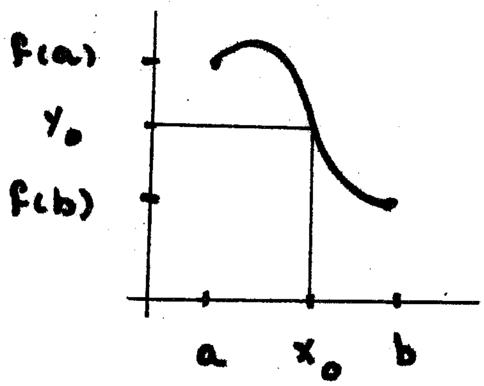
CONNECTEDNESS

RECALL FROM CALCULUS :

<u>INTERMEDIATE</u> <u>VALUE</u> <u>THEOREM</u> : LET $I \subseteq \mathbb{R}$ BE AN INTERVAL AND $f : I \to \mathbb{R}$ A CONTINUOUS FUNCTION. THEN, FOR ANY $a, b \in I$ AND ANY y_0 BETWEEN $f(a)$ AND $f(b)$, THERE IS AT LEAST ONE x_0 BETWEEN a AND b WITH $f(x_0) = y_0$.



NOTE THAT THE RESULT IS FALSE IF I IS NOT AN INTERVAL.

CONSEQUENCE OF A PROPERTY OF INTERVALS CALLED "CONNECTEDNESS" WHICH WE NOW DEFINE IN GENERAL.

A TOPOLOGICAL SPACE X IS <u>DISCONNECTED</u> IF IT CAN BE WRITTEN AS $X = H \cup K$, WHERE H AND K ARE DISJOINT, NONEMPTY, OPEN SETS IN X.

THEN $\{H, K\}$ IS A <u>DISCONNECTION</u> OF X.

NOTE THAT H AND K ARE BOTH CLOSED AS WELL AS OPEN.

X IS CONNECTED IF IT IS NOT DISCONNECTED

IFF IT CANNOT BE WRITTEN AS A DISJOINT UNION OF NONEMPTY OPEN SUBSETS

IFF IT CONTAINS NO PROPER SUBSET THAT IS BOTH OPEN AND CLOSED

LEMMA : A SUBSPACE X OF $\mathbb{R}$ IS CONNECTED IFF IT IS AN INTERVAL.

PROOF : $\Rightarrow$ SUPPOSE $X \subseteq \mathbb{R}$ IS CONNECTED

$X = \phi \Rightarrow X = (x_0, x_0)$ FOR ANY $x_0 \in \mathbb{R}$ SO ASSUME $X \neq \phi$.

THEN THERE IS SOME $x_0 \in X$. IF $X = \{x_0\}$, THEN $X = [x_0, x_0]$. THUS, ASSUME X HAS MORE THAN ONE POINT.

THUS, IF X WERE NOT AN INTERVAL THERE WOULD EXIST $x, y \in X$ WITH $x < y$ AND A z SATISFYING $x < z < y$ WITH $z \notin X$. BUT THEN

$$X = [X \cap (-\infty, z)] \cup [X \cap (z, \infty)]$$

WOULD BE A DISCONNECTION OF X AND THIS IS A CONTRADICTION.

$\Leftarrow$ NOW ASSUME $X \subseteq \mathbb{R}$ IS AN INTERVAL. CAN ASSUME X HAS MORE THAN ONE POINT ($(x_0, x_0) = \phi$ AND $[x_0, x_0] = \{x_0\}$ ARE BOTH CONNECTED).

SUPPOSE X WERE DISCONNECTED, I.E.,

$$X = H \cup K$$

WITH H AND K NONEMPTY, DISJOINT, OPEN (AND THEREFORE CLOSED) SUBSETS OF X.

<u>NOTE</u> : H AND K ARE BOTH INTERSECTIONS WITH X OF CLOSED SETS IN $\mathbb{R}$.

CHOOSE $x \in H$ AND $y \in K$. THEN $x \neq y$. WITHOUT LOSS OF GENERALITY, ASSUME

$$x < y \, .$$

X IS AN INTERVAL SO

$$[x, y] \subseteq X \, .$$

THUS, EVERY POINT OF $[x, y]$ IS IN EITHER H OR K. LET

$$z = \sup \{ t \in [x, y] : t \in H \} \, .$$

THEN

$$x \le z \le y$$

SO

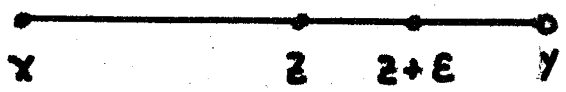
$$z \in X.$$

H CLOSED IN X AND z THE SUP OF A SET OF POINTS IN $H \Rightarrow$

$$\boxed{z \in H}$$

SO

$$x \le z < y.$$



FOR ALL $\varepsilon > 0$ FOR WHICH $z + \varepsilon < y$, WE MUST HAVE

$$z + \varepsilon \in K.$$

THUS, z IS AN ACCUMULATION POINT OF K. SINCE K IS CLOSED,

$$\boxed{z \in K}$$

AND SO $z \in H \cap K$, WHICH IS A CONTRADICTION. $\square$

THE INTERMEDIATE VALUE THEOREM BASICALLY SAYS THAT THE CONTINUOUS IMAGE OF AN INTERVAL IS AN INTERVAL.

MORE GENERALLY,

<u>THEOREM</u>: IF X IS CONNECTED AND $f : X \to Y$ IS A CONTINUOUS MAP OF X ONTO Y, THEN Y IS CONNECTED.

<u>PROOF</u>: SUPPOSE $Y = H \cup K$, WHERE H AND K ARE NONEMPTY, DISJOINT OPEN SETS IN Y. THEN $f^{-1}(H)$ AND $f^{-1}(K)$ ARE OPEN IN X (CONTINUITY), DISJOINT (BECAUSE $H \cap K = \emptyset$) AND NONEMPTY (BECAUSE f IS ONTO, $H \neq \emptyset$, $K \neq \emptyset$ AND $Y = H \cup K$). THUS, $\{f^{-1}(H), f^{-1}(K)\}$ IS A DISCONNECTION OF X AND THIS IS IMPOSSIBLE. THUS, Y MUST BE CONNECTED. $\square$

<u>EXERCISE</u> 51: LET X BE A TOPOLOGICAL SPACE AND Y A SUBSPACE OF X THAT IS CONNECTED (IN ITS RELATIVE TOPOLOGY). SHOW THAT IF Z IS ANY SUBSPACE OF X WITH $Y \subseteq Z \subseteq \bar{Y}$, THEN Z IS CONNECTED (IN PARTICULAR, $\bar{Y}$ IS CONNECTED).
<u>HINT</u>: IF Z WERE NOT CONNECTED THERE WOULD EXIST CLOSED SETS H AND K <u>IN X</u> WHOSE UNION CONTAINS Z AND WHOSE INTERSECTIONS WITH Z ARE NONEMPTY AND DISJOINT.

A SPACE X IS <u>PATHWISE</u> <u>CONNECTED</u> IF, FOR ANY $x_0, x_1 \in X$ $\exists$ CONTINUOUS $\alpha : [0,1] \to X$ WITH $\alpha(0) = x_0$ AND $\alpha(1) = x_1$ (α IS CALLED A CONTINUOUS <u>PATH</u> FROM x_0 TO x_1 IN X).

EXAMPLES :

1. $\mathbb{R}^n$ IS PATHWISE CONNECTED : $x_0, x_1 \in \mathbb{R}^n$

$$\alpha(t) = (1-t)x_0 + t x_1$$

2. ANY CONVEX SUBSPACE OF $\mathbb{R}^n$ IS PATHWISE CONNECTED (SAME ARGUMENT AS FOR $\mathbb{R}^n$)

3. S^n IS PATHWISE CONNECTED (USE # 1 AND THE INVERSE OF STEREOGRAPHIC PROJECTION : EXERCISE 52)

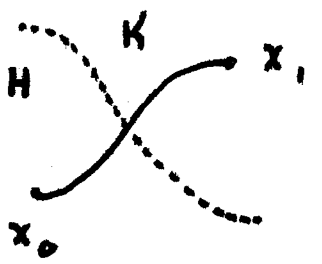
4. PUNCTURED EUCLIDEAN SPACE $\mathbb{R}^n - \{p\}$ (SOME $p \in \mathbb{R}^n$) IS PATHWISE CONNECTED (USE TWO LINE SEGMENTS, IF NECESSARY)

THEOREM : A PATHWISE CONNECTED SPACE IS CONNECTED.

PROOF : SUPPOSE X IS PATHWISE CONNECTED, BUT

$$X = H \cup K$$

WHERE H, K ARE DISJOINT, NONEMPTY AND OPEN. CHOOSE $x_0 \in H$ AND $x_1 \in K$. $\exists$ CONTINUOUS $\alpha : [0,1] \to X$ WITH $\alpha(0) = x_0$ AND $\alpha(1) = x_1$.



$[0,1]$ CONNECTED $\Rightarrow \alpha([0,1])$ CONNECTED, BUT

$$\alpha([0,1]) = (\alpha([0,1]) \cap H) \cup (\alpha([0,1]) \cap K)$$

WHICH IS A CONTRADICTION. $\square$

ALL OF THE FOLLOWING ARE CONNECTED :

1. $\mathbb{R}^n$
2. CONVEX SUBSPACES OF $\mathbb{R}^n$
3. S^n
4. $\mathbb{R}^n - \{p\}$
5. PROJECTIVE SPACES (CONTINUOUS IMAGES OF SPHERES)
6. $SU(2)$

<u>NOTE</u> : $O(n)$ IS <u>NOT</u> CONNECTED ($\det = -1$ AND $\det = 1$), BUT WE WILL SEE LATER THAT $SO(n)$, $U(n)$ AND $SU(n)$ ARE ALL CONNECTED.

<u>LEMMA</u> : LET X BE A TOPOLOGICAL SPACE AND $x_0 \in X$. SUPPOSE THAT, FOR EACH $x_1 \in X$ $\exists$ PATH IN X FROM x_0 TO x_1. THEN X IS PATHWISE CONNECTED.

<u>PROOF</u> : $x_1, x_2 \in X$. CHOOSE

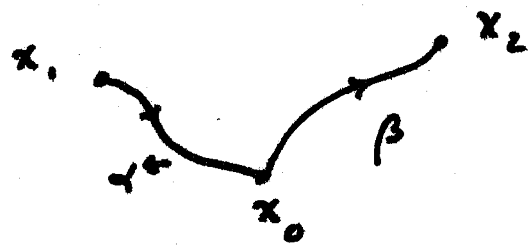
$$\alpha : [0,1] \to X \qquad \alpha(0) = x_0 \qquad \alpha(1) = x_1$$

$$\beta : [0,1] \to X \qquad \beta(0) = x_0 \qquad \beta(1) = x_2$$

DEFINE

$$\overleftarrow{\alpha} : [0,1] \to X \qquad \overleftarrow{\alpha}(t) = \alpha(1-t)$$

$\alpha^{\leftarrow}$ IS CONTINUOUS, $\alpha^{\leftarrow}(0) = \alpha(1) = x_1$, $\alpha^{\leftarrow}(1) = \alpha(0) = x_0$



NOW DEFINE

$$\alpha^{\leftarrow}\beta : [0,1] \to X$$

BY

$$(\alpha^{\leftarrow}\beta)(t) = \begin{cases} \alpha^{\leftarrow}(2t), & 0 \le t \le \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

CONTINUOUS ON $[0, \frac{1}{2}]$, CONTINUOUS ON $[\frac{1}{2}, 1]$ AND

$$\alpha^{\leftarrow}(2(\tfrac{1}{2})) = \alpha^{\leftarrow}(1) = \alpha(0) = x_0$$

$$\beta(2(\tfrac{1}{2})-1) = \beta(0) = x_0$$

CONTINUITY OF $\alpha^{\leftarrow}\beta$ FOLLOWS EXERCISE 53 (BELOW)

$$(\alpha^{\leftarrow}\beta)(0) = \alpha^{\leftarrow}(0) = \alpha(1) = x_1$$

$$(\alpha^{\leftarrow}\beta)(1) = \beta(1) = x_2 \qquad \square$$

<u>NOTE</u> : THE IDEAS INTRODUCED IN THIS PROOF ARE USED TO DEFINE THE "FUNDAMENTAL GROUP" OF X IN ALGEBRAIC TOPOLOGY.

EXERCISE 53 (THE GLUING LEMMA): LET X AND Y BE TOPOLOGICAL SPACES AND ASSUME THAT $X = A_1 \cup A_2$, WHERE A_1 AND A_2 ARE OPEN (OR CLOSED) SETS IN X. SUPPOSE $f_1 : A_1 \to Y$ AND $f_2 : A_2 \to Y$ ARE CONTINUOUS AND THAT $f_1 | A_1 \cap A_2 = f_2 | A_1 \cap A_2$. SHOW THAT THE MAP $f : X \to Y$ GIVEN BY

$$f(x) = \begin{cases} f_1(x), & x \in A_1 \\ f_2(x), & x \in A_2 \end{cases}$$

IS WELL-DEFINED AND CONTINUOUS. HINT: FOR ANY SUBSET S OF Y, $f^{-1}(S) = f_1^{-1}(S) \cup f_2^{-1}(S)$.

EXERCISE 54: LET $\alpha : [0,1] \to X$ AND $\beta : [0,1] \to X$ BE TWO PATHS IN X FROM $x_0 = \alpha(0) = \beta(0)$ TO $x_1 = \alpha(1) = \beta(1)$. WE SAY THAT α IS HOMOTOPIC TO β AND WRITE $\alpha \simeq \beta$ IF THERE IS A CONTINUOUS MAP $F : [0,1] \times [0,1] \to X$, CALLED A HOMOTOPY FROM α TO β, SATISFYING

$$F(0,t) = x_0 \quad \text{AND} \quad F(1,t) = x_1 \qquad \forall t \in [0,1]$$

AND

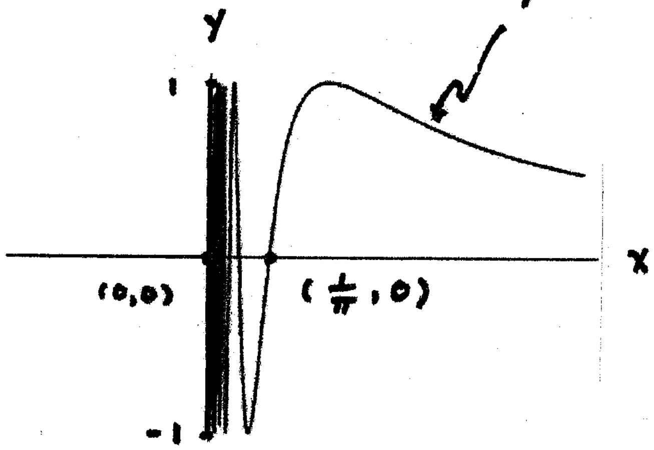
$$F(s,0) = \alpha(s) \quad \text{AND} \quad F(s,1) = \beta(s) \qquad \forall s \in [0,1]$$

DRAW A PICTURE TO ILLUSTRATE WHAT THIS REALLY MEANS (INTUITIVELY). THEN PROVE THAT $\simeq$ IS AN EQUIVALENCE RELATION ON THE SET OF ALL PATHS IN X FROM x_0 TO x_1.

PATHWISE CONNECTED $\Rightarrow$ CONNECTED

$\not\Leftarrow$

EXAMPLE :



$$X = \left\{ (0,y) \in \mathbb{R}^2 : -1 \leq y \leq 1 \right\} \cup \left\{ (x, \sin\tfrac{1}{x}) : x > 0 \right\}$$

WITH ITS SUBSPACE TOPOLOGY FROM $\mathbb{R}^2$.

THE SECOND PIECE IS CONNECTED (CONTINUOUS IMAGE OF $(0,\infty)$) AND X IS ITS CLOSURE SO EXERCISE 51 $\Rightarrow X$ IS CONNECTED.

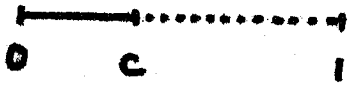
NOT PATHWISE CONNECTED : SUPPOSE $\alpha : [0,1] \to X$ IS A PATH FROM $\alpha(0) = (\frac{1}{\pi}, 0)$ TO $\alpha(1) = (0,0)$.

LET

$$c = \text{INF} \left\{ t \in [0,1] : \alpha(t) = (0,y), \text{ SOME } y \in [-1,1] \right\}$$

THEN

$$0 < c \leq 1$$



NOTE THAT $\alpha([0,c])$ CONTAINS EXACTLY ONE POINT OF THE y-AXIS $(\alpha(c))$

HOWEVER, <u>EVERY</u> POINT $(0,y)$, $-1 \leq y \leq 1$, IS IN THE CLOSURE OF $\alpha([0,c])$ (<u>EXERCISE</u> 55)

THUS, $\alpha([0,c])$ CANNOT BE CLOSED WHICH CONTRADICTS THE FACT THAT IT IS THE CONTINUOUS IMAGE OF A COMPACT SET.

WE NEED TO SHOW THAT THIS SORT OF THING CANNOT OCCUR FOR THE SPACES WE ARE MOST INTERESTED IN (I.E., LOCALLY EUCLIDEAN SPACES).

A SPACE X IS SAID TO BE <u>LOCALLY CONNECTED</u> (RESPECTIVELY, <u>LOCALLY PATHWISE CONNECTED</u>) IF WHENEVER $x \in X$ AND V IS AN OPEN SET CONTAINING x, THEN THERE IS AN OPEN SET U IN X WITH $x \in U \subseteq V$ SUCH THAT U, WITH ITS RELATIVE TOPOLOGY, IS CONNECTED (RESPECTIVELY, PATHWISE CONNECTED).

EXAMPLES: ANY LOCALLY EUCLIDEAN SPACE; IN PARTICULAR, ANY TOPOLOGICAL MANIFOLD.

THEOREM: A CONNECTED, LOCALLY PATHWISE CONNECTED SPACE IS PATHWISE CONNECTED.

PROOF: LET X BE CONNECTED AND LOCALLY PATHWISE CONNECTED. FIX SOME $x_0 \in X$.

WE'LL SHOW THAT ANY $x_1 \in X$ CAN BE CONNECTED TO x_0 BY A PATH AND USE THE PREVIOUS LEMMA (PAGE 7).

LET

$$H = \text{SET OF ALL } x_1 \in X \text{ FOR WHICH } \exists \text{ CONTINUOUS } \alpha : [0,1] \to X \text{ WITH } \alpha(0) = x_0 \text{ AND } \alpha(1) = x_1.$$

WANT TO SHOW THAT $H = X$.

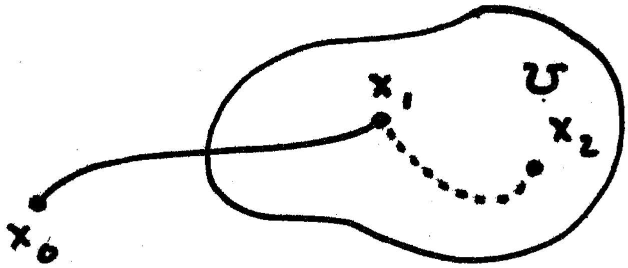
$H \neq \emptyset$ SINCE $x_0 \in H$ $\left(\alpha(t) = x_0 \ \forall t \in [0,1] \right)$.

WILL SUFFICE TO SHOW THAT H IS BOTH OPEN AND CLOSED (BECAUSE X IS CONNECTED).

(1) H IS OPEN IN X

LET $x_1 \in H$ BE ARBITRARY. X LOCALLY PATHWISE CONNECTED SO $\exists$ OPEN SET U IN X WITH $x_1 \in U$ AND SUCH THAT, IN ITS RELATIVE TOPOLOGY, U IS PATHWISE CONNECTED.

<u>CLAIM</u>: $U \subseteq H$ (SO H IS OPEN IN X)



LET $x_2 \in U$ BE ARBITRARY. CHOOSE

$$\alpha : [0,1] \to X \ , \ \alpha(0) = x_0 \ , \ \alpha(1) = x_1$$

$$\beta : [0,1] \to U \ , \ \beta(0) = x_1 \ , \ \beta(1) = x_2$$

β IS ALSO CONTINUOUS AS A MAP INTO X SO BOTH α AND β ARE PATHS IN X. DEFINE

$$\alpha\beta : [0,1] \to X$$

$$(\alpha\beta)(t) = \begin{cases} \alpha(2t) \ , & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) \ , & \frac{1}{2} \le t \le 1 \end{cases}$$

THEN $\alpha\beta$ IS A PATH IN X FROM x_0 TO x_2 SO $x_2 \in H$. THUS,

$$U \subseteq H.$$

(2) H IS CLOSED IN X

WE SHOW THAT $\bar{H} = H$. $H \subseteq \bar{H}$ IS ALWAYS TRUE SO WE NEED ONLY PROVE THAT $\bar{H} \subseteq H$.

LET $x_2 \in \bar{H}$ BE ARBITRARY. CHOOSE A PATHWISE CONNECTED OPEN SET U CONTAINING x_2.

$$U \cap H \neq \emptyset$$

SELECT $x_1 \in U \cap H$.

$x_1 \in H \Rightarrow \exists$ PATH FROM x_0 TO x_1

$x_1 \in U \Rightarrow \exists$ PATH FROM x_1 TO x_2

THUS, JUST AS IN THE PREVIOUS ARGUMENT, THERE IS A PATH FROM x_0 TO x_2 SO $x_2 \in H$. THUS, $\bar{H} \subseteq H$ AND THE PROOF IS COMPLETE. $\square$

<u>COROLLARY</u> : A LOCALLY EUCLIDEAN SPACE IS CONNECTED IF AND ONLY IF IT IS PATHWISE CONNECTED.

TO DEAL WITH PRODUCTS OF CONNECTED SPACES WE NEED A LITTLE LEMMA :

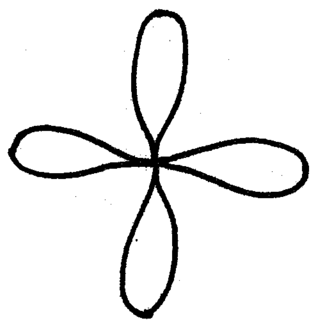
LEMMA : LET X BE A TOPOLOGICAL SPACE AND $\{X_\alpha : \alpha \in \mathcal{A}\}$ A FAMILY OF CONNECTED SUBSPACES OF X WITH

$$\bigcup_{\alpha \in \mathcal{A}} X_\alpha = X$$

AND

$$\bigcap_{\alpha \in \mathcal{A}} X_\alpha \neq \phi .$$

THEN X IS CONNECTED.



PROOF : SUPPOSE $X = H \cup K$, WHERE H AND K ARE DISJOINT OPEN SETS. WE'LL SHOW THAT ONE OF THEM MUST BE EMPTY.

EACH X_α IS CONNECTED AND CONTAINED IN $H \cup K$ SO EACH X_α MUST BE ENTIRELY CONTAINED H OR ENTIRELY CONTAINED IN K (OTHERWISE $(X_\alpha \cap H) \cup (X_\alpha \cap K)$ WOULD BE A DISCONNECTION OF X_α).

SINCE $\bigcap X_\alpha \neq \phi$ AND $H \cap K = \phi$, ALL OF THE X_α MUST BE CONTAINED IN THE SAME ONE OF THESE SETS, SAY, $X_\alpha \subseteq H \ \forall \alpha \in \mathcal{A}$.

BUT THEN

$$X = \bigcup_{\alpha \in A} X_\alpha \subseteq H$$

SO $K = \emptyset$. $\square$

<u>THEOREM</u>: LET $X_1, \dots, X_k$ BE TOPOLOGICAL SPACES AND $X = X_1 \times \cdots \times X_k$ THE PRODUCT SPACE. THEN X IS CONNECTED IFF EVERY X_i, $i = 1, \dots, k$, IS CONNECTED.

<u>PROOF</u>: X CONNECTED $\Rightarrow$ EACH X_i CONNECTED BECAUSE THE PROJECTION

$$\pi_i : X \to X_i$$

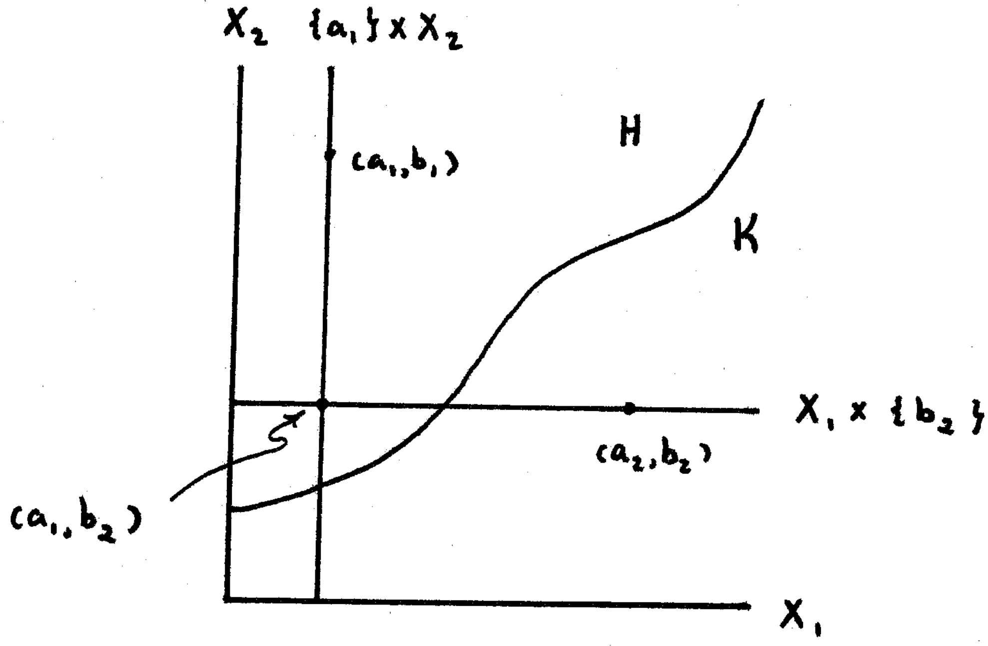
MAPS X CONTINUOUSLY ONTO X_i.

FOR $\Leftarrow$ ASSUME $X_1, \dots, X_k$ CONNECTED AND SHOW X CONNECTED. BY INDUCTION IT IS ENOUGH TO PROVE THIS WHEN $k = 2$.

SO, ASSUME X_1 AND X_2 CONNECTED. TO SEE THAT $X_1 \times X_2$ IS CONNECTED SUPPOSE IT IS NOT, I.E., SUPPOSE

$$X_1 \times X_2 = H \cup K$$

H, K NONEMPTY, DISJOINT, OPEN SETS



CHOOSE $(a_1, b_1) \in H$ AND $(a_2, b_2) \in K$ AND CONSIDER THE SUBSPACES

$$\{a_1\} \times X_2 \cong X_2$$

AND

$$X_1 \times \{b_2\} \cong X_1$$

BOTH ARE CONNECTED AND $(\{a_1\} \times X_2) \cap (X_1 \times \{b_2\}) \neq \emptyset$ BECAUSE (a_1, b_2) IS IN IT.

THUS, $(\{a_1\} \times X_2) \cup (X_1 \times \{b_2\})$ MUST BE CONNECTED AND THIS IS IMPOSSIBLE SINCE IT INTERSECTS BOTH H AND K. $\square$

MORE EXAMPLES OF CONNECTED SPACES:

1. $S^1 \times \mathbb{R}$ (CYLINDER)
2. $S^1 \times S^1$ (TORUS)
3. $S^1 \times S^1 \times \cdots \times S^1$ (HIGHER DIMENSIONAL TORI)

SUPPOSE NOW THAT X IS <u>NOT</u> CONNECTED. WE'LL SHOW THAT X SPLITS INTO A DISJOINT UNION OF MAXIMAL CONNECTED SUBSPACES CALLED "CONNECTED COMPONENTS".

FOR EACH $p \in X$ DEFINE

$C(p)$ = UNION OF ALL THE CONNECTED SUBSPACES OF X CONTAINING p

<u>NOTE</u>: THERE IS AT LEAST ONE SUCH, NAMELY, $\{p\}$

= <u>CONNECTED COMPONENT OF</u> p IN X.

<u>EXERCISE</u> 56:

1. $C(p)$ IS A CONNECTED SUBSPACE OF X
2. IF $p \neq q$, THEN EITHER $C(p) = C(q)$ OR $C(p) \cap C(q) = \phi$
3. $C(P)$ IS CLOSED IN X

THUS, X IS THE DISJOINT UNION OF ITS COMPONENTS:

$$X = \bigsqcup_{p \in X} C(p)$$

EXERCISE 57 : LET X BE A TOPOLOGICAL SPACE AND $\{X_\alpha : \alpha \in \Lambda\}$ A FAMILY OF PATHWISE CONNECTED SUBSPACES OF X WITH $\bigcup_{\alpha \in \Lambda} X_\alpha = X$ AND $\bigcap_{\alpha \in \Lambda} X_\alpha \neq \emptyset$. SHOW THAT X IS PATHWISE CONNECTED.

EXERCISE 58 : SHOW THAT $X_1 \times \cdots \times X_k$ IS PATHWISE CONNECTED IF AND ONLY IF EACH X_i, $i = 1, \ldots, k$, IS PATHWISE CONNECTED.

EXERCISE 59 : A SPACE X IS SAID TO BE TOTALLY DISCONNECTED IF $C(p) = \{p\}$ FOR EACH $p \in X$. FIND A NONDISCRETE EXAMPLE.

EXERCISE 60 : SHOW THAT IN A LOCALLY CONNECTED SPACE (E.G., A TOPOLOGICAL MANIFOLD) THE COMPONENTS ARE OPEN AS WELL AS CLOSED, BUT THIS IS NOT TRUE IN GENERAL.