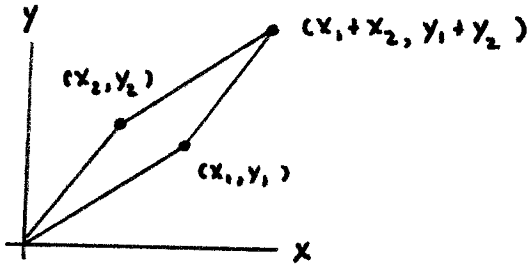
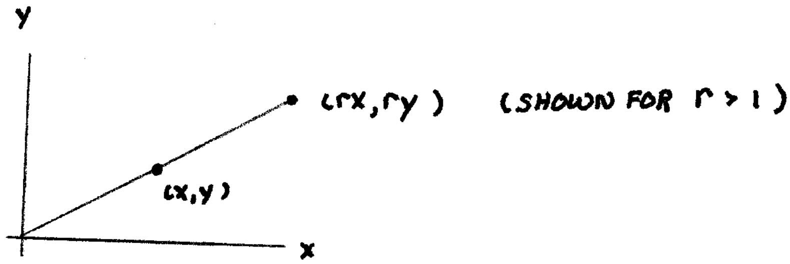
COMPLEX NUMBERS

POINTS (x,y) IN THE PLANE CAN BE THOUGHT OF AS "VECTORS" BECAUSE THEY CAN BE ADDED (COORDINATEWISE / PARALLELOGRAM LAW)



AND MULTIPLIED BY REAL NUMBERS (COORDINATEWISE / SCALED).



IT IS ALSO POSSIBLE TO DEFINE A MULTIPLICATION FOR POINTS IN THE PLANE.

$$(x_1,y_1)(x_2,y_2) = (x_1x_2 - y_1y_2 \ , \ x_1y_2 + x_2y_1)$$

NOTE : THIS MAY LOOK STRANGE, BUT ONE CAN PROVE THAT IT IS THE ONLY WAY TO DEFINE MULTIPLICATION IF ONE WOULD LIKE TO HAVE ALL OF THE "USUAL" PROPERTIES OF REAL NUMBER ARITHMETIC.

WITH THESE ALGEBRAIC OPERATIONS THE POINTS IN THE PLANE ARE CALLED COMPLEX NUMBERS AND THE PLANE ITSELF IS CALLED THE COMPLEX PLANE AND DENOTED

$$\mathbb{C}$$

MORE COMMON NOTATION :

NOTICE THAT

$$(x,y) = x(1,0) + y(0,1)$$

$(1,0)(a,b) = (a,b)$ (SO $(1,0)$ "BEHAVES LIKE" 1)

$(0,1)^2 = (0,1)(0,1) = (-1,0) = (-1)(1,0)$ (SO $(0,1)$ "BEHAVES LIKE" A SQUARE ROOT OF -1)

IDENTIFYING $(1,0)$ WITH 1 AND DEFINING

$$i = (0,1)$$

ANY COMPLEX NUMBER IS WRITTEN

$$x + yi$$

WHERE

$$i^2 = -1 .$$

WITH THIS NOTATION, ADDITION AND MULTIPLICATION LOOK LIKE THIS :

IF $z_1 = x_1 + y_1 i$ AND $z_2 = x_2 + y_2 i$, THEN

$$\begin{aligned} z_1 + z_2 &= (x_1 + y_1 i) + (x_2 + y_2 i) \\ &= (x_1 + x_2) + (y_1 + y_2) i \end{aligned}$$

$$\begin{aligned} z_1 z_2 &= (x_1 + y_1 i)(x_2 + y_2 i) \\ &= (x_1 x_2 - y_1 y_2) + (x_1 y_2 - x_2 y_1) i \end{aligned}$$

(JUST THE USUAL ALGEBRA, BUT $i^2 = -1$).

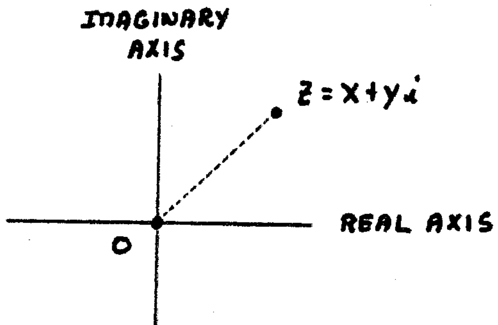
SOME NOTATION AND TERMINOLOGY :

$$z = x + yi$$

$$x = Re(z) = \underline{\text{REAL PART}} \text{ OF } z$$

$$y = Im(z) = \underline{\text{IMAGINARY PART}} \text{ OF } z$$

$\mathbb{C}$



- $\bar{z} = x - yi = \underline{\text{CONJUGATE}} \text{ OF } z$

$$|z| = \sqrt{x^2+y^2} = \underline{\text{MODULUS}} \text{ OF } z$$

A FEW IMPORTANT FACTS :

1. $z\bar{z} = |z|^2$

PROOF : $z\bar{z} = (x+yi)(x-yi) = x^2 - (yi)^2 = x^2 - y^2 i^2 = x^2+y^2 = |z|^2$

2. $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

PROOF :
$$\overline{z_1 z_2} = \overline{(x_1+y_1 i)(x_2+y_2 i)} = \overline{(x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i}$$
$$= (x_1x_2 - y_1y_2) - (x_1y_2 + x_2y_1)i$$

$$\bar{z}_1\bar{z}_2 = (x_1 - y_1 i)(x_2 - y_2 i) = (x_1x_2 - y_1y_2) - (x_1y_2 + x_2y_1)i$$
$$= \overline{z_1 z_2}$$

3. EXERCISE 1 : PROVE THAT

(a) $z_1 + z_2 = z_2 + z_1$

(b) $z_1 z_2 = z_2 z_1$

(c) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(d) $\bar{\bar{z}} = z$

(e) $|\bar{z}| = |z|$

4. $|z_1 z_2| = |z_1||z_2|$

PROOF : $|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 z_2)(\bar{z}_1 \bar{z}_2)$

$= (z_1 \bar{z}_1)(z_2 \bar{z}_2) = |z_1|^2 |z_2|^2$

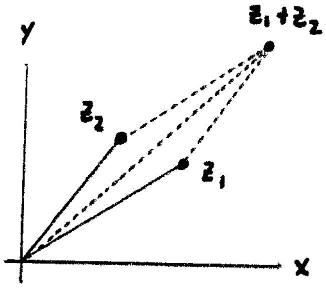
$= (|z_1||z_2|)^2$

SO

$|z_1 z_2| = |z_1||z_2|$

5. (TRIANGLE INEQUALITY) $|z_1 + z_2| \le |z_1| + |z_2|$

PROOF :



$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2})$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

$= |z_1|^2 + |z_2|^2 + (z_1\bar{z}_2 + \overline{z_1\bar{z}_2})$

$= |z_1|^2 + |z_2|^2 + 2\,Re\,(z_1\bar{z}_2)$

$\le |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$

$|z_1+z_2|^2 \le |z_1|^2 + 2|z_1||z_2| + |z_2|^2$

$|z_1+z_2|^2 \le (|z_1| + |z_2|)^2$

$|z_1+z_2| \le |z_1| + |z_2|$

DIVISION : E.G.,

$$\frac{1+i}{2-3i} = \frac{1+i}{2-3i}\,\frac{2+3i}{2+3i} = \frac{(1+i)(2+3i)}{|2-3i|^2} = \frac{-1+5i}{13}$$

$$= -\frac{1}{13} + \frac{5}{13}i$$

AND, IN GENERAL,

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\,\frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1\bar{z}_2}{|z_2|^2} \qquad \text{ETC.}$$

(PROVIDED $z_2 \neq 0$)

EXERCISES :

2. WRITE EACH OF THE FOLLOWING IN THE FORM $x+yi$:

(a) $(\sqrt{2}-i)-i(1-\sqrt{2}i)$ ANS. $-2i$ $(0+(-2)i)$

(b) $(1-i)^4$ ANS. -4 $(-4+0i)$

(c) $\dfrac{1+2i}{3-4i}$ ANS. $-\frac{1}{5}+\frac{2}{5}i$

(d) $\dfrac{5}{(1-i)(2-i)(3-i)}$ ANS. $\frac{1}{2}i$

3. SHOW THAT $z = 1+i$ SATISFIES $z^2 - 2z + 2 = 0$.

4. PROVE THAT

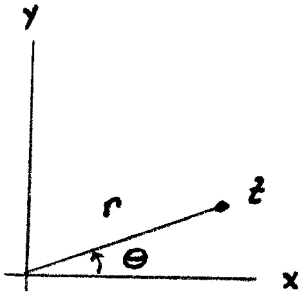
(a) $\operatorname{Re}(iz) = -\operatorname{Im}(z)$

(b) $\operatorname{Im}(iz) = \operatorname{Re}(z)$

(c) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ (FOR $z_2 \neq 0$)

(d) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (FOR $z_2 \neq 0$)

POLAR FORM : FOR $z = x + yi \neq 0$,



$$z = x + yi = r\cos\theta + i r\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

NOTE : $r = |z| > 0$

θ IS DETERMINED ONLY UP TO INTEGER MULTIPLES OF 2π

E.G., $1+i = \sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$$= \sqrt{2}\,(\cos(\tfrac{\pi}{4} + 2k\pi) + i\sin(\tfrac{\pi}{4} + 2k\pi)),\ k = 0, \pm1, \pm2, \ldots$$

ANY SUCH VALUE OF θ IS AN **ARGUMENT** OF z AND THE SET OF ALL OF THESE ARGUMENTS IS DENOTED

$$\mathrm{ARG}(z).$$

THE UNIQUE VALUE IN THE INTERVAL $-\pi < \theta \le \pi$ IS THE **PRINCIPAL VALUE** OF THE ARGUMENT AND IS DENOTED

$$\arg(z).$$

E.G., $\mathrm{ARG}(1+i) = \{ \frac{\pi}{4} + 2k\pi : k = 0, \pm1, \pm2, \ldots \}$ AND

$\arg(1+i) = \frac{\pi}{4}$.

$$\mathrm{ARG}(z) = \{ \arg(z) + 2k\pi : k = 0, \pm1, \pm2, \ldots \}$$

PRODUCTS AND QUOTIENTS :

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$\Rightarrow$

$$z_1 z_2 = r_1 r_2 \left(\cos\theta_1 + i\sin\theta_1\right)\left(\cos\theta_2 + i\sin\theta_2\right)$$

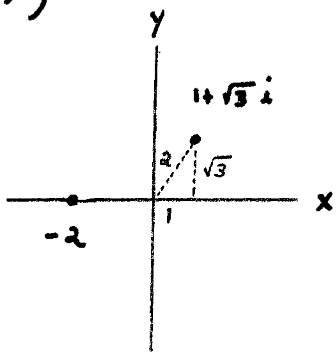
$$= r_1 r_2 \left((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right)$$

$$= r_1 r_2 \left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$

(MULTIPLY THE MODULI AND ADD THE ARGUMENTS)

SIMILARLY,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$$



E.G., $\quad z_1 = -2 = 2(\cos\pi + i\sin\pi)$

$$z_2 = 1+\sqrt{3}\,i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\Rightarrow \quad z_1 z_2 = (2)(2)\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= 4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -2-2\sqrt{3}\,i$$

AND

$$\frac{z_1}{z_2} = \frac{2}{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

SPECIAL CASE : $\quad z = r(\cos\theta + i\sin\theta) \Rightarrow$

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

AND, MORE GENERALLY,

$$z^n = r^n(\cos n\theta + i\sin n\theta),$$

$n = 0, 1, 2, \cdots$

IN PARTICULAR, IF $|z| = 1$,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(De Moivre's FORMULA)

APPLICATION (DOUBLE ANGLE FORMULAS):

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

BUT ALSO,

$$(\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + (2\sin\theta\cos\theta)i$$

SO

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \quad \text{AND} \quad \sin 2\theta = 2\sin\theta\cos\theta$$

MORE IMPORTANT APPLICATION (ROOTS):

EVERY NONZERO COMPLEX NUMBER HAS n DISTINCT n^{TH} ROOTS AND WE NEED TO COMPUTE THEM. FIRST, AN EXAMPLE.

FIND THE THREE CUBE ROOTS OF $1+\sqrt{3}\,i$, I.E., THE THREE SOLUTIONS TO

$$z^3 = 1+\sqrt{3}\,i$$

WRITE

$$\begin{aligned} 1+\sqrt{3}\,i &= 2\left(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\right) \\ &= 2\left(\cos\left(\tfrac{\pi}{3}+2k\pi\right) + i\sin\left(\tfrac{\pi}{3}+2k\pi\right)\right) \end{aligned}$$

NOW WRITE

$$z = r(\cos\theta + i\sin\theta)$$

THEN $z^3 = r^3 (\cos 3\theta + i \sin 3\theta)$ SO

$$r^3(\cos 3\theta + i \sin 3\theta) = 2\left(\cos(\tfrac{\pi}{3} + 2k\pi) + i \sin(\tfrac{\pi}{3} + 2k\pi)\right)$$

IMPLIES

$$r^3 = 2 \quad \text{AND} \quad 3\theta = \frac{\pi}{3} + 2k\pi$$

SO

$$r = \sqrt[3]{2} \qquad \text{(THE USUAL POSITIVE CUBE ROOT OF 2)}$$

AND

$$\theta = \frac{\pi}{9} + \frac{2k\pi}{3}$$

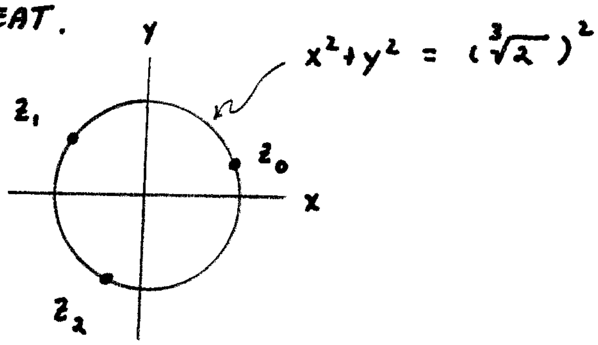
CHOOSING $k = 0, 1, 2$ GIVES

$$z_0 = \sqrt[3]{2}\left(\cos\frac{\pi}{9} + i \sin\frac{\pi}{9}\right)$$
$$z_1 = \sqrt[3]{2}\left(\cos\frac{7\pi}{9} + i \sin\frac{7\pi}{9}\right)$$
$$z_2 = \sqrt[3]{2}\left(\cos\frac{13\pi}{9} + i \sin\frac{13\pi}{9}\right)$$

NOTE THAT $k = 3$ WOULD GIVE

$$\sqrt[3]{2}\left(\cos(\tfrac{\pi}{9} + 2\pi) + i \sin(\tfrac{\pi}{9} + 2\pi)\right) = z_0$$

SO THEY BEGIN TO REPEAT.



THREE EQUALLY SPACED (BY $\frac{2\pi}{3}$)
POINTS ON THE CIRCLE OF RADIUS $\sqrt[3]{2}$
ABOUT THE ORIGIN.

IN GENERAL, TO FIND THE n DISTINCT n^{TH} ROOTS OF

$$w = R(\cos\phi + i\sin\phi)$$

WRITE

$$z = r(\cos\theta + i\sin\theta).$$

THEN $z^n = w$ IMPLIES

$$r^n(\cos n\theta + i\sin n\theta) = R(\cos(\phi + 2k\pi) + i\sin(\phi + 2k\pi))$$

SO

$$r = \sqrt[n]{R}$$

$$\theta = \frac{\phi}{n} + \frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n-1$$

THE SYMBOL

$$\sqrt[n]{w} \qquad (\text{OR } w^{\frac{1}{n}})$$

IS USED TO STAND FOR THE SET OF <u>ALL</u> OF THESE VALUES

$$\sqrt[n]{w} = \sqrt[n]{R}\left(\cos\left(\frac{\phi}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\phi}{n} + \frac{2k\pi}{n}\right)\right)$$

$$k = 0, 1, \ldots, n-1$$

(n EQUALLY SPACED (BY $\frac{2\pi}{n}$) POINTS ON THE CIRCLE OF RADIUS $\sqrt[n]{R}$ ABOUT O)

THE <u>PRINCIPAL</u> n^{TH} <u>ROOT</u> OF w IS OBTAINED BY TAKING $\phi = \arg(w)$ AND $k = 0$, E.G., FOR $w = 1 + \sqrt{3}\,i$, THE PRINCIPAL CUBE ROOT IS $\sqrt[3]{2}\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)$.

<u>CAUTION</u>: THIS NOTATION IS STANDARD, BUT INVOLVES SOME AMBIGUITY ($\sqrt[n]{w}$ VERSUS $\sqrt[n]{R}$).

SPECIAL CASE (THE n^{TH} ROOTS OF UNITY) :

APPLYING THIS TO $1 = 1(\cos 0 + i \sin 0)$ GIVES THE SOLUTIONS TO $z^n = 1$ AS

$$\omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \quad , \ k = 0, 1, \ldots, n-1$$

NOTE THAT

$$\omega_0 = 1.$$

ω_1 IS USUALLY WRITTEN SIMPLY

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

THEN

$$\omega_k = \omega^k \qquad \text{FOR } k = 2, 3, \ldots, n-1$$

SO THE ROOTS ARE

$$1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$$

(n EQUALLY SPACED POINTS ON THE UNIT CIRCLE, STARTING WITH 1)

EXERCISES :

5. PERFORM THE FOLLOWING OPERATIONS BY FIRST WRITING THE COMPLEX NUMBERS IN POLAR FORM, BUT WRITE THE ANSWER IN THE FORM $x + yi$.

(a) $(\sqrt{3} - i)^6$ ANS. -64

(b) $i(1-\sqrt{3}i)(\sqrt{3}+i)$ ANS. $2 + 2\sqrt{3}i$

(c) $(-1 + i)^7$ ANS. $-8 - 8i$

6. EXPRESS ALL OF THE FOLLOWING ROOTS IN THE FORM $x+yi$ AND INDICATE WHICH IS THE PRINCIPAL ROOT.

(a) $\sqrt{2i}$

ANS. $1+i$ (PRINCIPAL)
$-1-i$

(b) $(-1)^{\frac{1}{3}}$

ANS. $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ (PRINCIPAL)
-1
$\frac{1}{2}-\frac{\sqrt{3}}{2}i$

(c) $\sqrt[4]{-16}$

ANS. $\sqrt{2}(1+i)$ (PRINCIPAL)
$\sqrt{2}(-1+i)$
$-\sqrt{2}(1+i)$
$-\sqrt{2}(-1+i)$

7. FIND ALL SOLUTIONS TO THE EQUATION $z^4+4=0$ AND USE THESE ROOTS TO FACTOR z^4+4 INTO THE PRODUCT OF TWO QUADRATICS WITH REAL COEFFICIENTS.

ANS. ROOTS: $1+i$, $-1+i$, $-1-i$, $1-i$
$z^4+4=$
$(z^2-2z+2)(z^2+2z+2)$

8. PROVE THAT

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

AND

$$\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$$

9. LET c BE ANY OF THE n^{TH} ROOTS OF 1 EXCEPT 1. SHOW THAT

$$1+c+c^2+\cdots+c^{n-1} = 0.$$

10. LET z BE A NONZERO COMPLEX NUMBER AND n A POSITIVE INTEGER. DEFINE

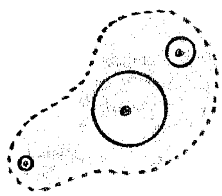
$$z^{-n} = \frac{1}{z^n}$$

SHOW THAT IF $z = r(\cos\theta + i\sin\theta)$, THEN

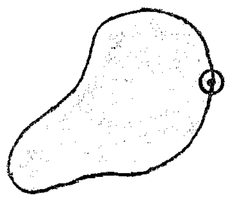
$$z^{-n} = r^{-n}(\cos n\theta - i\sin n\theta).$$

<u>SOME</u> <u>TECHNICAL</u> <u>TERMINOLOGY</u> (IN PREPARATION FOR OUR DISCUSSION OF COMPLEX-VALUED FUNCTIONS OF A COMPLEX VARIABLE):

A REGION R IN THE COMPLEX PLANE $\mathbb{C}$ IS SAID TO BE <u>OPEN</u> IF FOR EACH POINT z IN R THERE IS SOME (SUFFICIENTLY SMALL) DISC CENTERED AT z THAT IS ENTIRELY CONTAINED IN R, E.G.,

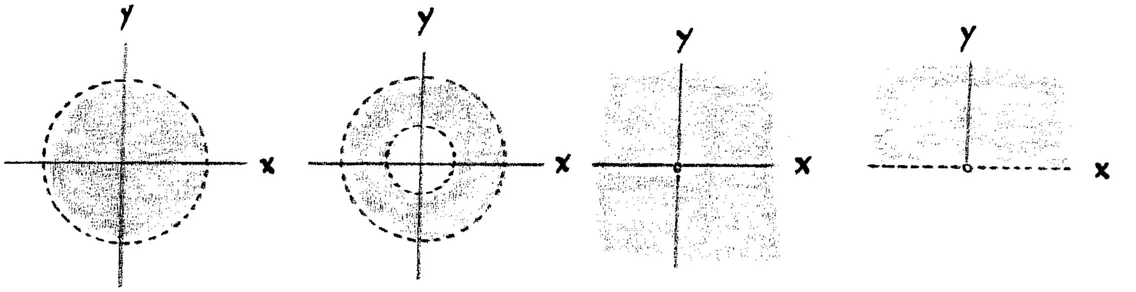


BUT NOT



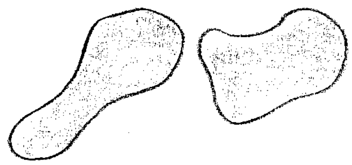
(INTUITIVELY, AN OPEN SET DOES NOT CONTAIN ANY OF ITS "BOUNDARY POINTS")

SOME COMMON EXAMPLES OF OPEN SETS:



UNIT DISC $|z| < 1$ | ANNULUS $\frac{1}{2} < |z| < 1$ | PUNCTURED PLANE $|z| \neq 0$ | UPPER HALF-PLANE $Im(z) > 0$

A REGION R IN $\mathbb{C}$ IS <u>CONNECTED</u> IF ANY TWO POINTS IN R CAN BE JOINED BY A CONTINUOUS CURVE CONTAINED ENTIRELY IN R, E.G., ANY OF THE ABOVE, BUT NOT



A <u>DOMAIN</u> IN $\mathbb{C}$ IS A REGION THAT IS BOTH OPEN AND CONNECTED.

<u>EXERCISE</u> 11: DESCRIBE EACH OF THE FOLLOWING REGIONS IN $\mathbb{C}$ AND DECIDE WHETHER OR NOT THEY ARE DOMAINS.

(a) $Re(z) > 1$

(b) $0 \leq \arg(z) \leq \frac{\pi}{4}$

(c) $Im(z) = 1$

(d) $|z - 2 + i| < 1$

(e) $|2z + 3| > 4$

SOLUTIONS TO THE EXERCISES :

1. (a) $z_1 + z_2 = (x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2) i$

$$= (x_2 + x_1) + (y_2 + y_1) i$$

$$= (x_2 + y_2 i) + (x_1 + y_1 i)$$

$$= z_2 + z_1$$

(b) $z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i$

$$= (x_2 x_1 - y_2 y_1) + (x_2 y_1 + x_1 y_2) i$$

$$= (x_2 + y_2 i)(x_1 + y_1 i)$$

$$= z_2 z_1$$

(c) $\overline{z_1 + z_2} = \overline{(x_1 + y_1 i) + (x_2 + y_2 i)} = \overline{(x_1 + x_2) + (y_1 + y_2) i}$

$$= (x_1 + x_2) - (y_1 + y_2) i$$

$$= (x_1 - y_1 i) + (x_2 - y_2 i)$$

$$= \bar{z}_1 + \bar{z}_2$$

(d) $\bar{\bar{z}} = \overline{x - yi} = x - (-y) i = x + yi = z$

(e) $|\bar{z}| = |x - yi| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$

2. (a) $(\sqrt{2} - i) - i(1 - \sqrt{2} i) = \sqrt{2} - i - i + \sqrt{2} i^2 = \sqrt{2} - 2i - \sqrt{2} = -2i$

(b) $(1 - i)^4 = (1 - i)^2 (1 - i)^2 = (-2i)(-2i) = -4$

(c) $\dfrac{1 + 2i}{3 - 4i} = \dfrac{1 + 2i}{3 - 4i} \dfrac{3 + 4i}{3 + 4i} = \dfrac{-5 + 10i}{3^2 + 4^2} = -\dfrac{5}{25} + \dfrac{10}{25} i = -\dfrac{1}{5} + \dfrac{2}{5} i$

(d) $\dfrac{5}{(1 - i)(2 - i)(3 - i)} = \dfrac{5}{(1 - 3i)(3 - i)} = \dfrac{5}{-10i} = -\dfrac{1}{2i}$

$$= -\frac{1}{2i} \frac{-2i}{-2i} = \frac{2i}{4} = \frac{1}{2} i$$

3. $z = 1+i$ $\qquad z^2 = (1+i)(1+i) = 2i$

$$z^2 - 2z + 2 = 2i - 2(1+i) + 2 = 2i - 2 - 2i + 2 = 0$$

4. (a) $Re(iz) = Re(i(x+yi)) = Re(-y+xi) = -y = -Im(z)$

(b) $Im(iz) = Im(i(x+yi)) = Im(-y+xi) = x = Re(z)$

(c)
$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1\bar{z}_2}{|z_2|^2} = \frac{(x_1+y_1i)(x_2-y_2i)}{x_2^2+y_2^2}$$
$$= \frac{(x_1x_2+y_1y_2)+(-x_1y_2+x_2y_1)i}{x_2^2+y_2^2}$$
$$= \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + \frac{-x_1y_2+x_2y_1}{x_2^2+y_2^2}i$$

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{\bar{z}_1 z_2}{\bar{z}_2 z_2} = \frac{(x_1-y_1i)(x_2+y_2i)}{x_2^2+y_2^2}$$
$$= \frac{(x_1x_2+y_1y_2)+(x_1y_2-x_2y_1)i}{x_2^2+y_2^2}$$
$$= \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} - \frac{-x_1y_2+x_2y_1}{x_2^2+y_2^2}i = \overline{\left(\frac{z_1}{z_2}\right)}$$

(d)
$$\left|\frac{z_1}{z_2}\right|^2 = \frac{z_1}{z_2}\left(\overline{\frac{z_1}{z_2}}\right) = \frac{z_1}{z_2}\frac{\bar{z}_1}{\bar{z}_2} = \frac{z_1\bar{z}_1}{z_2\bar{z}_2}$$
$$= \frac{|z_1|^2}{|z_2|^2}$$
$$= \left(\frac{|z_1|}{|z_2|}\right)^2$$

so $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

5. (a) $$(\sqrt{3}-i)^6 = \left(2\left(\cos\left(-\tfrac{\pi}{6}\right)+i\sin\left(-\tfrac{\pi}{6}\right)\right)\right)^6$$
$$= 2^6\left(\cos 6\left(-\tfrac{\pi}{6}\right)+i\sin 6\left(-\tfrac{\pi}{6}\right)\right)$$
$$= 2^6\left(\cos(-\pi)+i\sin(-\pi)\right)$$
$$= 2^6(-1+0i)$$
$$= -64$$

(b) $$i = \cos\left(\tfrac{\pi}{2}\right)+i\sin\left(\tfrac{\pi}{2}\right)$$
$$1-\sqrt{3}i = 2\left(\cos\left(-\tfrac{\pi}{3}\right)+i\sin\left(-\tfrac{\pi}{3}\right)\right)$$
$$\sqrt{3}+i = 2\left(\cos\left(\tfrac{\pi}{6}\right)+i\sin\left(\tfrac{\pi}{6}\right)\right)$$

$$i(1-\sqrt{3}i)(\sqrt{3}+i) = (1)(2)(2)\left(\cos\left(\tfrac{\pi}{2}-\tfrac{\pi}{3}+\tfrac{\pi}{6}\right)+ i\sin\left(\tfrac{\pi}{2}-\tfrac{\pi}{3}+\tfrac{\pi}{6}\right)\right)$$
$$= 4\left(\cos\left(\tfrac{\pi}{3}\right)+i\sin\left(\tfrac{\pi}{3}\right)\right)$$
$$= 4\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right) = 2+2\sqrt{3}i$$

(c) $$(-1+i)^7 = \left(\sqrt{2}\left(\cos\tfrac{3\pi}{4}+i\sin\tfrac{3\pi}{4}\right)\right)^7$$
$$= (\sqrt{2})^7\left(\cos 7\left(\tfrac{3\pi}{4}\right)+i\sin 7\left(\tfrac{3\pi}{4}\right)\right)$$
$$= 8\sqrt{2}\left(\cos\tfrac{21\pi}{4}+i\sin\tfrac{21\pi}{4}\right)$$
$$= 8\sqrt{2}\left(\cos\left(\tfrac{5\pi}{4}+4\pi\right)+i\sin\left(\tfrac{5\pi}{4}+4\pi\right)\right)$$
$$= 8\sqrt{2}\left(\cos\tfrac{5\pi}{4}+i\sin\tfrac{5\pi}{4}\right)$$
$$= 8\sqrt{2}\left(-\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right) = -8-8i$$

6. (a) $2i = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2\left(\cos\left(\frac{\pi}{2}+2k\pi\right) + i\sin\left(\frac{\pi}{2}+2k\pi\right)\right)$

$$\sqrt{2i} = \sqrt{2}\left(\cos\left(\frac{\pi}{4}+\frac{2k\pi}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{2k\pi}{2}\right)\right)$$

$$= \sqrt{2}\left(\cos\left(\frac{\pi+4k\pi}{4}\right) + i\sin\left(\frac{\pi+4k\pi}{4}\right)\right)$$

$k=0$: $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= 1+i$ (PRINCIPAL VALUE)

$k=1$: $\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$= -1-i$

(b) $-1 = \cos\pi + i\sin\pi = \cos(\pi+2k\pi) + i\sin(\pi+2k\pi)$

$$(-1)^{\frac{1}{3}} = \cos\left(\frac{\pi+2k\pi}{3}\right) + i\sin\left(\frac{\pi+2k\pi}{3}\right)$$

$k=0$: $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ (PRINCIPAL VALUE)

$k=1$: $\cos\pi + i\sin\pi = -1$

$k=2$: $\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

(c) $-16 = 16(\cos\pi + i\sin\pi) = 16(\cos(\pi+2k\pi) +$
$i\sin(\pi+2k\pi))$

$$\sqrt[4]{-16} = \sqrt[4]{16}\left(\cos\frac{\pi+2k\pi}{4} + i\sin\frac{\pi+2k\pi}{4}\right)$$

$k=0$: $2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= \sqrt{2} + \sqrt{2}i$

$= \sqrt{2}(1+i)$ (PRINCIPAL VALUE)

$k=1$: $2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 2\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= \sqrt{2}(-1+i)$

$k=2$: $2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt{2}(-1-i)$

$k=3$: $2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \sqrt{2}(1-i)$

7. $z^4 + 4 = 0 \Rightarrow z^4 = -4 = 4(\cos\pi + i\sin\pi)$

$$= 4(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi))$$

$$z = (4(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)))^{\frac{1}{4}}$$
$$= \sqrt[4]{4}\left(\cos\frac{\pi + 2k\pi}{4} + i\sin\frac{\pi + 2k\pi}{4}\right)$$

$k = 0: \quad z_0 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$
$= 1 + i$

$k = 1: \quad z_1 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$
$= -1 + i$

$k = 2: \quad z_2 = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$
$= -1 - i$

$k = 3: \quad z_3 = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$
$= 1 - i$

THUS,

$$z^4 + 4 = (z - z_0)(z - z_1)(z - z_2)(z - z_3)$$
$$= ((z - z_0)(z - z_3))((z - z_1)(z - z_2))$$
$$= ((z - z_0)(z - \bar{z}_0))((z - z_1)(z - \bar{z}_1))$$
$$= (z^2 - (z_0 + \bar{z}_0)z + |z_0|^2)(z^2 - (z_1 + \bar{z}_1)z + |z_1|^2)$$
$$= (z^2 - 2z + 2)(z^2 + 2z + 2)$$

8. $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

BUT ALSO,

$$(\cos\theta + i\sin\theta)^3 = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^2 =$$

$$(\cos\theta + i\sin\theta)\left((\cos^2\theta - \sin^2\theta) + 2\sin\theta\cos\theta\, i\right) =$$

$$(\cos^3\theta - \cos\theta\sin^2\theta - 2\cos\theta\sin^2\theta) +$$

$$i\left(\cos^2\theta\sin\theta - \sin^3\theta + 2\sin\theta\cos^2\theta\right) =$$

$$\underbrace{(\cos^3\theta - 3\cos\theta\sin^2\theta)}_{\cos 3\theta} + i\,\underbrace{(3\sin\theta\cos^2\theta - \sin^3\theta)}_{\sin 3\theta}$$

9. FOR THIS PROBLEM WE WILL NEED THE FOLLOWING FORMULA FROM CALCULUS: FOR ANY $z \neq 1$ AND ANY POSITIVE INTEGER k,

$$1 + z + z^2 + \cdots + z^k = \frac{z^{k+1} - 1}{z - 1}$$

TO SEE WHY THIS IS TRUE JUST MULTIPLY THE LEFT-HAND SIDE BY $z - 1$ AND YOU'LL GET $z^{k+1} - 1$.

NOW LET C BE ANY n^{TH} ROOT OF UNITY EXCEPT 1. THEN

$$1 + c + c^2 + \cdots + c^{n-1} = \frac{c^{n-1+1} - 1}{c - 1} = \frac{c^n - 1}{c - 1} = \frac{1 - 1}{c - 1} = 0.$$

10. $z = r(\cos\theta + i\sin\theta)$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$z^{-n} = \frac{1}{z^n} = \frac{1}{r^n(\cos n\theta + i\sin n\theta)}$$

$$= \frac{1}{r^n}\ \frac{1}{\cos n\theta + i\sin n\theta}\ \frac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta}$$

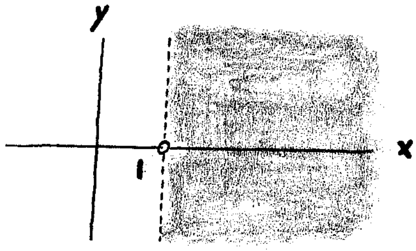
$$= r^{-n}\ \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta}$$

$$= r^{-n}(\cos n\theta - i\sin n\theta)$$

11. (a) $\mathrm{Re}(z) > 1$

$z = x + yi \Rightarrow \mathrm{Re}(z) = x$

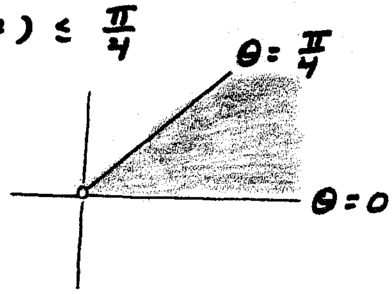
$x > 1$ IS AN OPEN HALF-PLANE



CONNECTED AND OPEN

I.E., A DOMAIN

(b) $0 \le \arg(z) \le \frac{\pi}{4}$



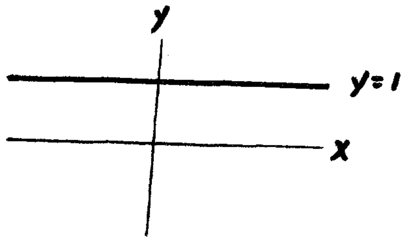
CONNECTED, BUT NOT OPEN

NOT A DOMAIN

(c) $\operatorname{Im}(z) = 1$

$z = x + yi \Rightarrow \operatorname{Im}(z) = y$

$y = 1$ IS A HORIZONTAL LINE



CONNECTED, BUT NOT OPEN

NOT A DOMAIN

(d) $|z - 2 + i| < 1$

$|x + yi - 2 + i| < 1$

$|(x-2) + (y+1)i| < 1$

$\sqrt{(x-2)^2 + (y+1)^2} < 1$

$(x-2)^2 + (y-(-1))^2 < 1$

INTERIOR OF THE CIRCLE OF RADIUS 1 WITH CENTER $(2,-1)$

CONNECTED AND OPEN, I.E., A DOMAIN

(e) $|2z + 3| > 4$

$|2(x+yi) + 3| > 4$

$|(2x+3) + 2yi| > 4$

$\sqrt{(2x+3)^2 + (2y)^2} > 4$

$(2x+3)^2 + (2y)^2 > 16$

$4(x + \frac{3}{2})^2 + 4y^2 > 16$

$(x - (-\frac{3}{2}))^2 + (y-0)^2 > 4^2$

EXTERIOR OF THE CIRCLE OF RADIUS 4 WITH CENTER $(-\frac{3}{2}, 0)$

CONNECTED AND OPEN, I.E., A DOMAIN