GENERAL TOPOLOGICAL SPACE AND, EVEN IN THOSE SPACES FOR WHICH IT DOES MAKE SENSE, "CLOSED AND BOUNDED" DOES NOT GENERALLY IMPLY "COMPACT".

<u>EXAMPLE</u> : LET $\ell_\infty(\mathbb{R})$ DENOTE THE SET OF ALL <u>BOUNDED</u> SEQUENCES

$$x = \{x_n\}_{n=1,2,\ldots} = \{x_1, x_2, \ldots\}$$

OF REAL NUMBERS. THIS IS A REAL VECTOR SPACE WITH THE OBVIOUS OPERATIONS :

$$x+y = \{x_n\} + \{y_n\} = \{x_n + y_n\} \quad \text{AND} \quad \alpha x = \alpha\{x_n\} = \{\alpha x_n\}$$

IT IS A NORMED VECTOR SPACE IF WE DEFINE

$$\|x\| = \|\{x_n\}\| = \sup_n |x_n|$$

DEFINE THE "DISTANCE" BETWEEN ANY TWO POINTS IN $\ell_\infty(\mathbb{R})$ BY

$$d(x,y) = \|y-x\| = \sup_n |y_n - x_n| .$$

I'LL LEAVE IT TO YOU TO SHOW THAT d HAS ALL OF THE PROPERTIES ONE WOULD EXPECT OF A REASONABLE NOTION OF "DISTANCE" :

$$d(x,y) \geq 0 \quad \text{AND} \quad d(x,y) = 0 \text{ IF AND ONLY IF } x = y$$

$$d(y,x) = d(x,y)$$

$$d(x,y) \leq d(x,z) + d(z,y) \quad \text{FOR ANY } z$$

(THIS WAS <u>EXERCISE</u> 9).

WITH THIS WE CAN DEFINE "OPEN BALLS", "OPEN SETS", "BOUNDED SETS", ETC. EXACTLY AS WE DID FOR $\mathbb{R}^n$.

NOW CONSIDER THE CLOSED UNIT BALL IN $\ell_\infty(\mathbb{R})$:

$$B = \{ x \in \ell_\infty(\mathbb{R}) : \|x\| \leq 1 \}$$

THIS IS SURELY CLOSED AND BOUNDED, BUT WE WILL NOW SHOW THAT IT IS <u>NOT</u> COMPACT:

THE FOLLOWING POINTS ARE ALL IN B:

$$\begin{aligned} e_1 &= (1, 0, 0, 0, \dots) \\ e_2 &= (0, 1, 0, 0, \dots) \\ e_3 &= (0, 0, 1, 0, \dots) \\ &\vdots \end{aligned}$$

AND THE DISTANCE BETWEEN ANY TWO OF THEM IS 1.

NOW CONSIDER THE OPEN COVER $\{ U_{\frac{1}{2}}(p) : p \in B \}$ OF B BY OPEN BALLS OF RADIUS $\frac{1}{2}$. IF THIS HAD A FINITE SUBCOVER, THEN INFINITELY MANY OF THE e_n, $n = 1, 2, \dots$ WOULD HAVE TO BE IN ONE OF THE $U_{\frac{1}{2}}(p)$. BUT THIS IS CLEARLY IMPOSSIBLE SINCE ANY TWO POINTS IN $U_{\frac{1}{2}}(p)$ HAVE A DISTANCE APART THAT IS < 1.

<u>THEOREM</u>: IF X IS COMPACT, Y IS HAUSDORFF AND $f : X \to Y$ IS A CONTINUOUS MAP OF X ONTO Y, THEN Y IS COMPACT (THE CONTINUOUS IMAGE OF A COMPACT SPACE IS COMPACT, PROVIDED IT'S HAUSDORFF)

<u>PROOF</u>: LET $\{U_\alpha\}_{\alpha \in A}$ BE AN OPEN COVER OF Y. THEN $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ IS AN OPEN COVER OF X. X IS COMPACT SO WE CAN SELECT A FINITE SUBCOVER $\{f^{-1}(U_{\alpha_1}), \dots, f^{-1}(U_{\alpha_k})\}$. SINCE f IS ONTO, $\{U_{\alpha_1}, \dots, U_{\alpha_k}\}$ COVERS Y. $\square$

<u>APPLICATION</u>: $\mathbb{RP}^n$ IS COMPACT.

<u>EXERCISE</u> 39: SHOW THAT $O(n)$ AND $SO(n)$ ARE COMPACT.
<u>HINT</u>: HEINE-BOREL.

A <u>TOPOLOGICAL PROPERTY</u> IS ONE WHICH, IF TRUE OF A TOPOLOGICAL SPACE X, IS NECESSARILY TRUE OF ANY SPACE Y HOMEOMORPHIC TO X.

<u>EXERCISE</u> 40: SHOW THAT "HAUSDORFF", "SECOND COUNTABLE" AND "COMPACT" ARE TOPOLOGICAL PROPERTIES.

<u>EXERCISE</u> 41: SHOW THAT $SU(2)$ IS COMPACT.

<u>EXERCISE</u> 42: SHOW THAT A CLOSED SUBSPACE OF A COMPACT SPACE IS COMPACT (MORE PRECISELY, IF X IS COMPACT AND A IS A CLOSED SUBSET OF X, THEN, WITH ITS RELATIVE TOPOLOGY FROM X, A IS COMPACT.)

<u>EXERCISE</u> 43 : SHOW THAT A COMPACT SUBSPACE A OF A HAUSDORFF SPACE Y IS CLOSED IN Y.

<u>THEOREM</u> : A CONTINUOUS BIJECTION FROM A COMPACT SPACE ONTO A HAUSDORFF SPACE IS A HOMEOMORPHISM.

<u>PROOF</u> : LET X BE COMPACT, Y HAUSDORFF AND $f : X \to Y$ CONTINUOUS, ONE-TO-ONE AND ONTO.

NEED TO SHOW THAT

$$f^{-1} : Y \to X$$

IS CONTINUOUS AND THIS IS EQUIVALENT TO SHOWING THAT $f : X \to Y$ IS AN OPEN MAP (<u>EXERCISE</u> 44).

U OPEN IN X $\Rightarrow$ $X - U$ CLOSED IN X

$\Rightarrow$ $X - U$ COMPACT (EXERCISE 42)

$\Rightarrow$ $f(X - U)$ COMPACT (THEOREM, PAGE 5)

$\Rightarrow$ $f(X) - f(U) = Y - f(U)$ CLOSED IN Y (EXERCISE 43)

$\Rightarrow$ $f(U)$ OPEN IN Y.

$\square$

THEOREM : $X_1, \dots, X_n$ HAUSDORFF SPACES. THEN

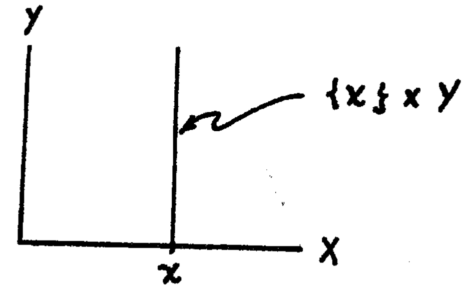
$$X_1 \times \cdots \times X_n \text{ COMPACT} \iff X_1, \dots, X_n \text{ ALL COMPACT}$$

PROOF : $\Rightarrow$ $X_1 \times \cdots \times X_n$ COMPACT AND $\pi_i : X_1 \times \cdots \times X_n \to X_i$ CONTINUOUS SURJECTION $\Rightarrow X_i$ COMPACT

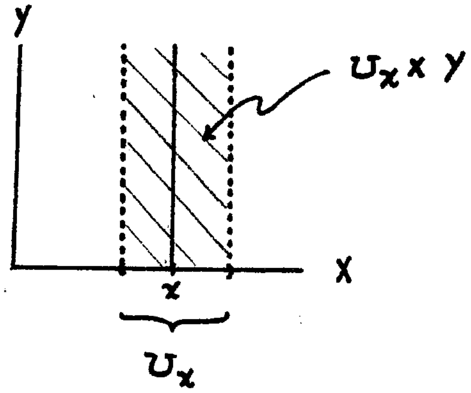
FOR $\Leftarrow$ IT IS ENOUGH (BY INDUCTION) TO PROVE THAT

$$X, Y \text{ COMPACT} \Rightarrow X \times Y \text{ COMPACT.}$$

ASSUME X, Y COMPACT AND LET $\mathcal{U}$ BE AN ARBITRARY OPEN COVER OF $X \times Y$. FIX AN $x \in X$ AND CONSIDER $\{x\} \times Y \subseteq X \times Y$.



CLAIM THAT $\exists$ OPEN SET U_x IN X CONTAINING x SUCH THAT $U_x \times Y$ IS COVERED BY FINITELY MANY ELEMENTS OF $\mathcal{U}$.



TO SEE THIS: FOR EACH $(x,y) \in \{x\} \times Y$ SELECT A BASIC OPEN SET $U_{(x,y)} \times V_{(x,y)}$ IN $X \times Y$ CONTAINING (x,y) AND CONTAINED IN SOME ELEMENT OF $\mathcal{U}$. THEN $\{V_{(x,y)} : y \in Y\}$ IS AN OPEN COVER OF Y. Y IS COMPACT SO SELECT A FINITE SUBCOVER $\{V_{(x,y_1)}, \dots, V_{(x,y_k)}\}$. LET

$$U_x = U_{(x,y_1)} \cap \cdots \cap U_{(x,y_k)}.$$

U_x IS OPEN IN X, CONTAINS x AND

$$U_x \times Y \subseteq (U_{(x,y_1)} \times V_{(x,y_1)}) \cup \cdots \cup (U_{(x,y_k)} \times V_{(x,y_k)}).$$

EACH $U_{(x,y_i)} \times V_{(x,y_i)}$, $i = 1, \dots, k$, IS CONTAINED IN SOME ELEMENT U_i OF $\mathcal{U}$ SO

$$U_x \times Y \subseteq U_1 \cup \cdots \cup U_k$$

AS REQUIRED.

THUS, FOR EACH $x \in X$ WE HAVE AN OPEN SET U_x IN X CONTAINING x AND A FINITE COLLECTION $\mathcal{U}_x$ OF ELEMENTS OF $\mathcal{U}$ COVERING $U_x \times Y$. $\{U_x : x \in X\}$ IS AN OPEN COVER OF X AND X IS COMPACT SO THERE IS A FINITE SUBCOVER $\{U_{x_1}, \dots, U_{x_\ell}\}$. $\mathcal{U}_{x_i}$ COVERS $U_{x_i} \times Y$ FOR $i = 1, \dots, \ell$ SO $\mathcal{U}_{x_1} \cup \cdots \cup \mathcal{U}_{x_\ell}$ COVERS $X \times Y$ AND SO IT'S A FINITE SUBCOVER OF $\mathcal{U}$. $\square$

MANY SPACES OF INTEREST (E.G., $\mathbb{R}^n$) WHICH FAIL TO BE COMPACT NEVERTHELESS HAVE A "LOCAL" VERSION OF THIS PROPERTY AND, MOREOVER, CAN BE "COMPACTIFIED" BY THE ADDITION OF A SINGLE POINT (THINK ABOUT THE PLANE $\mathbb{R}^2$ AND THE RIEMANN SPHERE S^2).

TO DISCUSS THIS WE NEED TO GENERALIZE SOME NOTIONS FROM REAL ANALYSIS:

X A TOPOLOGICAL SPACE AND $x \in X$

A <u>NEIGHBORHOOD</u> (<u>NBD</u>) OF x IN X IS ANY SUBSET OF X WHICH CONTAINS AN OPEN SET CONTAINING x.

IF $A \subseteq X$, THEN $x \in X$ IS AN <u>ACCUMULATION POINT</u> OF A IF EVERY NBD OF x IN X CONTAINS A POINT OF A OTHER THAN x.

E.G., IF $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \subseteq \mathbb{R}$ THEN 0 IS AN ACCUMULATION POINT OF A, BUT 1 IS NOT.

THE SET OF ALL ACCUMULATION POINTS OF A IS CALLED THE <u>DERIVED SET</u> OF A AND IS DENOTED A'.

THE <u>CLOSURE</u> OF A IN X IS $\bar{A} = cl_X A = A \cup A'$.

A SUBSET A OF X IS SAID TO BE <u>DENSE</u> IN X IF $\bar{A} = X$.

<u>EXERCISE</u> 44: LET X BE A TOPOLOGICAL SPACE AND A AND B SUBSETS OF X. SHOW THAT

(a) $\bar{\phi} = \phi$ AND $\bar{X} = X$

(b) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$

(c) A IS CLOSED IN X IFF $\bar{A} = A$

(d) $\bar{\bar{A}} = \bar{A}$

(e) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

(f) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

(g) FIND AN EXAMPLE TO SHOW THAT $\overline{A \cap B}$ NEED NOT EQUAL $\bar{A} \cap \bar{B}$.

(h) $\bar{A}$ IS THE INTERSECTION OF ALL CLOSED SUBSETS OF X CONTAINING A.

(i) IF X IS A SUBSPACE OF SOME $\mathbb{R}^n$ AND $A \subseteq X$, THEN EVERY POINT IN A' IS THE LIMIT OF SOME SEQUENCE OF POINTS IN A.

<u>EXERCISE</u> 45: A SPACE X IS SAID TO BE <u>SEPARABLE</u> IF IT HAS A COUNTABLE DENSE SUBSET. SHOW THAT ANY SECOND COUNTABLE SPACE IS SEPARABLE.

<u>EXERCISE</u> 46: SHOW THAT A PRODUCT OF TWO SEPARABLE SPACES IS SEPARABLE.

NOW, WE WILL SAY THAT A SUBSET U OF X IS RELATIVELY COMPACT (OR PRECOMPACT) IF $\overline{U}$ IS COMPACT.

A HAUSDORFF SPACE X IS LOCALLY COMPACT IF EACH POINT IN X HAS A RELATIVELY COMPACT NBD.

E.G., ANY $\mathbb{R}^n$ IS LOCALLY COMPACT, AS IS ANY LOCALLY EUCLIDEAN SPACE.

LEMMA: A LOCALLY COMPACT SPACE X HAS A BASIS CONSISTING OF RELATIVELY COMPACT OPEN SETS (IF X IS SECOND COUNTABLE, THEN IT HAS A COUNTABLE SUCH BASIS).

PROOF: LET $\mathcal{B}$ BE A BASIS FOR THE OPEN SETS IN X (AND TAKE $\mathcal{B}$ COUNTABLE IF X IS SECOND COUNTABLE). NOW LET

$$\mathcal{B}' = \{ U \in \mathcal{B} : \overline{U} \text{ compact} \}$$

WE SHOW THAT $\mathcal{B}'$ IS NOT ONLY NONEMPTY, BUT ACTUALLY IS A BASIS FOR X.

LET $U \subseteq X$ BE OPEN. WILL SUFFICE TO FIND, FOR EACH $x \in U$, A $W_x \in \mathcal{B}'$ WITH $x \in W_x \subseteq U$.

SELECT A NBD V OF x WITH $\bar{V}$ COMPACT (LOCAL COMPACTNESS). $U \cap V$ IS A NBD OF x SO $\exists$ $W \in \mathcal{B}$ WITH $x \in W \subseteq U \cap V$.

BUT

$$\bar{W} \subseteq \overline{U \cap V} \subseteq \bar{V} \quad \text{(EXERCISE 44(b))}$$

AND $\bar{W}$ IS CLOSED (EXERCISE 44(c)) SO $\bar{W}$ IS COMPACT, I.E., $W \in \mathcal{B}'$. MOREOVER,

$$x \in W \subseteq U \cap V \subseteq U$$

SO WE CAN TAKE $W_x = W$. $\square$

NOW WE SHOW THAT, FOR LOCALLY COMPACT SPACES, WE CAN MIMIC THE USUAL CONSTRUCTION OF THE "EXTENDED COMPLEX PLANE $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$"

RECALL THE PICTURE: $\mathbb{C}$ CAN BE IDENTIFIED WITH $S^2 - \{\text{NORTH POLE}\}$ BY STEREOGRAPHIC PROJECTION. ADDING THE NORTH POLE TO S^2 CORRESPONDS TO ADDING A "POINT AT INFINITY" TO $\mathbb{C}$. THE RESULT IS A COMPACT SPACE CONTAINING $\mathbb{C}$ AS A DENSE SUBSPACE.

LET X BE A LOCALLY COMPACT SPACE. SELECT SOME OBJECT THAT IS NOT AN ELEMENT OF X AND DENOTE IT

$$\infty .$$

DEFINE A TOPOLOGY ON THE SET

$$X^* = X \cup \{\infty\}$$

AS FOLLOWS :

> A BASIS FOR THE TOPOLOGY OF X^* CONSISTS OF ALL THE OPEN SETS IN X TOGETHER WITH ALL THE COMPLEMENTS IN X^* OF COMPACT SETS IN X.

<u>NOTE</u> : THE OPEN NBDS OF THE POINT AT INFINITY IN THE EXTENDED COMPLEX PLANE ARE JUST THE IMAGES UNDER STEREOGRAPHIC PROJECTION OF OPEN NBDS OF THE NORTH POLE IN S^2. THESE ARE COMPLEMENTS OF COMPACT SETS IN $\mathbb{C}$.

I WILL LEAVE IT TO YOU TO SHOW THAT THIS REALLY IS A TOPOLOGY ON X^* AND THAT THE RELATIVE TOPOLOGY ON X IN X^* IS JUST THE ORIGINAL TOPOLOGY OF X (<u>EXERCISE</u> 47).

I WILL SHOW THAT X^* IS

1. HAUSDORFF
2. COMPACT

TO SEE THAT X^* IS HAUSDORFF, LET x AND y BE DISTINCT POINTS IN X^*.

IF $x, y \in X \subseteq X^*$ THEN (SINCE X IS HAUSDORFF) $\exists$ OPEN SETS U_x AND U_y IN X WITH $x \in U_x$, $y \in U_y$ AND $U_x \cap U_y = \phi$. BUT U_x AND U_y ARE ALSO OPEN IN X^* SO, IN THIS CASE, WE ARE DONE.

THUS, IT SUFFICES TO SHOW THAT $x \in X \subseteq X^*$ AND $y = \infty$ CAN BE SEPARATED BY OPEN SETS IN X^*.

CHOOSE AN OPEN NBD V OF x IN X WITH $\bar{V}$ COMPACT (LOCAL COMPACTNESS). THEN

$$V \quad \text{AND} \quad X^* - \bar{V}$$

ARE OPEN SETS IN X^* WITH $x \in V$ AND $\infty \in X^* - \bar{V}$ AND

$$V \cap (X^* - \bar{V}) = \phi$$

AS REQUIRED.

FOR COMPACTNESS, LET $\mathcal{U} = \{ U_\alpha \}_{\alpha \in \mathcal{A}}$ BE AN OPEN COVER OF X^*. SELECT SOME $U_{\alpha_0} \in \mathcal{U}$ WITH $\infty \in U_{\alpha_0}$.

U_{α_0} CONTAINS THE COMPLEMENT OF SOME COMPACT SET C IN X. SELECT FINITELY MANY ELEMENTS $U_{\alpha_1}, \ldots, U_{\alpha_k}$ OF $\mathcal{U}$ THAT COVER C. THEN $\{U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ IS A FINITE SUBCOVER OF $\mathcal{U}$.

WITH THIS TOPOLOGY, $X^* = X \cup \{\infty\}$ IS THE <u>ONE-POINT COMPACTIFICATION</u> OF X.

<u>EXERCISE</u> 48: SHOW THAT X IS AN OPEN, DENSE SUBSET OF X^*.

<u>EXERCISE</u> 49: USE THE STEREOGRAPHIC PROJECTION MAP TO SHOW THAT THE ONE-POINT COMPACTIFICATION OF $\mathbb{R}^n$ IS HOMEOMORPHIC TO S^n.

<u>EXERCISE</u> 50: SHOW THAT ANY CONTINUOUS, REAL-VALUED FUNCTION f ON A COMPACT SPACE X ASSUMES A MAXIMUM AND MINIMUM VALUE (I.E., $\exists\, x_0, x_1 \in X$ SUCH THAT $f(x) \le f(x_0)$ AND $f(x) \geq f(x_1)$ FOR EVERY $x \in X$) IS THE SAME THING TRUE FOR LOCALLY COMPACT SPACES?