CONTINUITY

A POLYNOMIAL $P(x)$ HAS THE FOLLOWING DESIRABLE PROPERTY THAT WE WOULD LIKE TO CAPTURE WITH A DEFINITION: GIVEN AN x_0 IN $\mathbb{R}$, $P(x_0)$ IS DEFINED, $\lim_{x \to x_0} P(x)$ EXISTS, AND, IN FACT,

$$\lim_{x \to x_0} P(x) = P(x_0).$$

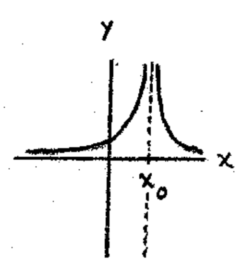
NOW LET f BE A FUNCTION AND x_0 A REAL NUMBER. WE SAY THAT f IS <u>CONTINUOUS</u> <u>AT</u> x_0 IF

(1) f IS DEFINED AT x_0 (I.E., $f(x_0)$ EXISTS)

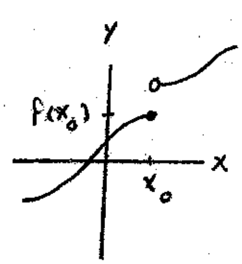
(2) $\lim_{x \to x_0} f(x)$ EXISTS (SO THAT f IS ACTUALLY DEFINED ON SOME INTERVAL ABOUT x_0)

(3) $\lim_{x \to x_0} f(x) = f(x_0)$.

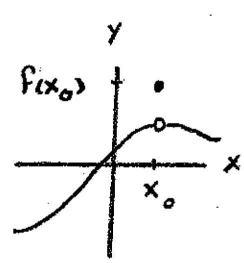
INTUITIVELY, THE DEFINITION IS INTENDED TO PROHIBIT THE FOLLOWING SORTS OF BEHAVIOR WHEREBY THE GRAPH OF f WOULD BE FORCED TO "BREAK" AT x_0.



f IS NOT DEFINED AT x_0



$f(x_0)$ EXISTS BUT $\lim_{x \to x_0} f(x)$ DOES NOT



$f(x_0)$ AND $\lim_{x \to x_0} f(x)$ BOTH EXIST, BUT THEY ARE NOT THE SAME

LET'S REPHRASE THE DEFINITION A COUPLE OF TIMES.

f IS CONTINUOUS AT $x_0 \iff \forall \varepsilon > 0 \ \exists \delta > 0$ S.T.

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

NOTE: THE EXPECTED "$0 < |x - x_0|$" HAS BEEN DROPPED SINCE f MUST BE DEFINED AT x_0 AND $|f(x_0) - f(x_0)| < \varepsilon$ IS SURELY SATISFIED.

f IS CONTINUOUS AT $x_0 \iff \forall \varepsilon > 0 \ \exists \delta > 0$ S.T.

$$f(U_\delta(x_0)) \subseteq U_\varepsilon(f(x_0))$$

SUPPOSE $f : A \to \mathbb{R}$, WHERE $A \subseteq \mathbb{R}$ HAS THE PROPERTY THAT EVERY $x \in A$ IS CONTAINED IN SOME $U_\delta(x) \subseteq A$ (SUCH SUBSETS OF $\mathbb{R}$ ARE CALLED OPEN SETS). IF f IS CONTINUOUS AT EVERY $x_0 \in A$, THEN WE WILL SIMPLY SAY THAT f IS CONTINUOUS ON A.

NOTE: WE WILL ALSO BE INTERESTED IN FUNCTIONS f DEFINED ON SUCH SETS AS $[a, b]$, $(a, b]$, $[a, b)$, $(-\infty, b]$ AND $[a, \infty)$, WHICH ARE NOT OPEN SETS BECAUSE THEY HAVE ENDPOINTS AT WHICH ONLY A LIMIT FROM ABOVE OR BELOW IS DEFINED. IN THIS CASE WE EXTEND OUR PREVIOUS DEFINITION A BIT, E.G., $f : [a, b] \to \mathbb{R}$ IS SAID TO BE CONTINUOUS ON $[a, b]$ IF IT IS CONTINUOUS ON (a, b) AND IF

$$\lim_{x \to a^+} f(x) = f(a)$$

AND

$$\lim_{x \to b^-} f(x) = f(b).$$

EXAMPLES :

1. A POLYNOMIAL $P(x)$ IS CONTINUOUS AT EVERY x_0 IN $\mathbb{R}$.

2. A RATIONAL FUNCTION $R(x) = \frac{P(x)}{Q(x)}$ IS CONTINUOUS AT EVERY x_0 IN $\mathbb{R}$ AT WHICH $Q(x_0) \neq 0$.

3. $f : [0,\infty) \to \mathbb{R}$ IS CONTINUOUS ON $[0,\infty)$ BECAUSE, IF $x_0 > 0$, $\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$, AND $\lim_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0$.

4. DEFINE $f : \mathbb{R} \to \mathbb{R}$ BY $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$.

   WE HAVE SHOWN THAT, FOR EVERY $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} f(x)$ DOES NOT EXIST SO f IS NOT CONTINUOUS AT ANY x_0.

5. DEFINE $f : \mathbb{R} \to \mathbb{R}$ BY $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$.

   WE HAVE SHOWN THAT, IF $x_0 \neq 0$, $\lim_{x \to x_0} f(x)$ DOES NOT EXIST, BUT
   $$\lim_{x \to 0} f(x) = 0 = f(0).$$
   THUS, f IS CONTINUOUS ONLY AT 0.

6. (DIRICHLET FUNCTION) DEFINE $f : \mathbb{R} \to \mathbb{R}$ BY
   $$f(x) = \begin{cases} \frac{1}{n}, & x = \frac{m}{n} \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$
   WHERE $\frac{m}{n}$ IS REDUCED AND $n > 0$. WE HAVE SHOWN THAT, FOR EVERY $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} f(x) = 0$ SO f IS CONTINUOUS ONLY AT THE IRRATIONAL NUMBERS.

7. WE STILL DO NOT HAVE PRECISE DEFINITIONS OF $\sin X$ AND $\cos X$ AND SO WE ARE NOT IN A POSITION TO GIVE COMPLETE PROOFS OF THEIR CONTINUITY. HOWEVER, WE WOULD LIKE TO CONTINUE OUR PROGRAM OF ASSUMING A FEW OF THEIR BASIC PROPERTIES TO PRODUCE EXAMPLES. HERE WE WILL TAKE IT FOR GRANTED THAT

$$\sin^2 x + \cos^2 x = 1$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\lim_{x \to 0} \cos x = 1$$

AND PROVE THAT $\cos X$ AND $\sin X$ ARE CONTINUOUS EVERYWHERE ON $\mathbb{R}$.

FIRST NOTE THAT $\cos 0 = 1$ BECAUSE, FOR ANY X,

$$\cos 0 = \cos(X - X) = \cos^2 X + \sin^2 X = 1.$$

THUS, $\lim_{X \to 0} \cos X = \cos 0$ SO $\cos X$ IS CONTINUOUS AT 0.

NEXT LET $X_0 \in \mathbb{R}$ BE ARBITRARY. NOTICE THAT

$$(\sin X - \sin X_0)^2 = \sin^2 X - 2 \sin X \sin X_0 + \sin^2 X_0$$

$$(\cos X - \cos X_0)^2 = \cos^2 X - 2 \cos X \cos X_0 + \cos^2 X_0$$

SO

$$(\sin X - \sin X_0)^2 + (\cos X - \cos X_0)^2 = 2 - 2(\cos X \cos X_0 + \sin X \sin X_0)$$

$$= 2(1 - \cos(X - X_0))$$

(IN PARTICULAR, $2(1 - \cos(X - X_0)) \geq 0$).

THUS,

$$(\sin X - \sin X_0)^2 \leq 2(1 - \cos(X - X_0))$$

AND

$$(\cos X - \cos X_0)^2 \leq 2(1 - \cos(X - X_0))$$

SO

$$|\sin x - \sin x_0| \leq \sqrt{2(1-\cos(x-x_0))}$$

AND

$$|\cos x - \cos x_0| \leq \sqrt{2(1-\cos(x-x_0))}$$

NOW LET $\varepsilon > 0$ BE GIVEN. THEN, BY CONTINUITY OF $\cos x$ AT 0, $\exists\, \delta > 0$ S.T. $|t-0| < \delta \Rightarrow |1-\cos t| < \frac{\varepsilon^2}{2}$. THEN

$$|x-x_0| < \delta \Rightarrow |1-\cos(x-x_0)| < \frac{\varepsilon^2}{2}$$

$$\Rightarrow \sqrt{2(1-\cos(x-x_0))} < \varepsilon$$

$$\Rightarrow |\sin x - \sin x_0| < \varepsilon \quad \text{AND}$$

$$|\cos x - \cos x_0| < \varepsilon$$

AS REQUIRED.

<u>EXERCISE</u> 1 : ASSUME THE FOLLOWING TWO PROPERTIES OF THE EXPONENTIAL FUNCTION AND PROVE FROM THEM THAT e^x IS CONTINUOUS EVERYWHERE ON $\mathbb{R}$.

$$e^x e^y = e^{x+y}$$

$$\lim_{x \to 0} e^x = 1$$

<u>EXERCISE</u> 2 : ASSUME THE FOLLOWING TWO PROPERTIES OF THE NATURAL LOGARITHM AND PROVE FROM THEM THAT $\ln x$ IS CONTINUOUS ON $(0, \infty)$.

$$\ln(xy) = \ln x + \ln y \quad (x, y > 0)$$

$$\lim_{x \to 1} \ln x = 0$$

LOTS OF OTHER EXAMPLES CAN BE CONSTRUCTED FROM THE FOLLOWING SIMPLE CONSEQUENCE OF OUR BASIC LIMIT THEOREM.

<u>THEOREM</u>: SUPPOSE f AND g ARE BOTH CONTINUOUS AT x_0 IN $\mathbb{R}$. THEN

(1) $f \pm g$ IS CONTINUOUS AT x_0.

(2) fg IS CONTINUOUS AT x_0.

(3) $\frac{f}{g}$ IS CONTINUOUS AT x_0 WHENEVER $g(x_0) \neq 0$.

<u>EXERCISE</u> 3: PROVE THE THEOREM. □

OF COURSE, (1) AND (2) EXTEND TO ARBITRARY FINITE SUMS AND PRODUCTS.

E.G.,

$$f(x) = x^3 e^x + \sin^2 x \cos x$$

IS CONTINUOUS ON ALL OF $\mathbb{R}$.

STILL MORE CAN BE CONSTRUCTED FROM THE FACT THAT COMPOSITIONS OF CONTINUOUS FUNCTIONS ARE CONTINUOUS, I.E.,

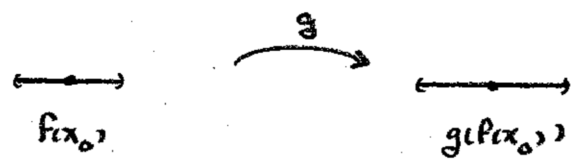
<u>THEOREM</u>: SUPPOSE f IS CONTINUOUS AT x_0 AND g IS CONTINUOUS AT $f(x_0)$. THEN $g \circ f$ IS CONTINUOUS AT x_0.

PROOF : WE WILL SHOW THAT $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ S.T.

$$(g \circ f)(U_\delta(x_0)) \subseteq U_\varepsilon(g(f(x_0))).$$

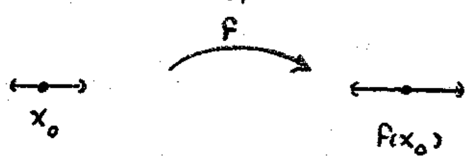
BUT g IS CONTINUOUS AT $f(x_0)$ SO $\exists\, \delta_1 > 0$ S.T.

$g(U_{\delta_1}(f(x_0))) \subseteq U_\varepsilon(g(f(x_0)))$.



BUT THEN f IS CONTINUOUS AT x_0 SO $\exists\, \delta > 0$ S.T.

$f(U_\delta(x_0)) \subseteq U_{\delta_1}(f(x_0))$.



CONSEQUENTLY,

$$\begin{aligned}(g \circ f)(U_\delta(x_0)) &= g(f(U_\delta(x_0))) \\ &\subseteq g(U_{\delta_1}(f(x_0))) \subseteq U_\varepsilon(g(f(x_0))).\end{aligned}$$ $\square$

E.G., $f(x) = x \sin(\frac{1}{x})$ IS CONTINUOUS ON $\mathbb{R} - \{0\}$.

A REAL NUMBER x_0 AT WHICH f IS NOT CONTINUOUS IS CALLED A <u>DISCONTINUITY</u> OF f.

E.G., $x_0 = 0$ IS A DISCONTINUITY OF $f(x) = x \sin(\frac{1}{x})$ BECAUSE f IS NOT DEFINED THERE.

NOTICE, HOWEVER, THAT

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$$

BECAUSE, FOR ANY $\varepsilon > 0$, IF WE TAKE $\delta = \varepsilon$, THEN

$$\begin{aligned} 0 < |x-0| < \delta = \varepsilon \Rightarrow |f(x) - 0| &= |x \sin(\tfrac{1}{x})| \\ &= |x|\,|\sin(\tfrac{1}{x})| \\ &\leq |x| \\ &< \varepsilon . \end{aligned}$$

THUS, IF WE DEFINE

$$F(x) = \begin{cases} x \sin(\frac{1}{x}) , & x \neq 0 \\ 0 , & x = 0 \end{cases}$$

WE FIND THAT F IS CONTINUOUS ON ALL OF $\mathbb{R}$.

HERE ARE THE WORDS WE USE TO DESCRIBE THIS.

A DISCONTINUITY x_0 OF f IS SAID TO BE <u>REMOVABLE</u> IF $\lim_{x\to x_0} f(x)$ EXISTS. IN THIS CASE THE FUNCTION

$$F(x) = \begin{cases} f(x) , & x \neq x_0 \\ \lim_{x\to x_0} f(x), & x = x_0 \end{cases}$$

IS CONTINUOUS AT x_0.

NOT ALL DISCONTINUITIES ARE REMOVABLE, OF COURSE, E.G., $x_0 = 0$ FOR $f(x) = \sin(\frac{1}{x})$, OR $f(x) = \frac{1}{x^2}$, OR $f(x) = \frac{|x|}{x}$.

EXERCISE 4: SHOW THAT, IF f IS CONTINUOUS AT x_0, THEN $|f|$ IS CONTINUOUS AT x_0 ($|f|$ IS DEFINED BY $|f|(x) = |f(x)|$). HINT: PROVE THAT

$$\big|\, |f(x)| - |f(x_0)| \,\big| \leq |f(x) - f(x_0)| \, .$$

WE NOW BEGIN TO INVESTIGATE SOME OF THE DEEPER PROPERTIES OF CONTINUOUS FUNCTIONS.

AN INTERVAL IN $\mathbb{R}$ IS A SUBSET I OF $\mathbb{R}$ WITH THE PROPERTY THAT IF $x_1, x_2 \in I$ WITH $x_1 < x_2$, THEN

$$(x_1, x_2) = \{ x \in \mathbb{R} : x_1 < x < x_2 \}$$

IS CONTAINED IN I, E.G.,

$$(a,b),\ [a,b],\ [a,b),\ (a,b],\ (-\infty, b],\ (a,\infty),\ \mathbb{R},\ \text{ETC.}$$

INTERMEDIATE VALUE THEOREM: LET $I \subseteq \mathbb{R}$ BE AN INTERVAL AND f A FUNCTION THAT IS CONTINUOUS ON I. LET $x_1, x_2 \in I$ WITH $x_1 < x_2$ AND $f(x_1) \neq f(x_2)$. THEN FOR EVERY L STRICTLY BETWEEN $f(x_1)$ AND $f(x_2)$ THERE IS A c IN (x_1, x_2) FOR WHICH

$$f(c) = L$$

PROOF: FIRST SUPPOSE $f(x_1) < f(x_2)$ SO

$$f(x_1) < L < f(x_2).$$

LET

$$A = \{ x \in [x_1, x_2] : f(x) \leq L \}$$

THEN $A \neq \emptyset$ (SINCE $x_1 \in A$) AND A IS BOUNDED FROM ABOVE (BY x_2) SO $\sup(A)$ EXISTS.

<u>EXERCISE</u> 5 : SHOW THAT $\sup(A) \in [x_1, x_2]$.

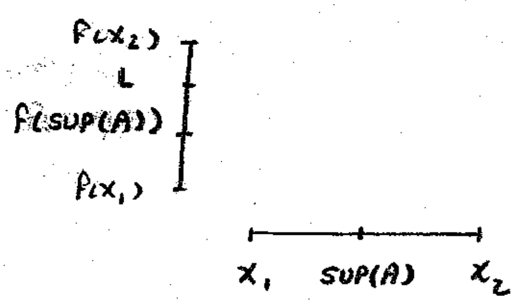
WE INTEND TO SHOW THAT

$$f(\sup(A)) = L$$

SO WE CAN TAKE $c = \sup(A)$.

<u>NOTE</u> : FROM THIS IT WILL FOLLOW THAT $c \in (x_1, x_2)$ SINCE $f(x_1) < L < f(x_2)$.

FIRST SUPPOSE $f(\sup(A)) < L$.



NOTE THAT $L < f(x_2) \implies \sup(A) < x_2$ (AT THIS POINT IT IS CONCEIVABLE THAT $\sup(A) = x_1$)

LET $\varepsilon = L - f(\sup(A))$, WHICH IS POSITIVE. SINCE f IS CONTINUOUS AT $\sup(A)$, $\lim_{x \to \sup(A)^+} f(x) = f(\sup(A))$.

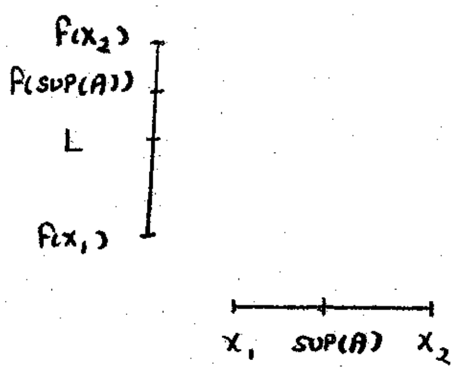
<u>NOTE</u> : WE DEAL ONLY WITH THE LIMIT FROM ABOVE JUST IN CASE $\sup(A) = x_1$ = LEFT-HAND ENDPOINT OF I.

THUS, THERE IS A $\delta > 0$, WHICH WE CAN TAKE TO BE LESS THAN $x_2 - \sup(A) > 0$, SUCH THAT

$$\begin{aligned} \sup(A) < x < \sup(A) + \delta &\Rightarrow |f(x) - f(\sup(A))| < \varepsilon. \\ &\Rightarrow f(x) - f(\sup(A)) < \varepsilon \\ &\Rightarrow f(x) < f(\sup(A)) + \varepsilon \\ &\Rightarrow f(x) < L. \end{aligned}$$

SINCE EVERY SUCH x IS IN $[x_1, x_2]$, IT MUST BE IN A SO, BEING GREATER THAN $\sup(A)$, THIS CONTRADICTS THE FACT THAT $\sup(A)$ IS AN UPPER BOUND FOR A. THUS, $f(\sup(A)) < L$ IS IMPOSSIBLE.

NEXT SUPPOSE $f(\sup(A)) > L$.



NOTE THAT $f(x_1) < L \Rightarrow \sup(A) > x_1$ (ALTHOUGH IT IS CONCEIVABLE THAT $\sup(A) = x_2$)

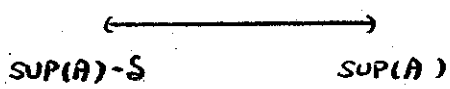
LET $\varepsilon = f(\sup(A)) - L > 0$. SINCE f IS CONTINUOUS AT $\sup(A)$,

$$\lim_{x \to \sup(A)^-} f(x) = f(\sup(A)).$$

THUS, $\exists\ \delta > 0$, WHICH WE CAN TAKE TO BE $< \sup(A) - x_1$, S.T.

$$\sup(A) - \delta < x < \sup(A) \Rightarrow |f(x) - f(\sup(A))| < \varepsilon$$
$$\Rightarrow -\varepsilon < f(x) - f(\sup(A))$$
$$\Rightarrow f(x) > f(\sup(A)) - \varepsilon$$
$$\Rightarrow f(x) > L.$$

SINCE ANY SUCH x IS IN $[x_1, x_2]$, IT IS NOT IN A SO EVERYTHING IN A, BEING LESS THAN $\sup(A)$ MUST ALSO BE LESS THAN x. BUT THEN EVERY x IN



IS AN UPPER BOUND FOR A THAT IS $< \sup(A)$ CONTRADICTING THE FACT THAT $\sup(A)$ IS THE LEAST UPPER BOUND FOR A. THUS, $f(\sup(A)) > L$ IS ALSO IMPOSSIBLE AND WE MUST HAVE

$$f(\sup(A)) = L.$$

EXERCISE 6: ALL OF THIS HAS ASSUMED THAT $f(x_1) < f(x_2)$. SHOW THAT IF $f(x_1) > f(x_2)$, THEN APPLYING WHAT WE HAVE JUST PROVED TO $-f$ FINISHES THE PROOF. $\square$

A FEW APPLICATIONS :

1. WE SHOW THAT THE POLYNOMIAL EQUATION
$$3x^4 + 7x^3 + 8x^2 + x - 3 = 0$$
HAS A REAL SOLUTION.

NOTE: POLYNOMIALS OF EVEN DEGREE NEED NOT HAVE REAL ROOTS (E.G., $x^2+1=0$) AND EVEN WHEN THEY DO THEY CAN BE QUITE A CHALLENGE TO FIND.

LET $f(x) = 3x^4 + 7x^3 + 8x^2 + x - 3$. THEN f IS CONTINUOUS ON $\mathbb{R}$. MOREOVER,

$$f(0) = -3$$

AND

$$f(1) = 16$$

SINCE $-3 < 0 < 16$ THE INTERMEDIATE VALUE THEOREM IMPLIES THAT THERE IS A c IN $(0,1)$ AT WHICH $f(c) = 0$.

2. THEOREM: A POLYNOMIAL OF ODD DEGREE WITH REAL COEFFICIENTS HAS AT LEAST ONE REAL ROOT.

PROOF: ASSUME THAT n IS ODD AND $a_n x^n + \cdots + a_1 x + a_0$ IS A POLYNOMIAL WITH $a_0, \ldots, a_n \in \mathbb{R}$ AND $a_n \neq 0$. WE SHOW THAT $a_n x^n + \cdots + a_1 x + a_0 = 0$ HAS A REAL SOLUTION. DIVIDING THROUGH BY $a_n \neq 0$ SHOWS THAT WE CAN ASSUME AT THE OUTSET THAT $a_n = 1$. MOREOVER, WE CAN ASSUME $a_0 \neq 0$ SINCE, IF $a_0 = 0$, $x = 0$ IS A SOLUTION. THUS, WE DEFINE

$$f(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

WHERE $a_0, \ldots, a_{n-1} \in \mathbb{R}$ AND $a_0 \neq 0$. f IS CONTINUOUS ON $\mathbb{R}$ AND, FOR $x \neq 0$, WE CAN WRITE

$$f(x) = x^n \left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)$$

NOTE THAT

$$\left| \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right| \le \frac{|a_{n-1}|}{|x|} + \cdots + \frac{|a_1|}{|x|^{n-1}} + \frac{|a_0|}{|x|^n}$$

LET $A = \max \{ 1, |a_{n-1}|, \ldots, |a_1|, |a_0| \}$. THEN

$$\left| \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right| \le \frac{A}{|x|} + \cdots + \frac{A}{|x|^{n-1}} + \frac{A}{|x|^n}$$

NOW CONSIDER ANY x SATISFYING $|x| > An$. THEN

$$|x| > An \implies \frac{|x|}{A} > n$$

$$\implies \frac{A}{|x|} < \frac{1}{n} .$$

SINCE $|x| > 1$, IT FOLLOWS THAT EACH OF

$$\frac{A}{|x|}, \ \cdots, \ \frac{A}{|x|^{n-1}}, \ \frac{A}{|x|^n}$$

IS $< \frac{1}{n}$. THUS,

$$\left| \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right| < \frac{1}{n} + \cdots + \frac{1}{n} + \frac{1}{n} = 1$$

AND THEREFORE

$$-1 < \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} < 1 .$$

SO

$$0 < 1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} .$$

THE CONCLUSION OF ALL THIS IS THAT

FOR ANY x WITH $|x| > A_n$,

$$f(x) = x^n \left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)$$

WHERE THE SECOND FACTOR IS POSITIVE. BUT n IS ODD SO CHOOSING A POSITIVE x_2 WITH $|x_2| > A_n$ GIVES

$$f(x_2) > 0$$

AND CHOOSING A NEGATIVE x_1 WITH $|x_1| > A_n$ GIVES

$$f(x_1) < 0.$$

SINCE $f(x_1) < 0 < f(x_2)$ THE INTERMEDIATE VALUE THEOREM IMPLIES THAT THERE IS A c IN (x_1, x_2) WITH $f(c) = 0$. □

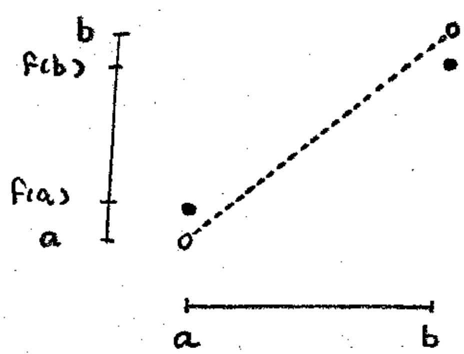
3. (<u>1-DIMENSIONAL BROUWER FIXED POINT THEOREM</u>)

<u>THEOREM</u>: SUPPOSE $a < b$ AND $f : [a,b] \to [a,b]$ IS CONTINUOUS. THEN THERE EXISTS AN x_0 IN $[a,b]$ FOR WHICH

$$f(x_0) = x_0$$

(x_0 IS CALLED A <u>FIXED POINT</u> OF f).

<u>PROOF</u>: IF $f(a) = a$ OR $f(b) = b$ WE ARE DONE SO ASSUME $f(a) > a$ AND $f(b) < b$.



DEFINE

$$g : [a,b] \to \mathbb{R}$$

BY

$$g(x) = x - f(x).$$

THEN g IS CONTINUOUS, $g(a) = a - f(a) < 0$ AND $g(b) = b - f(b) > 0$ SO $g(a) < 0 < g(b)$ AND THE INTERMEDIATE VALUE THEOREM IMPLIES THAT THERE IS AN x_0 IN (a,b) AT WHICH $g(x_0) = 0$, I.E., $f(x_0) = x_0$. $\square$

<u>NOTE</u> : THERE ARE HIGHER DIMENSIONAL VERSIONS OF THIS RESULT, BUT THEY ARE SUBSTANTIALLY HARDER TO PROVE (HAVE A LOOK AT ANY BOOK ON ALGEBRAIC TOPOLOGY).

<u>EXERCISE</u> 7 : SHOW THAT THE EQUATION

$$\cos x = x$$

HAS AT LEAST ONE SOLUTION.

<u>EXERCISE</u> 8 : IS THE CONVERSE OF THE INTERMEDIATE VALUE THEOREM TRUE? MORE PRECISELY : SUPPOSE I IS AN INTERVAL AND $f : I \to \mathbb{R}$ HAS THE PROPERTY THAT WHENEVER $x_1, x_2 \in I$ WITH $x_1 < x_2$ AND $f(x_1) \neq f(x_2)$, THEN, FOR EVERY L STRICTLY BETWEEN $f(x_1)$ AND $f(x_2)$, THERE IS A c IN (x_1, x_2) WITH $f(c) = L$. MUST f BE CONTINUOUS ON I?

IN PREPARATION FOR OUR NEXT MAJOR RESULT ON CONTINUITY WE WILL NEED TO RECORD A FEW DEFINITIONS.

LET $f : D \to \mathbb{R}$ BE A FUNCTION (NOT NECESSARILY CONTINUOUS) DEFINED ON SOME SUBSET D OF $\mathbb{R}$. WE SAY THAT f IS <u>BOUNDED ON</u> D (<u>BOUNDED FROM ABOVE ON</u> D, <u>BOUNDED FROM BELOW ON</u> D) IF ITS RANGE

$$f(D) = \{ f(x) : x \in D \} \subseteq \mathbb{R}$$

IS BOUNDED (BOUNDED FROM ABOVE, BOUNDED FROM BELOW). THUS, FOR EXAMPLE, f IS BOUNDED ON D IF THERE IS SOME CONSTANT $R \geq 0$ SUCH THAT $|f(x)| \leq R$ FOR ALL x IN D.

IF THERE IS SOME $x_1 \in D$ WITH THE PROPERTY THAT

$$f(x) \leq f(x_1)$$

FOR EVERY x IN D, THEN f IS SAID TO ACHIEVE A <u>MAXIMUM VALUE</u> OF $f(x_1)$ ON D AT x_1. SIMILARLY, IF THERE IS AN $x_2 \in D$ FOR WHICH

$$f(x) \geq f(x_2)$$

FOR EVERY $x \in D$, THEN f HAS A <u>MINIMUM VALUE</u> OF $f(x_2)$ ON D AT x_2.

THE CONTENT OF OUR NEXT FEW RESULTS IS THAT IF D IS A CLOSED BOUNDED INTERVAL $[a, b]$ AND f IS CONTINUOUS, THEN f IS BOUNDED AND, IN FACT, ACHIEVES BOTH A MAXIMUM AND A MINIMUM VALUE ON $[a, b]$.

<u>THEOREM</u>: IF $f : [a,b] \to \mathbb{R}$ IS CONTINUOUS, THEN f IS BOUNDED ON $[a,b]$.

<u>PROOF</u>: WE DEFINE

$$A = \{ x : a \le x \le b \text{ AND } f \text{ IS BOUNDED ON } [a,x] \}.$$

THEN $A \ne \emptyset$ SINCE $a \in A$ AND A IS BOUNDED FROM ABOVE (BY b). CONSEQUENTLY,

$$\alpha = \sup(A)$$

EXISTS AND $\alpha \le b$.

NOTE THAT $a < \alpha$: SINCE $\lim_{x \to a^+} f(x) = f(a)$, $\exists\ \delta > 0$ S.T.

$$\begin{aligned} a < x < a + \delta &\Rightarrow |f(x) - f(a)| < 1 \\ &\Rightarrow |f(x)| - |f(a)| < 1 \\ &\Rightarrow |f(x)| < |f(a)| + 1 \\ &\Rightarrow f \text{ IS BOUNDED ON } [a, a+\delta) \end{aligned}$$

THUS, f IS BOUNDED ON $[a,x]$ FOR ANY $a < x < a+\delta$ SO $x \in A$ AND THEREFORE $\alpha = \sup(A) > a$.

NEXT WE SHOW THAT, IN FACT, $\alpha = b$ AND THIS WE DO BY CONTRADICTION. THUS, SUPPOSE $\alpha < b$. THEN $a < \alpha < b$ AND, SINCE f IS CONTINUOUS AT α, $\exists\ \delta > 0$, WHICH WE CAN TAKE TO BE LESS MIN $(\alpha - a, b - \alpha)$, SUCH THAT

$$\alpha - \delta < x < \alpha + \delta \Rightarrow |f(x) - f(\alpha)| < 1$$

WHICH IMPLIES THAT f IS BOUNDED ON $(\alpha-\delta, \alpha+\delta)$ EXACTLY AS ABOVE. SINCE $\alpha = \sup(A)$ THERE IS A POINT x_1 OF A IN $(\alpha-\delta, \alpha)$. THUS, f IS BOUNDED ON $[a, x_1]$. BUT IF x_2 IS ANY POINT IN $(\alpha, \alpha+\delta)$, THEN f IS BOUNDED ON $[x_1, x_2]$ BECAUSE $[x_1, x_2] \subseteq (\alpha-\delta, \alpha+\delta)$. CONSEQUENTLY, f IS BOUNDED ON $[a, x_1] \cup [x_1, x_2] = [a, x_2]$ AND THIS MEANS $x_2 \in A$ WHICH CONTRADICTS THE FACT THAT α IS AN UPPER BOUND FOR A. THUS, $\alpha = b$.

WE CONCLUDE THAT f IS BOUNDED ON $[a, x]$ FOR EVERY $x < b$. BUT $\lim_{x \to b^-} f(x) = f(b)$ IMPLIES, EXACTLY AS FOR a ABOVE, THAT f IS BOUNDED ON $(b-\delta, b]$ FOR SOME $\delta > 0$. NOW, $b = \sup(A)$ IMPLIES THAT THERE IS AN ELEMENT x_3 OF A IN $(b-\delta, b)$ SO f IS BOUNDED ON $[a, x_3]$. BUT f IS ALSO BOUNDED ON $[x_3, b] \subseteq (b-\delta, b]$ SO IT IS BOUNDED ON $[a,b]$ AS REQUIRED. $\square$

<u>COROLLARY</u>: IF $f : [a,b] \to \mathbb{R}$ IS CONTINUOUS, THEN THERE EXISTS AN x_1 IN $[a,b]$ SUCH THAT

$$f(x) \leq f(x_1)$$

FOR ALL x IN $[a,b]$ AND THERE EXISTS AN x_2 IN $[a,b]$ SUCH THAT

$$f(x) \geq f(x_2)$$

FOR ALL x IN $[a,b]$.

PROOF: WE WILL PROVE THE FIRST STATEMENT AND LEAVE THE SECOND AS AN EXERCISE.

BY THE THEOREM,

$$\{ f(x) : a \le x \le b \}$$

IS BOUNDED. IT IS CERTAINLY NONEMPTY, SO IT HAS A LEAST UPPER BOUND α. THUS,

$$f(x) \le \alpha$$

FOR EVERY x IN $[a,b]$. WE SHOW THAT THERE IS AN x_1 IN $[a,b]$ SUCH THAT $\alpha = f(x_1)$. SUPPOSE TO THE CONTRARY THAT $\alpha \neq f(x)$ FOR EVERY x IN $[a,b]$. THUS, $\alpha - f(x) \neq 0$ FOR EVERY x IN $[a,b]$ SO

$$g(x) = \frac{1}{\alpha - f(x)}$$

IS CONTINUOUS ON $[a,b]$. ACCORDING TO THE THEOREM, g MUST BE BOUNDED ON $[a,b]$.

HOWEVER, SINCE $\alpha = \sup \{ f(x) : a \le x \le b \}$ WE CAN FIND, FOR ANY $\varepsilon > 0$, AN x IN $[a,b]$ WITH $f(x) > \alpha - \varepsilon$, I.E.,

$$\alpha - f(x) < \varepsilon$$

$$\frac{1}{\alpha - f(x)} > \frac{1}{\varepsilon}$$

$$g(x) > \frac{1}{\varepsilon}.$$

SINCE $\varepsilon > 0$ IS ARBITRARY THIS CONTRADICTS THE FACT THAT g MUST BE BOUNDED ON $[a,b]$ AND SO THERE MUST EXIST AN x_1 IN $[a,b]$ WITH $\alpha = f(x_1)$.

EXERCISE 9: COMPLETE THE PROOF BY SHOWING THAT THERE IS AN x_2 IN $[a,b]$ SATISFYING

$$f(x) \geq f(x_2)$$

FOR EVERY x IN $[a,b]$. $\square$

KNOWING THAT A FUNCTION ACHIEVES MAXIMUM AND MINIMUM VALUES SOMEWHERE IS NICE, BUT THE REAL APPLICATIONS OF THIS WILL REQUIRE THAT WE FIND OUT WHERE THESE OCCUR. WE WILL RETURN TO THIS IN THE NEXT SECTION (WHERE WE WILL HAVE TO ASSUME MORE THAN JUST CONTINUITY OF THE FUNCTION).

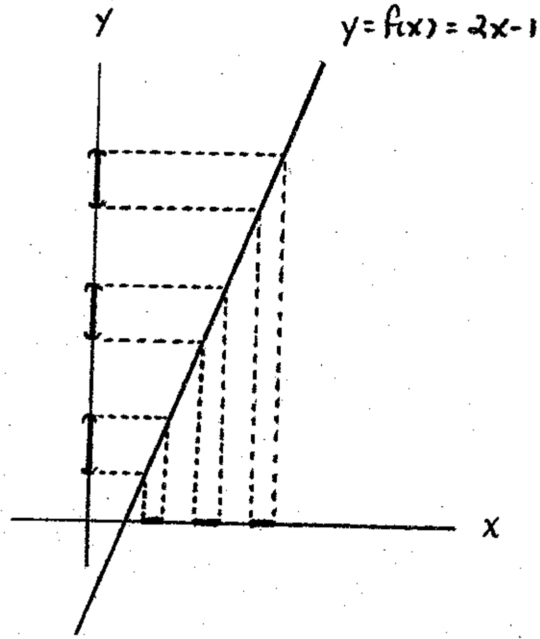
CONTINUOUS FUNCTIONS ON CLOSED, BOUNDED INTERVALS $[a,b]$ HAVE LOTS OF SPECIAL PROPERTIES (AS WE HAVE JUST SEEN IN THE THEOREM AND COROLLARY). WE NOW DESCRIBE ANOTHER IMPORTANT SUCH PROPERTY.

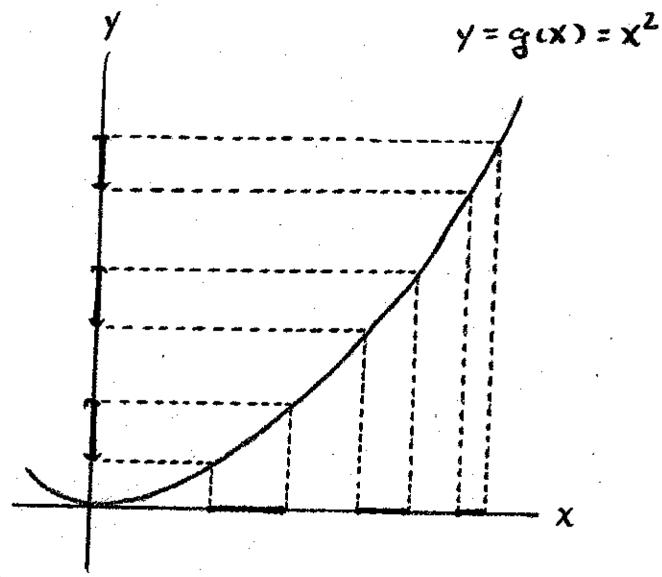
AS MOTIVATION FOR THE CONCEPT WE ARE ABOUT TO INTRODUCE WE WOULD LIKE TO CONSIDER TWO SIMPLE FUNCTIONS.

$$f : \mathbb{R} \to \mathbb{R} \qquad\qquad g : \mathbb{R} \to \mathbb{R}$$

$$f(x) = 2x - 1 \qquad\qquad g(x) = x^2$$

BOTH ARE, OF COURSE, CONTINUOUS ON $\mathbb{R}$, BUT THERE IS A DIFFERENCE. HERE ARE SOME PICTURES.





HERE WE HAVE FIXED SOME $\varepsilon > 0$ AND, FOR THREE POINTS IN THE DOMAIN OF EACH, DRAWN THE ε-NBD OF THE IMAGE POINT AND INDICATED A δ-NBD THAT THE FUNCTION MAPS INTO IT. THE POINT IS THAT IT APPEARS THAT, FOR f, THE SAME δ WORKS FOR ALL THE POINTS, BUT, FOR g, AS THE SELECTED POINT $x_0 \to \infty$ THE REQUISITE $\delta \to 0$.

NOTE: FOR FUTURE REFERENCE NOTICE THAT, IF $g(x) = x^2$ WERE RESTRICTED TO SOME CLOSED, BOUNDED INTERVAL, THEN IT APPEARS THAT THERE WOULD BE A SMALLEST SUCH δ WHICH WOULD THEN WORK FOR EVERY x_0 IN THE INTERVAL.

WE WILL PROVE ALL OF THIS SHORTLY, BUT FIRST THE APPROPRIATE DEFINITION.

A FUNCTION f IS <u>UNIFORMLY</u> <u>CONTINUOUS</u> <u>ON</u> <u>AN</u> <u>INTERVAL</u> I IF $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ SUCH THAT

$$x, y \in I \text{ AND } |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

(FOR A GIVEN $\varepsilon > 0$ THE SAME δ WORKS FOR ANY $y = x_0$).

<u>EXAMPLES</u> :

8. WE SHOW THAT

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = 2x - 1$$

IS UNIFORMLY CONTINUOUS ON $\mathbb{R}$.

LET $\varepsilon > 0$ BE GIVEN. TAKE $\delta = \frac{\varepsilon}{2}$. THEN

$$\begin{aligned} |x-y| < \delta = \frac{\varepsilon}{2} \implies |f(x) - f(y)| &= |(2x-1) - (2y-1)| \\ &= 2|x-y| \\ &< 2\left(\frac{\varepsilon}{2}\right) = \varepsilon. \end{aligned}$$

9. WE SHOW THAT

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = x^2$$

IS <u>NOT</u> UNIFORMLY CONTINUOUS ON $\mathbb{R}$.

MORE PRECISELY, WE TAKE $\varepsilon = 1$ (ANY OTHER VALUE WOULD DO JUST AS WELL) AND SHOW THAT, HOWEVER $\delta > 0$ IS CHOSEN, THERE EXISTS AN $x \in \mathbb{R}$ FOR WHICH $|x-y| < \delta \not\Rightarrow |f(x)-f(y)| < 1$.

THUS, LET $\delta > 0$ BE ARBITRARY. LET

$$x = \frac{1}{\delta}$$

AND

$$y = x + \frac{\delta}{2} = \frac{1}{\delta} + \frac{\delta}{2}.$$

THEN $|x-y| = \frac{\delta}{2} < \delta$, BUT

$$\begin{aligned} |f(x)-f(y)| &= |x^2-y^2| = |x+y||x-y| \\ &= \left| \frac{1}{\delta} + \frac{1}{\delta} + \frac{\delta}{2} \right| \left| \frac{\delta}{2} \right| \\ &= \left(\frac{2}{\delta} + \frac{\delta}{2} \right)\left(\frac{\delta}{2} \right) \\ &= 1 + \frac{\delta^2}{4} \\ &> 1. \end{aligned}$$

9. LET $R > 0$ BE SOME POSITIVE CONSTANT. WE SHOW THAT

$$f : [-R, R] \longrightarrow \mathbb{R}$$
$$f(x) = x^2$$

<u>IS</u> UNIFORMLY CONTINUOUS ON $[-R, R]$.

LET $\varepsilon > 0$ BE GIVEN. FOR ANY $x, y \in [-R, R]$,

$$|f(x)-f(y)| = |x^2-y^2| = |x+y||x-y| \le 2R|x-y|$$

SO WE CAN TAKE $\delta = \frac{\varepsilon}{2R}$. THEN

$$x, y \in [-R, R] \text{ AND } |x-y| < \delta = \frac{\varepsilon}{2R} \Rightarrow$$

$$|f(x) - f(y)| \leq 2R\,|x-y| < 2R\left(\frac{\varepsilon}{2R}\right) = \varepsilon .$$

10. DEFINE

$$f : (0, 1] \rightarrow \mathbb{R}$$

$$f(x) = \cos\left(\frac{1}{x}\right).$$

THEN f IS CONTINUOUS ON $(0, 1]$, BUT WE SHOW THAT IT IS NOT UNIFORMLY CONTINUOUS ON $(0, 1]$.

TAKE $\varepsilon = 1$. LET $\delta > 0$ BE ARBITRARY. THEN $\exists\, n \in \mathbb{N}$ S.T. $\frac{1}{n} < \delta$. NOTE THAT

$$\frac{1}{n\pi}, \ \frac{1}{(n+1)\pi} \in (0, 1]$$

AND

$$\left| \frac{1}{n\pi} - \frac{1}{(n+1)\pi} \right| = \frac{n\pi + \pi - n\pi}{n(n+1)\pi} = \frac{1}{n(n+1)} < \frac{1}{n} < \delta .$$

BUT

$$\left| f\left(\frac{1}{n\pi}\right) - f\left(\frac{1}{(n+1)\pi}\right) \right| = \left| \cos(n\pi) - \cos(n+1)\pi \right|$$

$$= |\pm 2| = 2 > 1 .$$

NOTE: WE WILL SOON SEE, HOWEVER, THAT $\cos\left(\frac{1}{x}\right)$ IS UNIFORMLY CONTINUOUS ON $[a, 1]$ FOR EVERY $0 < a < 1$.

EXERCISE 10: SHOW THAT IF f AND g ARE UNIFORMLY CONTINUOUS ON I, THEN $f+g$ AND $f-g$ ARE UNIFORMLY CONTINUOUS ON I.

THE PRODUCT OF TWO UNIFORMLY CONTINUOUS FUNCTIONS NEED NOT BE UNIFORMLY CONTINUOUS, HOWEVER. FOR EXAMPLE, $f(x) = x$ IS UNIFORMLY CONTINUOUS ON $\mathbb{R}$, BUT $f(x)f(x) = x^2$ IS NOT. HOWEVER,

EXERCISE 11 : SHOW THAT IF $I = [a,b]$ IS A CLOSED, BOUNDED INTERVAL AND $f, g : [a,b] \to \mathbb{R}$ ARE UNIFORMLY CONTINUOUS, THEN fg IS ALSO UNIFORMLY CONTINUOUS. HINT : YOU WILL NEED THE FACT THAT f AND g ARE BOUNDED ON $[a,b]$.

EXERCISE 12 : LET I BE ANY INTERVAL AND $g : I \to \mathbb{R}$ UNIFORMLY CONTINUOUS. ASSUME THAT g IS BOUNDED AWAY FROM 0, I.E., THAT THERE IS A POSITIVE CONSTANT M SUCH THAT $|g(x)| \geq M > 0$ FOR EVERY x IN I. SHOW THAT $\frac{1}{g}$ IS UNIFORMLY CONTINUOUS ON I.

EXERCISE 13 : LET I AND J BE INTERVALS, $f : I \to \mathbb{R}$ AND $g : J \to \mathbb{R}$ UNIFORMLY CONTINUOUS AND ASSUME $f(I) \subseteq J$. SHOW THAT $g \circ f$ IS UNIFORMLY CONTINUOUS ON I.

OUR FINAL RESULT ASSERTS THAT A CONTINUOUS FUNCTION ON A CLOSED, BOUNDED INTERVAL IS UNIFORMLY CONTINUOUS.

<u>THEOREM</u>: A CONTINUOUS FUNCTION $f : [a,b] \to \mathbb{R}$ IS UNIFORMLY CONTINUOUS ON $[a,b]$.

<u>PROOF</u>: LET $\varepsilon > 0$ BE GIVEN. DEFINE

$$A = \{ x \in [a,b] : \exists \delta > 0 \text{ S.T. } u, v \in [a, x] \text{ AND } |u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon \}$$

(THIS IS JUST THE SET OF x IN $[a,b]$ FOR WHICH WE CAN FIND AN APPROPRIATE $\delta > 0$ ON $[a,x]$). WE MUST SHOW THAT $b \in A$.

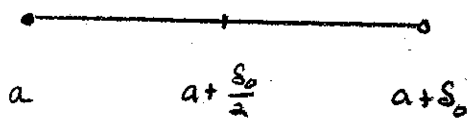
$A \neq \emptyset$ BECAUSE $a \in A$ AND A IS BOUNDED ABOVE (BY b) SO

$$\alpha = \sup(A)$$

EXISTS.

SINCE $a \in A$, $a \leq \alpha$ AND, SINCE b IS AN UPPER BOUND FOR A, $\alpha \leq b$. THUS, $\alpha \in [a,b]$.

WE CLAIM FIRST THAT, IN FACT, $a < \alpha$. TO SEE THIS NOTE THAT, BECAUSE f IS CONTINUOUS AT a, $\exists \delta_0 > 0$ S.T. $a \leq x < a + \delta_0 \Rightarrow |f(x) - f(a)| < \frac{\varepsilon}{2}$. WE CLAIM THAT $a + \frac{\delta_0}{2} \in A$ (SO $\alpha = \sup(A) \geq a + \frac{\delta_0}{2} > a$).



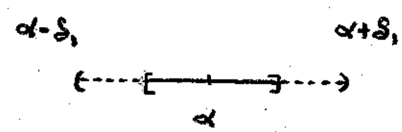
IN FACT, FOR ANY $u, v \in [a, a+\frac{\delta_0}{2}] \subseteq [a, a+\delta_0]$,

$$\begin{aligned} |f(u)-f(v)| &= |f(u)-f(a)+f(a)-f(v)| \\ &\leq |f(u)-f(a)| + |f(v)-f(a)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

SO $a+\frac{\delta_0}{2} \in A$.

THUS, $a < \alpha \leq b$ AND WE NOW CLAIM THAT, IN FACT, $\alpha = b$. SUPPOSE TO THE CONTRARY THAT $a < \alpha < b$. SINCE f IS CONTINUOUS AT α, $\exists\ \delta_1 > 0$, WHICH WE CAN TAKE TO BE LESS THAN MIN $(\alpha - a, b-\alpha)$ SUCH THAT

$$|x-\alpha| < \delta_1 \Rightarrow |f(x)-f(\alpha)| < \frac{\varepsilon}{2}$$



THE CLOSED INTERVAL $[\alpha-\frac{\delta_1}{2}, \alpha+\frac{\delta_1}{2}]$ IS CONTAINED IN $(\alpha-\delta_1, \alpha+\delta_1)$.

$$\begin{aligned} u, v \in [\alpha-\tfrac{\delta_1}{2}, \alpha+\tfrac{\delta_1}{2}] &\Rightarrow |u-\alpha| < \delta_1 \text{ AND } |v-\alpha| < \delta_1 \\ &\Rightarrow |f(u)-f(\alpha)| < \tfrac{\varepsilon}{2} \text{ AND } |f(v)-f(\alpha)| < \tfrac{\varepsilon}{2} \\ &\Rightarrow |f(u)-f(v)| = |f(u)-f(\alpha)+f(\alpha)-f(v)| \\ &\qquad \leq |f(u)-f(\alpha)| + |f(v)-f(\alpha)| \\ &\qquad < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

NOW, SINCE $\alpha = \sup(A)$, THERE IS SOME $x_0 \in A$ SATISFYING $\alpha - \frac{\delta_1}{2} < x_0 < \alpha$ AND, SINCE $[a, \alpha - \frac{\delta_1}{2}] \subseteq [a, x_0]$, THIS IMPLIES THAT $\alpha - \frac{\delta_1}{2} \in A$.

NOW WE INTEND TO ARRIVE AT A CONTRADICTION BY USING

$$[a, \alpha + \tfrac{\delta_1}{2}] = [a, \alpha - \tfrac{\delta_1}{2}] \cup [\alpha - \tfrac{\delta_1}{2}, \alpha + \tfrac{\delta_1}{2}]$$

TO SHOW THAT $\alpha + \frac{\delta_1}{2}$ IS IN A ($\alpha + \frac{\delta_1}{2} > \alpha = \sup(A)$ SO THIS IS IMPOSSIBLE). SINCE WE WILL NEED THE SAME ARGUMENT AGAIN AT THE END OF THE PROOF WE WILL PROVE A GENERAL LEMMA.

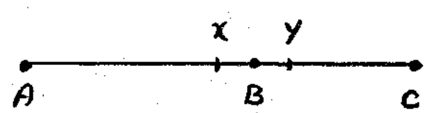
<u>LEMMA</u>: LET $A < B < C$ AND LET f BE CONTINUOUS ON $[A, C]$. LET $\varepsilon > 0$ BE GIVEN. SUPPOSE

(1) $\exists\, \delta_1 > 0$ S.T. $x, y \in [A, B]$ AND $|x - y| < \delta_1 \Rightarrow$
$$|f(x) - f(y)| < \varepsilon$$

(2) $\exists\, \delta_2 > 0$ S.T. $x, y \in [B, C]$ AND $|x - y| < \delta_2 \Rightarrow$
$$|f(x) - f(y)| < \varepsilon.$$

THEN $\exists\, \delta > 0$ S.T. $x, y \in [A, C]$ AND $|x - y| < \delta \Rightarrow$
$$|f(x) - f(y)| < \varepsilon.$$

<u>PROOF</u>: NOTICE THAT $\min(\delta_1, \delta_2)$ WILL NOT DO FOR δ BECAUSE IF $x \in [A, B]$ AND $y \in [B, C]$ WITH $|x - y| < \min(\delta_1, \delta_2)$ WE WOULD HAVE NO INFORMATION ABOUT $|f(x) - f(y)|$.



HOWEVER, f IS CONTINUOUS AT B SO $\exists\ \delta_3 > 0$, WHICH WE CAN TAKE TO BE LESS THAN $\min(B-A, C-B)$, S.T.

$$|x - B| < \delta_3 \Rightarrow |f(x) - f(B)| < \frac{\varepsilon}{2}.$$

THUS,

(3) $\quad |x-B| < \delta_3$ AND $|y-B| < \delta_3 \Rightarrow |f(x) - f(y)| < \varepsilon$.

NOW LET $\delta = \min(\delta_1, \delta_2, \delta_3)$. THEN IF $x, y \in [A, C]$ WITH $|x-y| < \delta$, THERE ARE THREE POSSIBILITIES.

(i) $x, y \in [A, B]$ SO $|f(x) - f(y)| < \varepsilon$ FOLLOWS FROM (1).

(ii) $x, y \in [B, C]$ SO $|f(x) - f(y)| < \varepsilon$ FOLLOWS FROM (2).

(iii) EITHER $x \in [A, B]$ AND $y \in [B, C]$, OR $y \in [A, B]$ AND $x \in [B, C]$. IN EITHER CASE, $|x-y| < \delta \Rightarrow |x - B| < \delta$ AND $|y-B| < \delta$ SO $|f(x) - f(y)| < \varepsilon$ FOLLOWS FROM (3).

$\square$

APPLYING THE LEMMA TO $a < \alpha - \frac{\delta_1}{2} < \alpha + \frac{\delta_1}{2}$ WE FIND THAT $\alpha + \frac{\delta_1}{2} \in A$, THUS ARRIVING AT OUR CONTRADICTION. THE CONCLUSION IS THAT

$$\alpha = \sup(A) = b.$$

WE COMPLETE THE PROOF BY SHOWING THAT $b \in A$.

THE ARGUMENT IS VIRTUALLY IDENTICAL. SINCE f IS CONTINUOUS AT b $\exists\ \delta_2 > 0$, WHICH WE CAN ASSUME IS LESS THAN $b-a$, SUCH THAT

$$b - \delta_2 < x \leq b \implies |f(x) - f(b)| < \frac{\varepsilon}{2} .$$

THUS,

$$x, y \in (b - \delta_2, b] \implies |f(x) - f(y)| < \varepsilon$$

BUT $b = \sup(A)$ SO THERE IS AN ELEMENT x_1 OF A SATISFYING $b - \delta_2 < x_1 < b$ (IF $x_1 = b$ WE ARE DONE). SINCE $[x_1, b] \subseteq (b - \delta_2, b]$ WE CAN APPLY THE LEMMA TO

$$[a,b] = [a, x_1] \cup [x_1, b]$$

TO CONCLUDE THAT $b \in A$, AS REQUIRED. $\square$

ADDITIONAL PROBLEMS:

14. ARE THE DISCONTINUITIES OF THE DIRICHLET FUNCTION REMOVABLE? JUSTIFY YOUR ANSWER.

15. GIVE AN EXAMPLE OF A FUNCTION f THAT IS CONTINUOUS NOWHERE, BUT FOR WHICH $|f|$ IS CONTINUOUS EVERYWHERE.

16. SUPPOSE $f : \mathbb{R} \to \mathbb{R}$ SATISFIES $f(x+y) = f(x) + f(y)$ FOR ALL $x, y \in \mathbb{R}$ AND IS CONTINUOUS AT $x=0$. SHOW THAT f IS CONTINUOUS ON $\mathbb{R}$.

17. SHOW THAT, IF I IS AN INTERVAL AND $f : I \to \mathbb{R}$ IS CONTINUOUS AND NONCONSTANT, THEN $f(I)$ IS AN INTERVAL AND THAT, IF $I = [a,b]$ IS CLOSED AND BOUNDED, THEN SO IS $f(I)$.

18. SUPPOSE THAT f AND g ARE CONTINUOUS ON SOME INTERVAL I. DEFINE $\max(f,g)$ AND $\min(f,g)$ ON I BY $(\max(f,g))(x) = \max(f(x), g(x))$ AND $(\min(f,g))(x) = \min(f(x), g(x))$. SHOW THAT BOTH ARE CONTINUOUS ON I.

19. SUPPOSE $f, g : \mathbb{R} \to \mathbb{R}$ ARE CONTINUOUS AND $f(x) = g(x)$ FOR ALL x IN SOME DENSE SUBSET D OF $\mathbb{R}$. SHOW THAT $f(x) = g(x)$ FOR ALL $x \in \mathbb{R}$.

20. SUPPOSE $f : [a,b] \to \mathbb{R}$ IS CONTINUOUS AND $f(x)$ IS RATIONAL FOR EVERY $x \in [a,b]$. WHAT CAN BE SAID ABOUT f?

21. SUPPOSE $f, g : [a,b] \to \mathbb{R}$ ARE CONTINUOUS, $f(a) < g(a)$, AND $f(b) > g(b)$. PROVE THAT $f(x) = g(x)$ FOR SOME x IN $[a,b]$.

22. PROVE THAT THERE DOES NOT EXIST A CONTINUOUS FUNCTION ON $\mathbb{R}$ WHICH TAKES ON EVERY VALUE EXACTLY TWICE.

23. GIVE AN EXAMPLE OF A FUNCTION $f : [0,\infty) \to \mathbb{R}$ THAT IS CONTINUOUS AND BOUNDED, BUT NOT UNIFORMLY CONTINUOUS ON $[0,\infty)$

24. SHOW THAT IF $f : [a,\infty) \to \mathbb{R}$ IS CONTINUOUS AND $\lim_{x\to\infty} f(x)$ EXISTS, THEN f IS UNIFORMLY CONTINUOUS ON $[a,\infty)$.

25. A FUNCTION $f : \mathbb{R} \to \mathbb{R}$ IS <u>PERIODIC</u> IF THERE IS A POSITIVE REAL NUMBER p SUCH THAT $f(x+p) = f(x)$ FOR EVERY $x \in \mathbb{R}$. SHOW THAT IF f IS PERIODIC AND CONTINUOUS ON $\mathbb{R}$, THEN f IS UNIFORMLY CONTINUOUS ON $\mathbb{R}$.

26. LET I BE AN INTERVAL. A FUNCTION $f : I \to \mathbb{R}$ IS SAID TO BE <u>LIPSCHITZ</u> ON I IF THERE IS A CONSTANT $M > 0$ SUCH THAT

$$\left| \frac{f(x) - f(y)}{x-y} \right| \le M$$

FOR ALL $x \neq y$ IN M. SHOW THAT SUCH A FUNCTION IS UNIFORMLY CONTINUOUS ON I. IS THE CONVERSE TRUE ?

SOLUTIONS TO THE EXERCISES :

1. WE ASSUME $e^x e^y = e^{x+y}$ AND $\lim_{x\to 0} e^x = 1$

CLAIM : e^x IS CONTINUOUS AT EVERY $x_0 \in \mathbb{R}$.

FIRST NOTE THAT $e^0 = e^{0+0} = e^0 e^0$ SO $e^0(e^0 - 1) = 0$ AND EITHER $e^0 = 0$ OR $e^0 = 1$. BUT $e^0 = 0$ WOULD IMPLY $e^x = e^{x+0} = e^x e^0 = 0$ AND THIS CONTRADICTS $\lim_{x\to 0} e^x = 1$.

THUS,

$$e^0 = 1 = \lim_{x\to 0} e^x$$

SO e^x IS CONTINUOUS AT $x = 0$.

NOW LET $x_0 \in \mathbb{R}$ BE ARBITRARY. NOTE THAT

$$\begin{aligned} |e^x - e^{x_0}| &= |e^{(x-x_0)+x_0} - e^{x_0}| \\ &= |e^{(x-x_0)} e^{x_0} - e^{x_0}| \\ &= |e^{x_0}|\,|e^{x-x_0} - 1| \end{aligned}$$

NOTE THAT $e^{x_0} \neq 0$ SINCE $e^{x_0} = 0 \Rightarrow e^x - e^{x_0} = 0 \Rightarrow e^x = 0$ FOR EVERY x AND THIS CONTRADICTS $\lim_{x\to 0} e^x = 1$.

NOW LET $\varepsilon > 0$ BE GIVEN. CHOOSE $\delta > 0$ S.T. $|t - 0| < \delta \Rightarrow |e^t - 1| < \frac{\varepsilon}{|e^{x_0}|}$. THEN

$$\begin{aligned} |x - x_0| < \delta &\Rightarrow |e^{x-x_0} - 1| < \frac{\varepsilon}{|e^{x_0}|} \\ &\Rightarrow |e^x - e^{x_0}| = |e^{x_0}|\,|e^{x-x_0} - 1| \\ &\qquad < |e^{x_0}| \left(\frac{\varepsilon}{|e^{x_0}|} \right) = \varepsilon. \end{aligned}$$

2. WE ASSUME THAT

$$\ln(xy) = \ln x + \ln y \qquad (x, y > 0)$$

$$\lim_{x \to 1} \ln x = 0$$

CLAIM: $\ln x$ IS CONTINUOUS FOR ALL $x > 0$.

NOTE THAT $\ln 1 = 0$ BECAUSE $\ln 1 = \ln(1 \cdot 1) = \ln 1 + \ln 1$. THUS,

$$\lim_{x \to 1} \ln x = 0 = \ln 1$$

SO $\ln x$ IS CONTINUOUS AT $x = 1$.

NOW, LET $x_0 \in (0, \infty)$. NOTE THAT

$$\begin{aligned} |\ln x - \ln x_0| &= \left|\ln\left(\tfrac{x}{x_0} x_0\right) - \ln x_0\right| \\ &= \left|\ln\left(\tfrac{x}{x_0}\right) + \ln x_0 - \ln x_0\right| \\ &= \left|\ln\left(\tfrac{x}{x_0}\right) - 0\right| \end{aligned}$$

AND

$$|x - x_0| = |x_0| \left|\frac{x}{x_0} - 1\right| \Rightarrow \left|\frac{x}{x_0} - 1\right| = \frac{|x - x_0|}{x_0}$$

LET $\varepsilon > 0$ BE GIVEN.

CHOOSE $\delta_1 > 0$ S.T. $|t - 1| < \delta_1 \Rightarrow |\ln t - 0| < \varepsilon$. LET

$$\delta = x_0 \delta_1 .$$

THEN FOR $x \in (0, \infty)$,

$$\begin{aligned} |x - x_0| < \delta &\Rightarrow \left|\frac{x}{x_0} - 1\right| = \frac{|x - x_0|}{x_0} < \frac{\delta}{x_0} = \frac{x_0 \delta_1}{x_0} = \delta_1 \\ &\Rightarrow \left|\ln\left(\tfrac{x}{x_0}\right) - 0\right| < \varepsilon \\ &\Rightarrow |\ln x - \ln x_0| < \varepsilon . \end{aligned}$$

3. $\lim\limits_{x\to x_0} f(x) = f(x_0)$ AND $\lim\limits_{x\to x_0} g(x) = g(x_0) \Rightarrow$

$$\lim_{x\to x_0}(f\pm g)(x) = \lim_{x\to x_0}(f(x)\pm g(x))$$
$$= \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x)$$
$$= f(x_0) \pm g(x_0)$$
$$= (f\pm g)(x_0)$$

$$\lim_{x\to x_0}(fg)(x) = \lim_{x\to x_0}(f(x)g(x)) = (\lim_{x\to x_0} f(x))(\lim_{x\to x_0} g(x))$$
$$= f(x_0)g(x_0)$$
$$= (fg)(x_0)$$

IF $g(x_0) \neq 0$, THEN $g(x) \neq 0$ ON SOME NBD OF x_0 SO

$$\lim_{x\to x_0}\left(\frac{f}{g}\right)(x) = \lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)} = \frac{f(x_0)}{g(x_0)}$$
$$= \left(\frac{f}{g}\right)(x_0).$$

4. BEGIN BY NOTING THAT

$$\big|\,|f(x)| - |f(x_0)|\,\big| = \begin{cases} |f(x)| - |f(x_0)|, & |f(x)| \geq |f(x_0)| \\ |f(x_0)| - |f(x)|, & |f(x)| \leq |f(x_0)| \end{cases}$$

BUT

$$|f(x)| - |f(x_0)| \leq |f(x) - f(x_0)|$$

AND

$$|f(x_0)| - |f(x)| \leq |f(x_0) - f(x)| = |f(x) - f(x_0)|$$

SO

$$\big|\,|f(x)| - |f(x_0)|\,\big| \leq |f(x) - f(x_0)|.$$

NOW LET $\varepsilon > 0$ BE GIVEN. CHOOSE $\delta > 0$ S.T.

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

THEN

$$\big|\, |f(x)| - |f(x_0)| \,\big| \le |f(x) - f(x_0)| < \varepsilon$$

SO $|f|$ IS CONTINUOUS AT x_0

5. x_2 IS AN UPPER BOUND FOR A SO $\sup(A) \le x_2$. SINCE $x_1 \in A$, $x_1 \le \sup(A)$. THUS, $x_1 \le \sup(A) \le x_2$.

6. $f(x_1) > f(x_2) \Rightarrow -f(x_1) < -f(x_2)$

$$\Rightarrow (-f)(x_1) < (-f)(x_2)$$

NOW, IF $f(x_1) > L > f(x_2)$, THEN $-f(x_1) < -L < -f(x_2)$

SO

$$(-f)(x_1) < -L < (-f)(x_2)$$

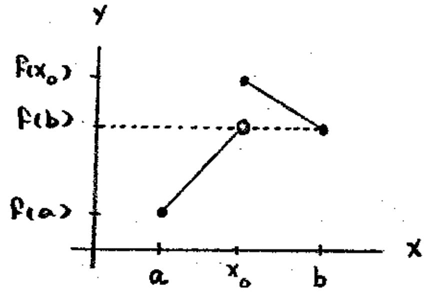
BY THE RESULT ALREADY PROVED THERE IS A c IN (x_1, x_2) FOR WHICH $(-f)(c) = -L$ SO

$$-f(c) = -L$$
$$f(c) = L.$$

7. $\cos x = x$

LET $f(x) = \cos x - x$. THEN f IS CONTINUOUS ON $\mathbb{R}$, $f(0) = \cos 0 - 0 = 1 > 0$ AND $f(\frac{\pi}{2}) = \cos\frac{\pi}{2} - \frac{\pi}{2} = -\frac{\pi}{2} < 0$. THE INTERMEDIATE VALUE THEOREM THEN GIVES A $c \in (0, \frac{\pi}{2})$ AT WHICH $f(c) = 0$ SO $\cos c = c$.

8. THE CONVERSE OF THE INTERMEDIATE VALUE THEOREM IS NOT TRUE, E.G.,



SATISFIES THE INTERMEDIATE VALUE PROPERTY ON $[a,b]$, BUT IS NOT CONTINUOUS ON $[a,b]$. MANY OTHER EXAMPLES ARE POSSIBLE.

9. SINCE $\{f(x) : a \le x \le b\}$ IS NONEMPTY AND BOUNDED, $\beta = \inf \{f(x) : a \le x \le b\}$ EXISTS. THUS, $f(x) \geq \beta$ FOR EVERY $x \in [a,b]$. WE SHOW THAT THERE IS AN $x_2 \in [a,b]$ AT WHICH $f(x_2) = \beta$. SUPPOSE NOT. THEN

$$h(x) = \frac{1}{f(x)-\beta}$$

IS CONTINUOUS ON $[a,b]$ AND SO MUST BE BOUNDED ON $[a,b]$. HOWEVER, $\forall\ \varepsilon > 0\ \exists\ x \in [a,b]$ WITH

$$f(x) < \beta + \varepsilon$$

I.E.,

$$f(x) - \beta < \varepsilon$$

$$\frac{1}{f(x)-\beta} > \frac{1}{\varepsilon}$$

$$h(x) > \frac{1}{\varepsilon}$$

SINCE $\varepsilon > 0$ IS ARBITRARY THIS CONTRADICTS THE FACT THAT h IS BOUNDED ON $[a,b]$.

10. $f, g : I \to \mathbb{R}$ UNIFORMLY CONTINUOUS ON I.

CLAIM: $f+g$ UNIFORMLY CONTINUOUS ON I.

LET $\varepsilon > 0$ BE GIVEN. $\exists\ \delta_1 > 0$ AND $\delta_2 > 0$ S.T.

$$x, y \in I \text{ AND } |x-y| < \delta_1 \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$$

$$x, y \in I \text{ AND } |x-y| < \delta_2 \Rightarrow |g(x) - g(y)| < \frac{\varepsilon}{2}$$

LET $\delta = \min(\delta_1, \delta_2)$. THEN

$$x, y \in I \text{ AND } |x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2} \text{ AND } |g(x) - g(y)| < \frac{\varepsilon}{2}$$

$$\Rightarrow |(f(x) + g(x)) - (f(y) + g(y))| =$$

$$|(f(x) - f(y)) + (g(x) - g(y))| \le$$

$$|f(x) - f(y)| + |g(x) - g(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

FOR $f-g$ THE ARGUMENT IS EXACTLY THE SAME USING

$$|(f(x) - g(x)) - (f(y) - g(y))| = |(f(x) - f(y)) + (g(y) - g(x))|$$

$$\le |f(x) - f(y)| + |g(y) - g(x)| =$$

$$|f(x) - f(y)| + |g(x) - g(y)|.$$

11. $f, g : [a,b] \to \mathbb{R}$ UNIFORMLY CONTINUOUS ON $[a,b]$. SUPPOSE

$|f(x)| \le M_1$ AND $|g(x)| \le M_2$ ON $[a,b]$.

NOW LET $\varepsilon > 0$ BE GIVEN. FOR $x, y \in [a,b]$,

$$|(fg)(x) - (fg)(y)| = |f(x)g(x) - f(y)g(y)|$$

$$= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|$$

$$\le |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)|$$

$$\le M_1 |g(x) - g(y)| + M_2 |f(x) - f(y)|$$

NOW CHOOSE $\delta_1 > 0$ AND $\delta_2 > 0$ S.T. FOR $x, y \in [a,b]$

$$|x-y| < \delta_1 \Rightarrow |g(x)-g(y)| < \frac{\varepsilon}{2M_1}$$

AND

$$|x-y| < \delta_2 \Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{2M_2} .$$

LET $\delta = \min(\delta_1, \delta_2)$. THEN

$$x, y \in [a,b] \text{ AND } |x-y| < \delta \Rightarrow |(fg)(x)-(fg)(y)| \le$$

$$M_1 |g(x)-g(y)| + M_2 |f(x)-f(y)| <$$

$$M_1 \left(\frac{\varepsilon}{2M_1}\right) + M_2 \left(\frac{\varepsilon}{2M_2}\right) = \varepsilon .$$

12. I AN INTERVAL AND $g : I \to \mathbb{R}$ UNIFORMLY CONTINUOUS ON I SATISFYING $|g(x)| \geq M > 0$ FOR SOME POSITIVE CONSTANT M.

FOR $x, y \in I$,

$$\left|\left(\frac{1}{g}\right)(x) - \left(\frac{1}{g}\right)(y)\right| = \left|\frac{1}{g(x)} - \frac{1}{g(y)}\right|$$

$$= \left|\frac{g(y)-g(x)}{g(x)g(y)}\right|$$

$$= \frac{|g(x)-g(y)|}{|g(x)||g(y)|}$$

$$\le \frac{1}{M^2} |g(x)-g(y)|$$

NOW LET $\varepsilon > 0$ BE GIVEN. CHOOSE $\delta > 0$ S.T.

$$x, y \in I \text{ AND } |x-y| < \delta \Rightarrow |g(x)-g(y)| < M^2 \varepsilon .$$

THEN

$$x, y \in I \text{ AND } |x-y| < \delta \Rightarrow \left|\left(\frac{1}{g}\right)(x) - \left(\frac{1}{g}\right)(y)\right| \le \frac{1}{M^2} |g(x)-g(y)|$$

$$< \frac{1}{M^2}(M^2 \varepsilon) = \varepsilon .$$

13. I, J INTERVALS, $f: I \to \mathbb{R}$ UNIFORMLY CONTINUOUS ON I WITH $f(I) \subseteq J$ AND $g: J \to \mathbb{R}$ UNIFORMLY CONTINUOUS ON J.

CLAIM: $g \circ f : I \to \mathbb{R}$ UNIFORMLY CONTINUOUS ON I

LET $\varepsilon > 0$ BE GIVEN. CHOOSE $\delta_1 > 0$ S.T.

$$u, v \in J \text{ AND } |u - v| < \delta_1 \Rightarrow |g(u) - g(v)| < \varepsilon$$

NOW CHOOSE $\delta > 0$ S.T.

$$x, y \in I \text{ AND } |x - y| < \delta \Rightarrow |f(x) - f(y)| < \delta_1 .$$

THEN

$$\begin{aligned} x, y \in I \text{ AND } |x - y| < \delta &\Rightarrow f(x), f(y) \in J \text{ AND } |f(x) - f(y)| < \delta_1 \\ &\Rightarrow |g(f(x)) - g(f(y))| < \varepsilon \\ &\Rightarrow |(g \circ f)(x) - (g \circ f)(y)| < \varepsilon . \end{aligned}$$