COMPLEX FUNCTIONS AND MAPPINGS

A COMPLEX-VALUED FUNCTION OF A COMPLEX VARIABLE IS A RULE f WHICH ASSIGNS TO EACH COMPLEX NUMBER z IN SOME REGION R OF THE COMPLEX PLANE A UNIQUE COMPLEX NUMBER

$$w = f(z).$$

EXAMPLES :

I. POLYNOMIALS : $P(z) = a_n z^n + \dots + a_1 z + a_0$

E.G., $P(z) = 5z^2 - 2iz + i$

II. RATIONAL FUNCTIONS : $R(z) = \dfrac{P(z)}{Q(z)}$, WHERE $P(z)$ AND $Q(z)$ ARE POLYNOMIALS

E.G., $R(z) = \dfrac{z^3+1}{5z^2-2iz+i}$

III. FRACTIONAL LINEAR TRANSFORMATIONS (SPECIAL CASE OF #2):

$$f(z) = \frac{az+b}{cz+d}$$

WHERE $ad-bc \neq 0$.

E.G., $f(z) = \dfrac{z-i}{-iz+1}$

(MORE EXAMPLES SHORTLY)

NOTE THAT IF $z = x + yi$, THEN

$$f(z) = z^2 = (x^2 - y^2) + 2xyi$$

IN GENERAL, ANY $f(z)$ CAN BE WRITTEN IN TERMS OF REAL AND IMAGINARY PARTS

$$f(z) = u(x,y) + i v(x,y)$$

ALTERNATIVELY, IF $z = r(\cos\theta + i\sin\theta)$,

$$f(z) = u(r,\theta) + i v(r,\theta)$$

E.G.,

$$f(z) = z^2 = r^2\cos 2\theta + i r^2 \sin 2\theta$$

<u>NOTE</u>: AT THIS POINT WE INTEND TO INTRODUCE A NUMBER OF ADDITIONAL EXAMPLES. THESE WILL HAVE VERY FAMILIAR NAMES, BUT RATHER ODD DEFINITIONS. THE REAL JUSTIFICATION FOR THESE DEFINITIONS WILL COME A BIT LATER WHEN WE WILL SEE THAT THERE IS ONLY ONE "REASONABLE" WAY TO DEFINE COMPLEX ANALOGUES OF THE FAMILIAR FUNCTIONS FROM CALCULUS (e^x, $\sin x$, ...). FOR THE TIME BEING WE WILL BE CONTENT WITH (1) NOTING THAT OUR NEW DEFINITIONS AGREE WITH THE FAMILIAR FUNCTIONS WHEN z IS REAL, AND (2) SHOWING THAT THE NEW FUNCTIONS HAVE MANY OF THE SAME PROPERTIES AS THE OLD ONES.

EXAMPLES (CONTINUED):

IV. COMPLEX EXPONENTIAL FUNCTION: FOR ANY $z = x + yi$ DEFINE

$$e^z = \exp(z) = e^x(\cos y + i \sin y)$$

NOTE: IF z IS REAL (I.E., $y = 0$) THIS REDUCES TO e^x.

A FEW SIMPLE PROPERTIES:

(a) $e^{z_1} e^{z_2} = e^{z_1 + z_2}$

PROOF:
$$\begin{aligned} e^{z_1} e^{z_2} &= (e^{x_1}(\cos y_1 + i \sin y_1))(e^{x_2}(\cos y_2 + i \sin y_2)) \\ &= e^{x_1} e^{x_2} (\cos(y_1 + y_2) + i \sin(y_1 + y_2)) \\ &= e^{x_1 + x_2} (\cos(y_1 + y_2) + i \sin(y_1 + y_2)) \\ &= e^{(x_1 + x_2) + i(y_1 + y_2)} \\ &= e^{z_1 + z_2} \end{aligned}$$

(b) $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

EXERCISE 12: PROVE (b)

(c) e^z IS PERIODIC WITH PERIOD $2\pi i$

PROOF:
$$\begin{aligned} e^{z + 2\pi i} &= e^{x + (y + 2\pi) i} = e^x(\cos(y + 2\pi) + i \sin(y + 2\pi)) \\ &= e^x(\cos y + i \sin y) \\ &= e^z \end{aligned}$$

EXERCISES :

13. EVALUATE $\exp\left(\frac{2+\pi i}{4}\right)$ ANS. $\sqrt{\frac{e}{2}}\,(1+i)$

14. SHOW THAT

$$|e^{z^2}| = e^{x^2-y^2}$$

15. SHOW THAT

$$\exp(\bar{z}) = \overline{\exp(z)}$$

16. SHOW THAT e^z IS REAL ONLY WHEN $\text{Im}(z) = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

NOTE : FROM THE DEFINITION OF e^z WITH $z = \theta i$ $(= 0 + \theta i)$ WE GET

$$e^{\theta i} = \cos\theta + i\sin\theta$$

SO THE POLAR FORM OF A COMPLEX NUMBER CAN NOW BE WRITTEN

$$z = r(\cos\theta + i\sin\theta) = re^{\theta i}$$

WE WILL ALMOST ALWAYS OPT FOR THIS SIMPLER NOTATION, E.G., WE WRITE

$$z^n = (re^{\theta i})^n = r^n e^{n\theta i}$$

V. COMPLEX TRIGONOMETRIC FUNCTIONS :

AS MOTIVATION FOR THE NEXT DEFINITION CONSIDER THE FOLLOWING :

FOR ANY REAL NUMBER x,

$$e^{ix} = \cos x + i \sin x$$
$$e^{-ix} = \cos x - i \sin x$$

$\Rightarrow$

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \text{ AND } \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

THUS, WE DEFINE, FOR ANY COMPLEX NUMBER z,

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \text{ AND } \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

$\tan z$, $\cot z$, $\sec z$ AND $\csc z$ ARE THEN DEFINED BY THE USUAL FORMULAS, E.G., $\tan z = \dfrac{\sin z}{\cos z}$, ETC.

NOTE : WHEN z IS REAL THESE REDUCE TO THE USUAL TRIGONOMETRIC FUNCTIONS.

A FEW SIMPLE PROPERTIES :

1. $$\sin z = \sin x \cosh y + i \cos x \sinh y$$
$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

PROOF : $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2i}(e^{i(x+yi)} - e^{-i(x+yi)})$

$$= \frac{1}{2i}(e^{-y+xi} - e^{y-xi}) = \frac{1}{2i}(e^{-y}e^{xi} - e^{y}e^{-xi})$$

$$= \frac{1}{2i}\left(e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)\right)$$

$$= \frac{1}{2i}\left(\cos x \, (e^{-y}-e^{y}) + i \sin x \, (e^{-y}+e^{y}) \right)$$

$$= \sin x \left(\tfrac{1}{2} (e^{y}+e^{-y})\right) + i \cos x \left(\tfrac{1}{2} (e^{y}-e^{-y})\right)$$

$$= \sin x \cosh y + i \cos x \sinh y$$

AND THE PROOF FOR $\cos z$ IS SIMILAR.

2. $\cos^2 z + \sin^2 z = 1$

PROOF : $\cos^2 z + \sin^2 z = \left(\frac{1}{2}(e^{iz}+e^{-iz})\right)^2 + \left(\frac{1}{2i}(e^{iz}-e^{-iz})\right)^2$

$$= \tfrac{1}{4}(e^{2iz}+2+e^{-2iz}) - \tfrac{1}{4}(e^{2iz}-2+e^{-2iz})$$

$$= \tfrac{1}{2} + \tfrac{1}{2} = 1$$

<u>EXERCISES</u> :

17. SHOW THAT

$$e^{iz} = \cos z + i \sin z$$

FOR ANY COMPLEX z.

18. SHOW THAT

$$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

19. SHOW THAT

(a) $\quad |\sin z|^2 = \sin^2 x + \sinh^2 y$

(b) $\quad |\cos z|^2 = \cos^2 x + \sinh^2 y$

20. SHOW THAT, FOR ANY REAL NUMBER y,

(a) $$\cos(iy) = \cosh y$$

(b) $$\sin(iy) = i \sinh y$$

21. SHOW THAT

(a) $\sin z = 0 \Rightarrow z = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

(b) $\cos z = 0 \Rightarrow z = \frac{\pi}{2} + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

VI. COMPLEX HYPERBOLIC FUNCTIONS :

THESE ARE DEFINED IN THE SAME WAY AS THE REAL HYPERBOLIC FUNCTIONS.

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z})$$

$$\cosh z = \frac{1}{2}(e^{z} + e^{-z})$$

EXERCISES :

22. SHOW THAT

(a) $\sinh z = \sinh x \cos y + i \cosh x \sin y$

(b) $\cosh z = \cosh x \cos y + i \sinh x \sin y$

23. FIND ALL z SATISFYING

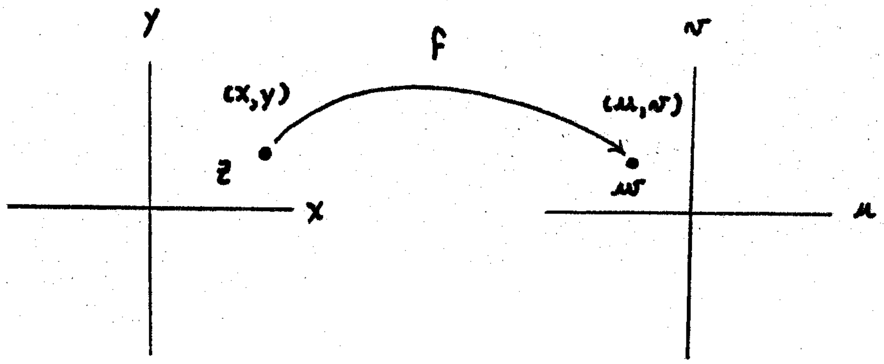
$$\cosh z = \frac{1}{2}$$

ANS. $\left(\pm \frac{\pi}{3} + 2n\pi\right) i$

$n = 0, \pm 1, \pm 2, \ldots$

OUR LIST OF EXAMPLES DOES NOT YET INCLUDE COMPLEX ANALOGUES OF SOME FAMILIAR FUNCTIONS FROM CALCULUS (E.G., $\sqrt{x}$, $\ln x, \ldots$). THESE ARE MORE SUBTLE AND WE WILL RETURN TO THEM SOON.

SINCE THE DOMAIN AND RANGE ARE BOTH 2-DIMENSIONAL THERE IS NO MEANINGFUL NOTION OF A "GRAPH" FOR A COMPLEX-VALUED FUNCTION OF A COMPLEX VARIABLE. THESE MUST BE REGARDED AS "MAPPINGS":



z - PLANE $\qquad\qquad$ w - PLANE

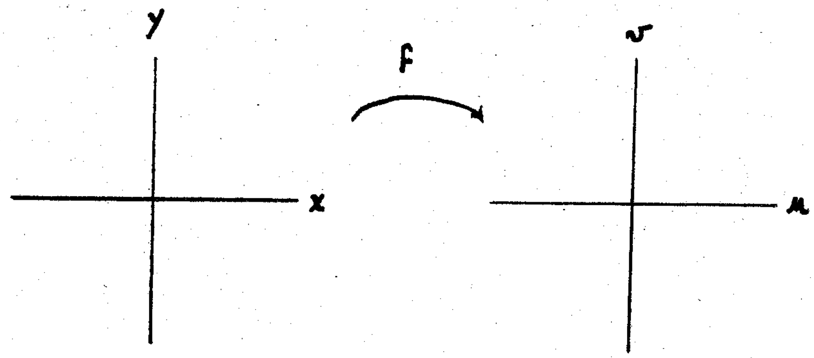
$$w = f(z)$$

WE STUDY THESE GEOMETRICALLY BY COMPUTING THE IMAGES UNDER f OF VARIOUS CURVES AND REGIONS IN z-PLANE.

<u>NOTE</u>: THIS PROCEDURE, ALTHOUGH IT CAN BECOME SOMEWHAT INVOLVED, IS ESSENTIAL NOT ONLY TO UNDERSTANDING THE NATURE OF COMPLEX FUNCTIONS, BUT ALSO TO THE TECHNIQUE OF "CONFORMAL MAPPING" THAT HAS APPLICATIONS TO FLUID FLOW, ELECTROSTATICS, HEAT FLOW, ...

WE WILL CONSIDER TWO EXAMPLES IN DETAIL.

MAPPING PROPERTIES OF $w = f(z) = z^2$:



Z-PLANE $\qquad\qquad$ w-PLANE

POLAR COORDINATES :

$$z = re^{i\theta} \qquad\qquad w = \rho e^{i\phi}$$

$$\begin{aligned} f(z) &= (x^2 - y^2) + 2xyi \\ &= r^2 e^{2\theta i} \end{aligned}$$

$$(x, y) \longrightarrow (u, v) = (x^2 - y^2,\ 2xy)$$

$$re^{i\theta} \longrightarrow \rho e^{i\phi} = r^2 e^{2\theta i}$$

IMAGE OF A POINT : SQUARE THE MODULUS AND DOUBLE THE ARGUMENT

$\pm z$ HAVE THE SAME IMAGE

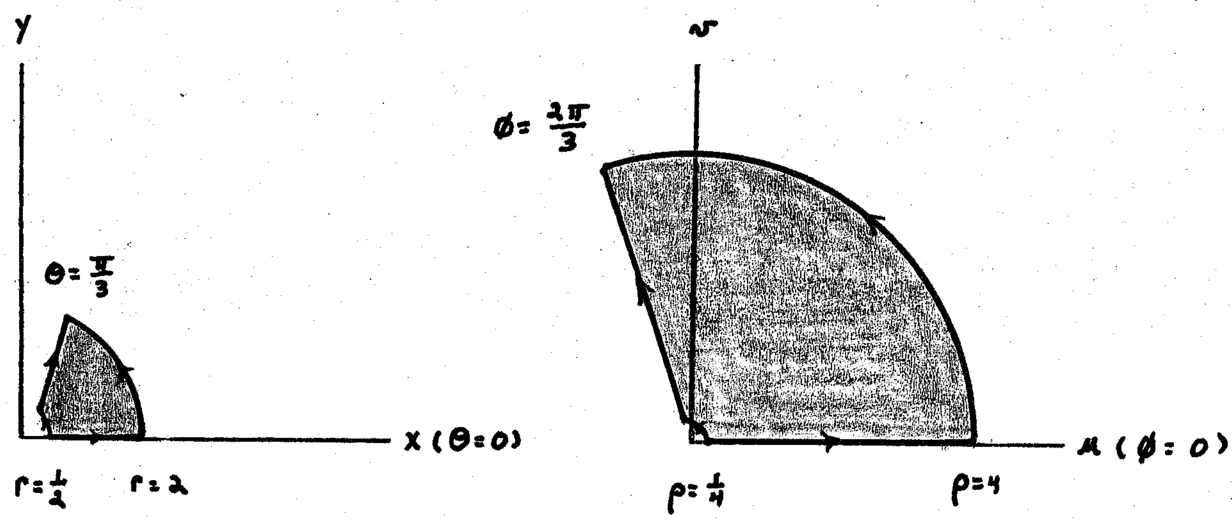
NOW WE LOOK AT IMAGES OF SOME CURVES AND REGIONS :

A. $r = r_0$ (θ VARYING) $\longrightarrow$ $\rho = r_0^2$, $\phi = 2\theta$

CIRCLE OF RADIUS r_0^2 , TRAVERSED IN THE SAME DIRECTION, BUT TWICE AS FAST

B. $\theta = \theta_0$ (r VARYING) $\longrightarrow$ $\phi = 2\theta_0$, $\rho = r^2$

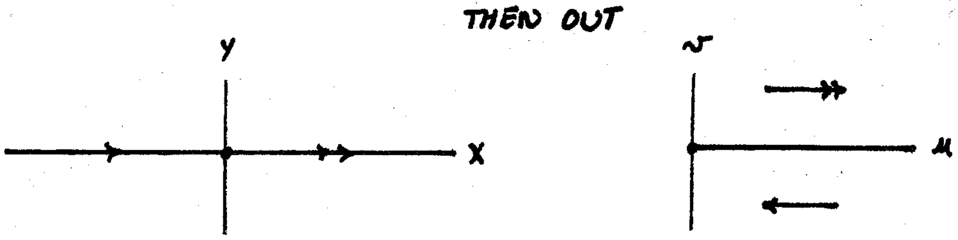
E.G.,



C. HORIZONTAL LINES : $y = c$

$y = 0$: $(x, 0) \longrightarrow (u, v) = (x^2, 0)$

x-AXIS $\longrightarrow$ POSITIVE u-AXIS, TRAVERSED TWICE AS X VARIES FROM $-\infty$ TO ∞, FIRST IN, THEN OUT

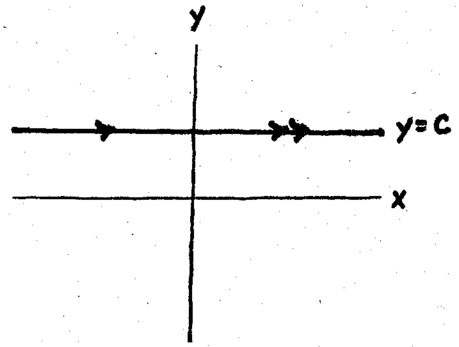


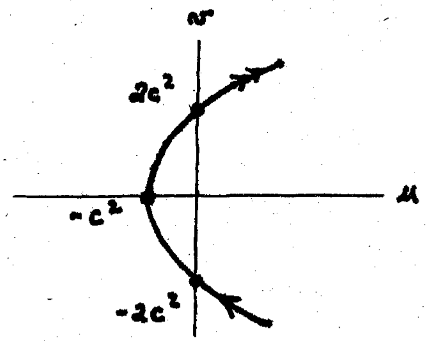
$y = c > 0$: $(x, c) \longrightarrow (u, v) = (x^2 - c^2, 2cx)$

$$v = 2cx \Rightarrow x = \frac{v}{2c}$$

$$u = x^2 - c^2 = \frac{v^2}{4c^2} - c^2$$

$u = \frac{1}{4c^2} v^2 - c^2$: PARABOLA AROUND u-AXIS, INTERCEPTS AT $(-c^2, 0)$ AND $(0, \pm 2c^2)$

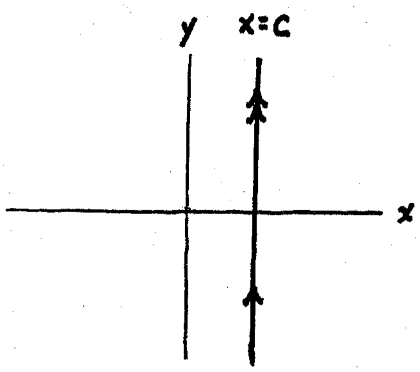


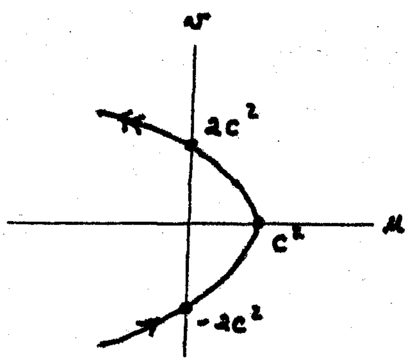


NOTE : $y = -c$ $(c > 0)$ HAS THE SAME IMAGE, BUT IT IS TRAVERSED IN THE OPPOSITE DIRECTION.

AS $c \to 0$ THE PARABOLAS PINCH OFF TO THE POSITIVE u-AXIS

D. VERTICAL LINES : SAME ANALYSIS AS ABOVE GIVES



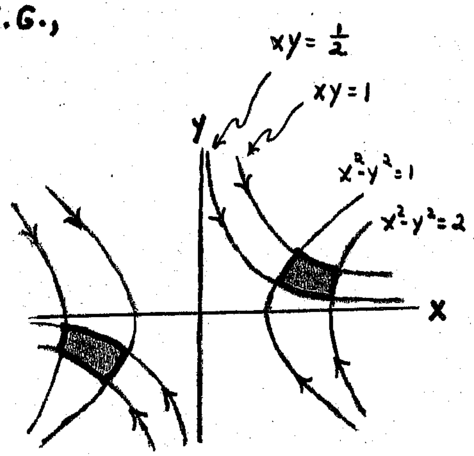


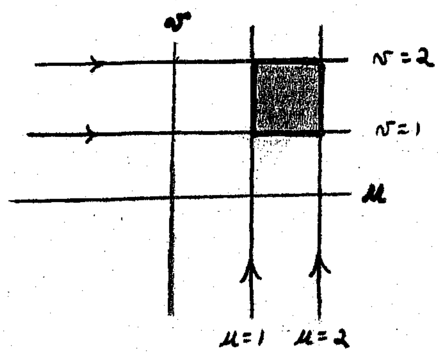
E. HYPERBOLAS

SINCE $(x,y) \longrightarrow (u,v) = (x^2-y^2,\ 2xy)$

$$x^2 - y^2 = C \quad \rightarrow \quad u = C$$
$$xy = C \quad \rightarrow \quad v = 2C$$

E.G.,

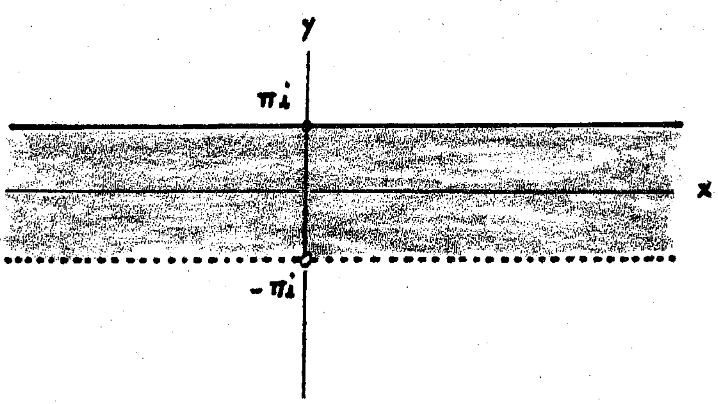




MAPPING PROPERTIES OF $w = \exp(z) = e^z$:

e^z IS PERIODIC WITH PERIOD $2\pi i$ (PAGE 3) SO ITS BEHAVIOR REPEATS OUTSIDE OF THE <u>FUNDAMENTAL</u> <u>REGION</u>

$$-\pi < \mathrm{Im}(z) \leq \pi$$



THERE IS NO VALUE OF z FOR WHICH $e^z = 0$:

PROOF: $e^z = 0 \Rightarrow e^x \cos y + i e^x \sin y = 0 + 0i$

$$\Rightarrow e^x \cos y = 0 \quad \text{AND} \quad e^x \sin y = 0$$

$$\Rightarrow \cos y = 0 \quad \text{AND} \quad \sin y = 0$$

AND THIS IS IMPOSSIBLE

HOWEVER, ANY NONZERO w IS THE IMAGE OF INFINITELY MANY z:

PROOF: LET $w = \rho e^{i\phi}$ $(\rho \neq 0)$. THEN

$$e^z = w$$

$$e^x(\cos y + i \sin y) = \rho e^{i\phi}$$

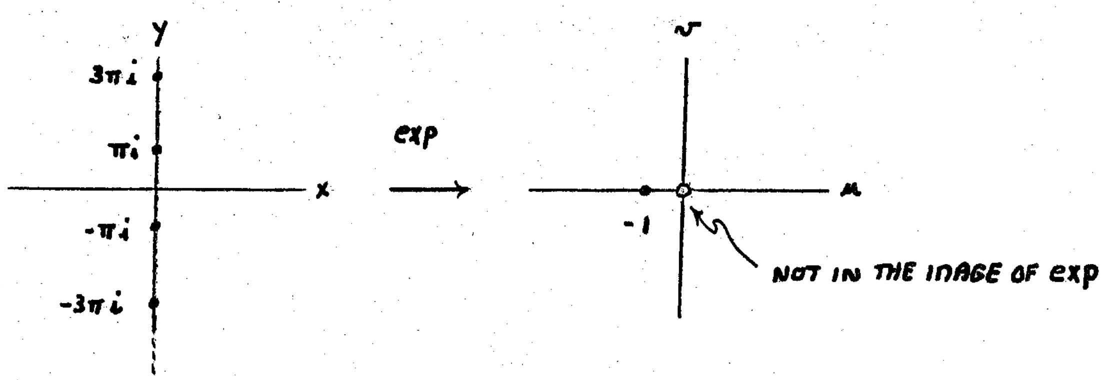
$$\Rightarrow e^x = \rho \quad \text{AND} \quad \cos y + i \sin y = \cos\phi + i \sin\phi$$

$$\Rightarrow x = \ln\rho \quad \text{AND} \quad y = \phi + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

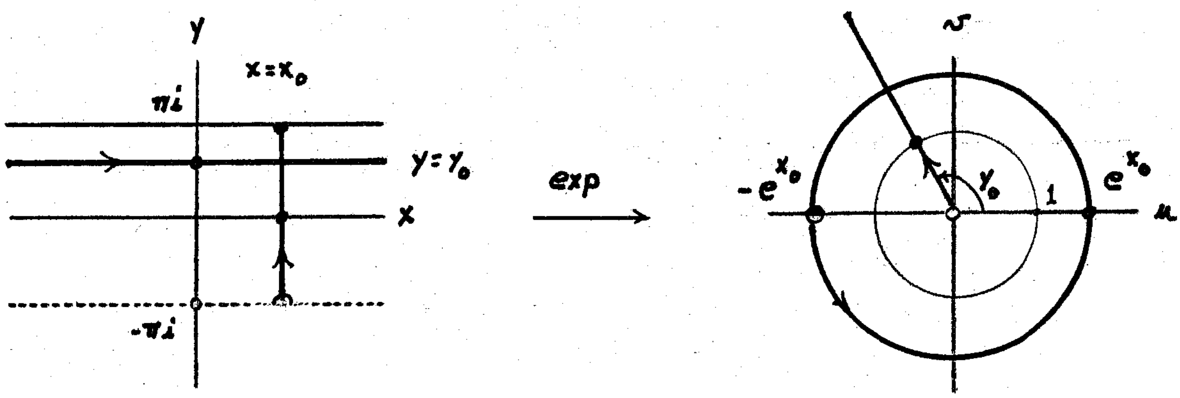
$$\Rightarrow z = \ln\rho + (\phi + 2k\pi) i$$

$$\boxed{\begin{array}{lcl} e^z = \rho e^{i\phi} & \Leftrightarrow & z = \ln\rho + (\phi + 2k\pi) i \\ \rho \neq 0 & & k = 0, \pm 1, \pm 2, \ldots \end{array}}$$

E.G., $e^z = -1 \; (= 1 e^{\pi i})$ WHEN $z = (2k+1)\pi i$



IMAGES OF HORIZONTAL AND VERTICAL LINES :



$x = x_0 \; : \; x_0 + yi \longrightarrow e^{x_0}(\cos y + i \sin y)$ (CIRCLE OF RADIUS e^{x_0} ABOUT 0)

$y = y_0 \; : \; x + y_0 i \longrightarrow e^{x}(\cos y_0 + i \sin y_0)$ (RAY INCLINED y_0 TO REAL AXIS)

NOTE THE PECULIAR HALF-OPEN DISC AT $-e^{x_0}$. THIS WILL BE IMPORTANT TO US SHORTLY.

EXERCISES :

24. FIX A REAL NUMBER θ_0 AND CONSIDER THE COMPLEX NUMBER $e^{i\theta_0}$ OF MODULUS 1. DEFINE $f : \mathbb{C} \to \mathbb{C}$ BY

$$f(z) = e^{i\theta_0} z$$

DESCRIBE THE MAPPING PROPERTIES OF THIS FUNCTION BY LOOKING AT IMAGES OF CIRCLES ABOUT $z = 0$ AND RAYS THROUGH $z = 0$.

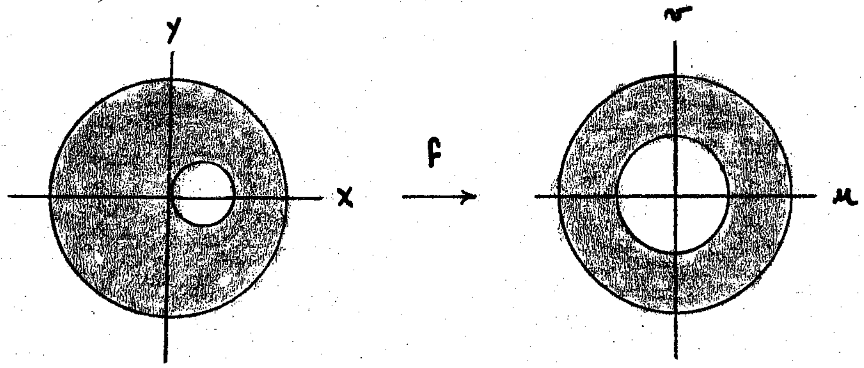
25. LET a BE A NONZERO REAL CONSTANT. SHOW THAT exp CARRIES THE STRAIGHT LINE $ay = x$ ONTO A SPIRAL.

26. DEFINE $f : \mathbb{C} - \{2\} \to \mathbb{C}$ BY

$$f(z) = \frac{2z-1}{z-2}$$

(a) SHOW THAT f MAPS THE UNIT CIRCLE $|z| = 1$ ONTO THE UNIT CIRCLE $|w| = 1$

(b) SHOW THAT f MAPS THE CIRCLE $|z - \frac{2}{5}| = \frac{2}{5}$ ONTO THE CIRCLE $|w| = \frac{1}{2}$.



<u>NOTE</u> : ONE CAN SHOW ALSO THAT THE REGION BETWEEN THE TWO CIRCLES IN THE z-PLANE IS MAPPED TO THE REGION BETWEEN THE TWO CIRCLES IN THE w-PLANE. A TYPICAL APPLICATION OF "CONFORMAL MAPPING" (TO ELECTROSTATICS) WOULD VIEW THE CIRCLES AS CROSS SECTIONS OF CHARGED CYLINDERS. SUPPOSE THE CYLINDERS IN THE z-PLANE ARE HELD AT SOME FIXED POTENTIALS AND THE PROBLEM IS TO DETERMINE THE POTENTIAL BETWEEN THEM. THE MAP f "MOVES" THE PROBLEM TO THE w-PLANE WHERE THE GEOMETRY MAKES IT EASY TO SOLVE AND THEN "MOVES" THE SOLUTION BACK WHERE YOU WANT IT. SEE PAGE 20 OF "ANALYTIC FUNCTIONS AND CONFORMAL MAPPINGS".

SOLUTIONS TO THE EXERCISES:

12. $$\frac{e^{z_1}}{e^{z_2}} = \frac{e^{x_1}(\cos y_1 + i\sin y_1)}{e^{x_2}(\cos y_2 + i\sin y_2)} = e^{x_1 - x_2}\,\frac{\cos y_1 + i\sin y_1}{\cos y_2 + i\sin y_2}\,\frac{\cos y_2 - i\sin y_2}{\cos y_2 - i\sin y_2}$$

$$= e^{x_1 - x_2}\,\frac{(\cos y_1\cos y_2 + \sin y_1\sin y_2) + i(\cos y_1\sin y_2 - \sin y_1\cos y_2)}{\cos^2 y_2 + \sin^2 y_2}$$

$$= e^{x_1 - x_2}(\cos(y_1 - y_2) + i\sin(y_1 - y_2))$$

$$= e^{z_1 - z_2}$$

13. $$\exp\left(\frac{2+\pi i}{4}\right) = e^{\frac{1}{2} + \frac{\pi}{4}i} = e^{\frac{1}{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= \sqrt{e}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{\frac{e}{2}}\,(1+i)$$

14. $$e^{z^2} = e^{x^2-y^2}(\cos 2xy + i\sin 2xy) \Rightarrow |e^{z^2}| = e^{x^2-y^2}$$

15. $$\exp(\bar{z}) = e^{x-yi} = e^{x}(\cos(-y) + i\sin(-y))$$

$$= e^{x}(\cos y - i\sin y)$$

$$= \overline{e^{x}(\cos y + i\sin y)} = \overline{\exp(z)}$$

16. e^{z} REAL $\Rightarrow$ $e^{x}\cos y + i e^{x}\sin y$ REAL $\Rightarrow$ $e^{x}\sin y = 0$

BUT $e^{x} \neq 0$ SO $\sin y = 0$, I.E., $y = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

THUS, $\operatorname{Im}(z) = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

17. $$\cos z + i\sin z = \frac{1}{2}(e^{iz} + e^{-iz}) + i\left(\frac{1}{2i}(e^{iz} - e^{-iz})\right)$$

$$= \frac{1}{2}(e^{iz} + e^{-iz} + e^{iz} - e^{-iz})$$

$$= e^{iz}$$

18. $\sin z_1 \cos z_2 + \cos z_1 \sin z_2 =$

$$\left(\tfrac{1}{2i}(e^{iz_1}-e^{-iz_1})\right)\left(\tfrac{1}{2}(e^{iz_2}+e^{-iz_2})\right)+\left(\tfrac{1}{2}(e^{iz_1}+e^{-iz_1})\right)\left(\tfrac{1}{2i}(e^{iz_2}-e^{-iz_2})\right)$$

$$= \tfrac{1}{4i}\Big(e^{i(z_1+z_2)} - e^{-i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{-i(z_1-z_2)} + e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}$$
$$- e^{i(z_1-z_2)} + e^{-i(z_1-z_2)} \Big)$$

$$= \tfrac{1}{4i}\left(2e^{i(z_1+z_2)} - 2e^{-i(z_1+z_2)} \right)$$

$$= \tfrac{1}{2i}\left(e^{i(z_1+z_2)} - e^{-i(z_1+z_2)} \right) = \sin(z_1+z_2)$$

19. (a) $\sin z = \sin x \cosh y + i \cos x \sinh y \Rightarrow$

$$|\sin z|^2 = (\sin x \cosh y)^2 + (\cos x \sinh y)^2$$
$$= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$
$$= \sin^2 x\,(1+\sinh^2 y) + \cos^2 x \sinh^2 y$$
$$= \sin^2 x + (\sin^2 x + \cos^2 x)\sinh^2 y$$
$$= \sin^2 x + \sinh^2 y$$

(b) $\cos z = \cos x \cosh y - i \sin x \sinh y \Rightarrow$

$$|\cos z|^2 = (\cos x \cosh y)^2 + (-\sin x \sinh y)^2$$
$$= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$$
$$= \cos^2 x\,(1+\sinh^2 y) + \sin^2 x \sinh^2 y$$
$$= \cos^2 x + (\cos^2 x + \sin^2 x)\sinh^2 y$$
$$= \cos^2 x + \sinh^2 y$$

20. (a) $\cos(iy) = \tfrac{1}{2}(e^{i(iy)} + e^{-i(iy)}) = \tfrac{1}{2}(e^{-y}+e^{y}) = \cosh y$

20. (b) $\sin(iy) = \frac{1}{2i}(e^{i(iy)} - e^{-i(iy)}) = \frac{1}{2i}(e^{-y} - e^{y})$

$$= -\frac{1}{i}\,\frac{1}{2}(e^{y} - e^{-y})$$

$$= -(-i)\sinh y \quad = i \sinh y$$

21. (a) $\sin z = 0 \Rightarrow \sin x \cosh y + i \cos x \sinh y = 0 + 0i \Rightarrow$

$$\sin x \cosh y = 0 \qquad \cos x \sinh y = 0$$

$$\sin x = 0$$

$$x = n\pi \Rightarrow \cos n\pi \sinh y = 0$$

$$(-1)^n \sinh y = 0$$

$$\sinh y = 0$$

$$y = 0$$

$$z = n\pi + 0i, \quad n = 0, \pm 1, \pm 2, \cdots$$

(b) $\cos z = 0 \Rightarrow \cos x \cosh y - i \sin x \sinh y = 0 + 0i \Rightarrow$

$$\cos x \cosh y = 0 \qquad \sin x \sinh y = 0$$

$$\cos x = 0$$

$$x = \frac{\pi}{2} + n\pi \Rightarrow \sin(\tfrac{\pi}{2} + n\pi)\sinh y = 0$$

$$(-1)^n \sinh y = 0$$

$$\sinh y = 0$$

$$y = 0$$

$$z = (\tfrac{\pi}{2} + n\pi) + 0i, \quad n = 0, \pm 1, \pm 2, \cdots$$

22. (a) $\sinh z = \frac{1}{2}(e^{z} - e^{-z})$

$$= \frac{1}{2}\left(e^{x}\cos y + i e^{x}\sin y - e^{-x}\cos(-y) - i e^{-x}\sin(-y)\right)$$

$$= \frac{1}{2}\left((e^{x}-e^{-x})\cos y + i\,(e^{x}+e^{-x})\sin y\right)$$

$$= \sinh x \cos y + i \cosh x \sin y$$

(b) $\cosh z = \frac{1}{2}(e^{z} + e^{-z})$

$$= \frac{1}{2}\left(e^{x}\cos y + i e^{x}\sin y + e^{-x}\cos(-y) + i e^{-x}\sin(-y)\right)$$

$$= \frac{1}{2}\left((e^{x}+e^{-x})\cos y + i\,(e^{x}-e^{-x})\sin y\right)$$

$$= \cosh x \cos y + i \sinh x \sin y$$

23. $\cosh z = \frac{1}{2} \Rightarrow \cosh x \cos y + i \sinh x \sin y = \frac{1}{2} + 0i \Rightarrow$

$$\cosh x \cos y = \frac{1}{2} \qquad \sinh x \sin y = 0$$

NOW, $\sinh x \sin y = 0 \Rightarrow \sinh x = 0$ OR $\sin y = 0$

$\Rightarrow \quad x = 0$ OR $y = n\pi$

CASE 1 : $x = 0$

$\cosh 0 \cos y = \frac{1}{2} \Rightarrow \cos y = \frac{1}{2} \Rightarrow y = \pm\frac{\pi}{3} + 2n\pi$ SO

$$z = 0 + \left(\pm\frac{\pi}{3} + 2n\pi\right)i \quad , n = 0, \pm 1, \pm 2, \ldots$$

CASE 2 : $y = n\pi$

$$\cosh x \cos n\pi = \frac{1}{2}$$

$$(-1)^{n}\cosh x = \frac{1}{2}$$

WHICH HAS NO SOLUTIONS BECAUSE $\cosh x \geq 1$ FOR ALL x

THUS,

$$z = \left(\pm\frac{\pi}{3} + 2n\pi\right)i \quad , n = 0, \pm 1, \pm 2, \ldots$$

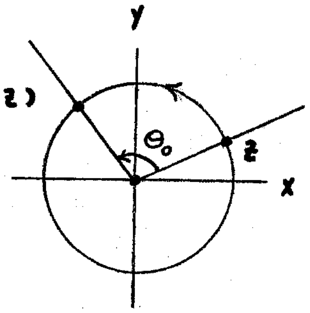
24. $f(z) = e^{i\theta_0} z \quad (\theta_0 \text{ FIXED})$

$f(0) = 0$ NOW SUPPOSE $z \neq 0$ AND WRITE $z = re^{i\theta}$. THEN

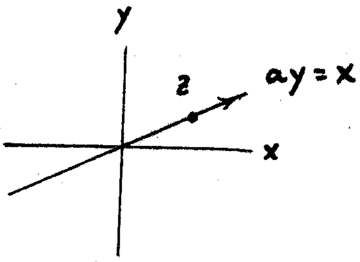
$$f(z) = e^{i\theta_0}(re^{i\theta}) = re^{i(\theta+\theta_0)}$$

$$= r(\cos(\theta+\theta_0) + i\sin(\theta+\theta_0))$$

THUS, f SIMPLY ROTATES ANY NONZERO z THROUGH AN ANGLE OF θ_0, CARRYING CIRCLES $|z| = r_0$ TO THEMSELVES AND ROTATING RAYS.



25. a = NONZERO REAL CONSTANT



ANY z ON $ay = x$ IS OF THE FORM $z = ay + yi$ SO

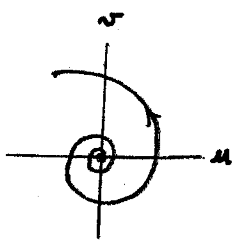
$$w = \exp(z) = e^{ay+yi} = e^{ay}(\cos y + i\sin y)$$

WRITING $w = \rho e^{i\phi} = \rho(\cos\phi + i\sin\phi)$, THIS GIVES

$$\rho = e^{ay}$$
$$\phi = y$$

SO

$$\rho = e^{a\phi}$$



WHICH IS THE POLAR EQUATION FOR A (LOGARITHMIC) SPIRAL IN THE w-PLANE (PICTURE ASSUMES $a > 0$).

26. $f : \mathbb{C} - \{2\} \to \mathbb{C}$

$$f(z) = \frac{2z-1}{z-2}$$

(a) $|z| = 1 \Rightarrow$

$$|w|^2 = w\bar{w} = f(z)\overline{f(z)} = \frac{2z-1}{z-2}\,\frac{2\bar{z}-1}{\bar{z}-2}$$

$$= \frac{4z\bar{z} - 2(z+\bar{z}) + 1}{z\bar{z} - 2(z+\bar{z}) + 4} = \frac{4 - 2(z+\bar{z}) + 1}{1 - 2(z+\bar{z}) + 4}$$

$$= \frac{5 - 2(z+\bar{z})}{5 - 2(z+\bar{z})} = 1$$

(b) $|z - \frac{2}{5}| = \frac{2}{5} \Rightarrow (z - \frac{2}{5})(\bar{z} - \frac{2}{5}) = \frac{4}{25}$

$$\Rightarrow z\bar{z} - \frac{2}{5}(z+\bar{z}) + \frac{4}{25} = \frac{4}{25}$$

$$\Rightarrow 5z\bar{z} = 2(z+\bar{z})$$

SO

$$|w|^2 = w\bar{w} = \frac{4z\bar{z} - 2(z+\bar{z}) + 1}{z\bar{z} - 2(z+\bar{z}) + 4} = \frac{4z\bar{z} - 5z\bar{z} + 1}{z\bar{z} - 5z\bar{z} + 4}$$

$$= \frac{1 - z\bar{z}}{4(1 - z\bar{z})} = \frac{1}{4}$$

AND

$$|w| = \frac{1}{2}$$