CONTINUOUS FUNCTIONS

RECALL: IF $p(x)$ IS A POLYNOMIAL, THEN FOR ANY a,

$$\lim_{x \to a} p(x) = p(a)$$

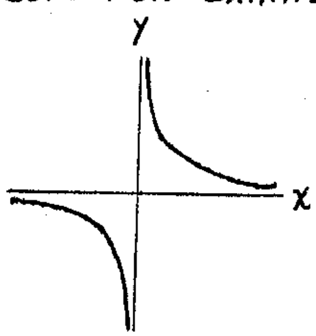
OR, IN A SOMEWHAT DIFFERENT FORM,

$$\lim_{x \to a} p(x) = p(\lim_{x \to a} x)$$

A GENERAL DEFINITION: A FUNCTION $f(x)$ IS SAID TO BE CONTINUOUS AT $x = a$ IF

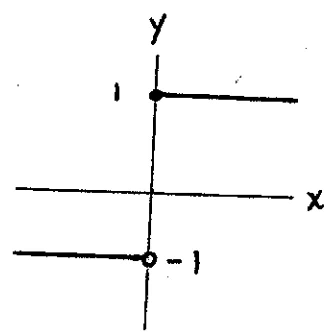
1. $f(x)$ IS DEFINED AT $x = a$ (I.E., $f(a)$ EXISTS)
2. $\lim_{x \to a} f(x)$ EXISTS.
3. $\lim_{x \to a} f(x) = f(a)$

SOME NON-EXAMPLES:



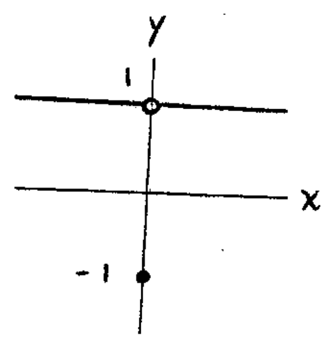
$f(x) = \dfrac{1}{x}$

$f(0)$ NOT DEFINED



$f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$

$f(0)$ DEFINED, BUT $\lim_{x \to 0} f(x)$ DNE



$f(x) = \begin{cases} 1, & x \neq 0 \\ -1, & x = 0 \end{cases}$

$f(0)$ DEFINED AND $\lim_{x \to 0} f(x)$ EXISTS, BUT THEY'RE NOT THE SAME

INTUITIVELY, $f(x)$ IS CONTINUOUS AT $x = a$ IF THE GRAPH OF $f(x)$ <u>DOES NOT</u> "BREAK" AT $x = a$.

IF $f(x)$ IS NOT CONTINUOUS AT $x = a$ (I.E., IF THE GRAPH OF $f(x)$ <u>DOES</u> BREAK AT $x = a$), THEN $x = a$ IS A <u>DISCONTINUITY</u> OF $f(x)$.

<u>NOTE</u> : IF $x = a$ IS AN ENDPOINT FOR THE DOMAIN OF $f(x)$, THEN " $\lim_{x\to a} f(x)$ " IN THE DEFINITION IS REPLACED BY THE APPROPRIATE ONE-SIDED LIMIT, E.G.,

$f(x) = \sqrt{x}$ IS DEFINED ON $[0,\infty)$

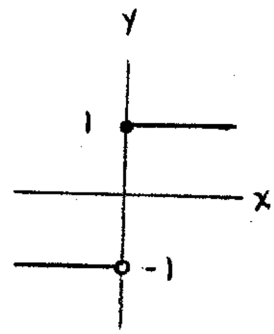
AND IS CONTINUOUS AT $x = 0$ BECAUSE

$$f(0) = 0$$

AND

$$\lim_{x\to 0^+} f(x) = 0.$$

A FUNCTION LIKE $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$



IS SAID TO BE <u>CONTINUOUS FROM THE RIGHT</u> AT $x = 0$ BECAUSE

$$f(0) = 1$$

AND

$$\lim_{x\to 0^+} f(x) = 1$$

A FUNCTION IS SAID TO BE CONTINUOUS IF IT IS CONTINUOUS AT $x = a$ FOR EVERY a IN ITS DOMAIN.

EXAMPLES : POLYNOMIALS ARE CONTINUOUS (EVERYWHERE)

RATIONAL FUNCTIONS ARE CONTINUOUS (WHERE THEIR DENOMINATORS ARE NOT ZERO)

SUMS OF CONTINUOUS FUNCTIONS ARE CONTINUOUS

PRODUCTS OF CONTINUOUS FUNCTIONS ARE CONTINUOUS

QUOTIENTS OF CONTINUOUS FUNCTIONS ARE CONTINUOUS (WHERE THEIR DENOMINATORS ARE NOT ZERO)

COMPOSITIONS OF CONTINUOUS FUNCTIONS ARE CONTINUOUS

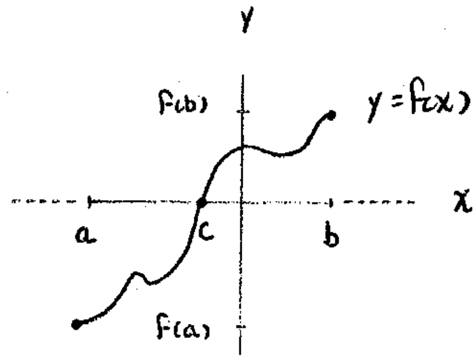
E.G., $f(x) = \left| \dfrac{(x+3)(x^3-2)}{x^2+4} \right|$ IS CONTINUOUS EVERYWHERE

BECAUSE $x+3$, x^3-2 AND x^2+4 ARE POLYNOMIALS, x^2+4 IS NEVER ZERO AND THE ABSOLUTE VALUE FUNCTION IS CONTINUOUS EVERYWHERE.

THINK OF CONTINUOUS FUNCTIONS $f(x)$ INTUITIVELY AS THOSE FOR WHICH

1. LIMITS ARE EASY ($\lim_{x \to a} f(x) = f(a)$)
2. LIMITS COME INSIDE THE FUNCTION ($\lim_{x \to a} f(x) = f(\lim_{x \to a} x)$)
3. THE GRAPHS DON'T BREAK.

INTERMEDIATE VALUE THEOREM (SPECIAL CASE):



- $f(x)$ CONTINUOUS ON $[a, b]$.
- $f(a)$ AND $f(b)$ HAVE OPPOSITE SIGN.
- THEN THERE IS AT LEAST ONE c IN (a, b) AT WHICH

$$f(c) = 0.$$

EXAMPLE: SOMEDAY WHEN YOU'VE GOT NOTHING BETTER TO DO TRY TO SOLVE THE EQUATION

$$x^3 + 3x - 2 = 0.$$

IT'S PRETTY HARD. DOES THE EQUATION EVEN HAVE SOLUTIONS?

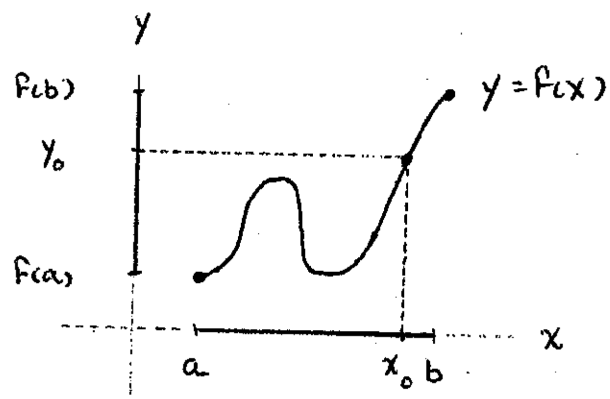
YES! IN FACT, THERE'S A SOLUTION IN $(0, 1)$. TO SEE THIS CONSIDER $f(x) = x^3 + 3x - 2$. POLYNOMIAL. CONTINUOUS EVERYWHERE AND, IN PARTICULAR, ON $[0, 1]$.

$$f(0) = -2 < 0$$

$$f(1) = 2 > 0$$

SO THE RESULT ABOVE SAYS THAT FOR AT LEAST ONE c IN $(0, 1)$, $f(c) = 0$, I.E., $c^3 + 3c - 2 = 0$.

INTERMEDIATE VALUE THEOREM (GENERAL CASE):



- $f(x)$ CONTINUOUS ON $[a,b]$.
- $f(a) \neq f(b)$.
- THEN, FOR EVERY y_0 BETWEEN $f(a)$ AND $f(b)$ THERE IS AT LEAST ONE x_0 IN (a,b) AT WHICH

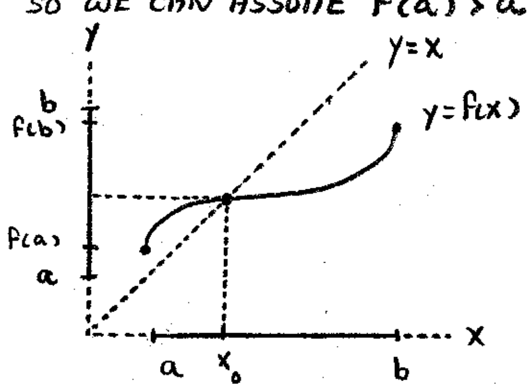
$$f(x_0) = y_0.$$

APPLICATION: THE 1-DIMENSIONAL BROUWER FIXED POINT THEOREM

SUPPOSE f IS A CONTINUOUS FUNCTION FROM $[a,b]$ TO $[a,b]$. WE WILL SHOW THAT THERE IS AT LEAST ONE x_0 IN $[a,b]$ FOR WHICH

$$f(x_0) = x_0 \qquad \text{(I.E., } x_0 \text{ IS A FIXED POINT OF } f\text{)}$$

IF $f(a) = a$ WE'RE DONE ($x_0 = a$) AND IF $f(b) = b$ WE'RE DONE ($x_0 = b$) SO WE CAN ASSUME $f(a) > a$ AND $f(b) < b$.



NOTICE THAT $g(x) = f(x) - x$ IS CONTINUOUS ON $[a,b]$,

$$g(a) = f(a) - a > 0$$

AND $g(b) = f(b) - b < 0$.

THUS, THERE IS AN x_0 IN (a,b) AT WHICH

$$g(x_0) = f(x_0) - x_0 = 0. \qquad \square$$