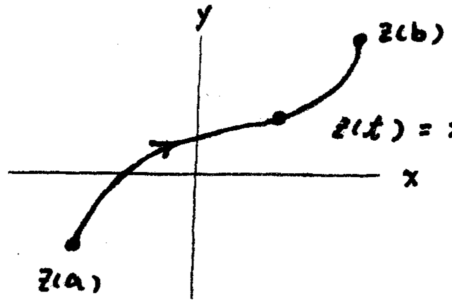
CONTOUR INTEGRALS :

LET

$$w = f(z) = u(x,y) + i\,v(x,y)$$

BE CONTINUOUS (NOT NECESSARILY ANALYTIC) ON A DOMAIN D.

SUCH FUNCTIONS ARE INTEGRATED OVER CERTAIN CURVES IN D CALLED " CONTOURS " WHICH WE NOW INTRODUCE.



$z(t) = x(t) + i\,y(t)$, $a \le t \le b$

$z(t)$ IS SMOOTH IF $z'(t) = x'(t) + i\,y'(t)$ IS CONTINUOUS AND NEVER ZERO.

E.G., $z(t) = \cos t + i \sin t = e^{it}$, $0 \le t \le 2\pi$

$$\begin{aligned} z'(t) &= -\sin t + i \cos t \quad \text{(NEVER ZERO)} \\ &= i(\cos t + i \sin t) \\ &= i e^{it} \end{aligned}$$

EXERCISE 59 : SHOW, MORE GENERALLY, THAT

$$z(t) = e^{z_0 t}$$

IMPLIES

$$z'(t) = z_0 e^{z_0 t}$$

FOR ANY COMPLEX NUMBER z_0.

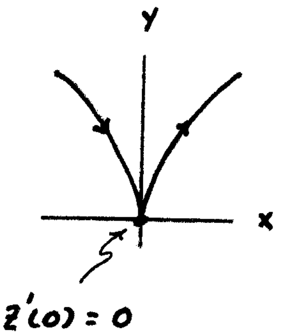
REASON FOR ASSUMING $z'(t)$ NONZERO, E.G.,

$$z(t) = t^3 + t^2 i \quad , \quad -\infty < t < \infty$$

$$z'(t) = 3t^2 + 2t i$$

$$z'(0) = 0$$

$$x = t^3 \; , \; y = t^2 \quad \Rightarrow \quad y = x^{\frac{2}{3}}$$



A <u>PIECEWISE</u> <u>SMOOTH</u> (<u>PWS</u>) <u>CURVE</u>, OR <u>CONTOUR</u>, IS ONE LIKE THIS THAT CAN BE SPLIT INTO A FINITE NUMBER OF SMOOTH PIECES.

$$z(t) = x(t) + i\,y(t) \; , \; a \le t \le b$$

IS <u>CLOSED</u> IF $z(a) = z(b)$. IT IS <u>SIMPLE</u> IF IT DOES NOT INTERSECT ITSELF ($t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$), EXCEPT AT ITS ENDPOINTS IF IT HAPPENS TO BE CLOSED.

IF $C : z(t) = x(t) + i y(t)$, $a \le t \le b$, IS A SMOOTH CURVE AND $f(z)$ IS CONTINUOUS ON SOME DOMAIN D CONTAINING C, THEN THE <u>CONTOUR</u> (OR <u>LINE</u>) <u>INTEGRAL</u> OF $f(z)$ OVER C IS DEFINED BY

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

NOTES: THE REASON FOR THE $z'(t)$ LIES IN THE CHAIN RULE. IT MAKES THE VALUE OF THE INTEGRAL INDEPENDENT OF THE CHOICE OF PARAMETRIZATION FOR C. ALL THAT IS REALLY NECESSARY FOR THE INTEGRAL TO BE DEFINED IS THAT $f(z(t))$ BE PIECEWISE CONTINUOUS ON $[a,b]$. IF C IS ONLY PIECEWISE SMOOTH, THEN THE INTEGRAL IS DEFINED BY INTEGRATING OVER EACH SMOOTH PIECE AND ADDING THE RESULTS.

IF C HAPPENS TO BE CLOSED THE INTEGRAL IS GENERALLY WRITTEN

$$\oint_C f(z)dz \,.$$

A CONTOUR INTEGRAL IS REALLY JUST TWO "ORDINARY" LINE INTEGRALS FROM CALCULUS:

$$\begin{aligned}
\int_C f(z)dz &= \int_a^b f(z(t))z'(t)dt \\
&= \int_a^b [u(x(t),y(t)) + i\,v(x(t),y(t))][x'(t)+y'(t)i]dt \\
&= \int_a^b \left(u(x(t),y(t))\frac{dx}{dt} - v(x(t),y(t))\frac{dy}{dt}\right)dt \\
&\qquad + i\int_a^b \left(u(x(t),y(t))\frac{dy}{dt} + v(x(t),y(t))\frac{dx}{dt}\right)dt \\
&= \int_C u\,dx - v\,dy \;+\; i\int_C u\,dy + v\,dx
\end{aligned}$$

SO ALL OF THE USUAL PROPERTIES OF LINE INTEGRALS ARE STILL VALID, E.G., IF $-C$ DENOTES THE CURVE C TRAVERSED IN THE OPPOSITE DIRECTION, THEN

$$\int_{-C} f(z)dz = -\int_C f(z)dz \,.$$

EXAMPLES :

1. $f(z) = z^2$

C = UNIT CIRCLE TRAVERSED ONCE, COUNTERCLOCKWISE

$z(t) = e^{it}, \quad 0 \le t \le 2\pi$

$z'(t) = i e^{it}$ (SEE PAGE 1)

$$\oint_C z^2 dz = \int_0^{2\pi} (z(t))^2 z'(t) dt = \int_0^{2\pi} (e^{it})^2 (i e^{it}) dt$$

$$= \int_0^{2\pi} i e^{3ti} dt$$

$$= \int_0^{2\pi} i (\cos 3t + i \sin 3t) dt$$

$$= \int_0^{2\pi} -\sin 3t + i \cos 3t \, dt$$

$$= \frac{1}{3} \cos 3t + \frac{1}{3} \sin 3t \, i \Big|_0^{2\pi} = 0$$

NOTE : $\frac{1}{3} \cos 3t + \frac{1}{3} \sin 3t \, i = \frac{1}{3} e^{3ti}$ SO OUR CALCULATION SHOWS

$$\int_0^{2\pi} i e^{3ti} dt = i \left[\frac{1}{3i} e^{3ti} \right]_0^{2\pi}$$

IT WILL SIMPLIFY MANY FUTURE CALCULATIONS TO NOTICE THAT, MORE GENERALLY, FOR ANY NONZERO z_0,

$$\boxed{\int_a^b e^{z_0 t} dt = \frac{1}{z_0} e^{z_0 t} \Big|_a^b}$$

THE PROOF IS NOT DIFFICULT, BUT INVOLVES THE SOMEWHAT NASTY INTEGRALS OF $e^{x_0 t} \cos y_0 t$ AND $e^{x_0 t} \sin y_0 t$ SO I WILL LEAVE IT FOR YOU TO THINK ABOUT IF YOU'RE INTERESTED.

FOR FUTURE REFERENCE :

$f(z) = z^2$ IS ANALYTIC ON AND INSIDE THE UNIT CIRCLE C AND $\oint_C f(z)dz = 0$. WE WILL SEE LATER THAT THERE IS SOMETHING MUCH MORE GENERAL GOING ON HERE.

CAUCHY - GOURSAT THEOREM : IF $f(z)$ IS ANALYTIC ON AND INSIDE A SIMPLE CLOSED CONTOUR C, THEN

$$\oint_C f(z)dz = 0.$$

2. $f(z) = \frac{1}{z}$

C = CIRCLE OF RADIUS 2 ABOUT $z=0$, TRAVERSED ONCE, COUNTERCLOCKWISE

$z(t) = 2e^{it}$, $0 \le t \le 2\pi$

$z'(t) = 2i\,e^{it}$

$$\oint_C \frac{1}{z}dz = \int_0^{2\pi} \frac{1}{z(t)} z'(t)dt = \int_0^{2\pi} \frac{1}{2e^{it}} (2ie^{it})dt$$

$$= \int_0^{2\pi} i\,dt = it\Big|_0^{2\pi} = 2\pi i$$

3. $f(z) = \frac{1}{z^2}$

C AS IN EXAMPLE 2 ABOVE.

$$\oint_C \frac{1}{z^2}dz = \int_0^{2\pi} \frac{1}{(z(t))^2} z'(t)dt = \int_0^{2\pi} \frac{1}{4e^{2ti}} (2ie^{it})dt$$

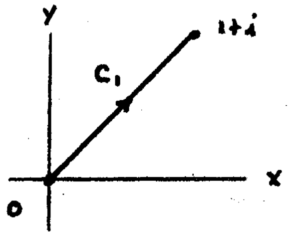
$$= \int_0^{2\pi} \frac{i}{2} e^{-it} dt = \frac{i}{2}\left[\frac{1}{-i} e^{-it} \right]_0^{2\pi}$$

$$= 0$$

4. $f(z) = Re(z)$

WE INTEGRATE OVER TWO CONTOURS, EACH FROM $z = 0$ TO $z = 1+i$

(a)



$C_1 : \; z(t) = t + ti \;, \; 0 \le t \le 1$

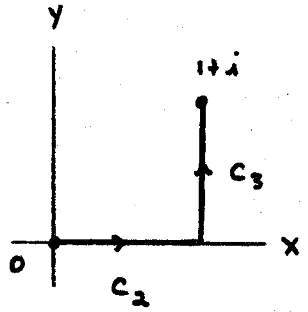
$z'(t) = 1 + i$

$$\int_{C_1} Re(z)dz = \int_0^1 Re(z(t))z'(t)dt$$

$$= \int_0^1 t\,(1+i)\,dt = (1+i)\tfrac{1}{2}t^2\Big|_0^1$$

$$= \tfrac{1}{2}(1+i)$$

(b)



$$C_2 + C_3 \; : \; z(t) = \begin{cases} t + 0i \,, & 0 \le t \le 1 \\ 1 + ti \,, & 0 \le t \le 1 \end{cases}$$

$$\int_{C_2+C_3} Re(z)dz = \int_{C_2} Re(z)dz + \int_{C_3} Re(z)dz$$

$$= \int_0^1 t(1)dt + \int_0^1 (1)(i)\,dt$$

$$= \tfrac{1}{2}t^2\Big|_0^1 + ti\,\Big|_0^1$$

$$= \tfrac{1}{2} + i$$

__MORAL__ : CONTOUR INTEGRALS ARE GENERALLY __PATH__ __DEPENDENT__ .

__EXERCISE 60__ : WHAT WILL THE CAUCHY-GOURSAT THEOREM SAY ABOUT THIS, HOWEVER, IF $f(z)$ IS ENTIRE ?

5. THIS EXAMPLE IS A BIT MORE SUBTLE. NOTICE FIRST THAT THE SYMBOL

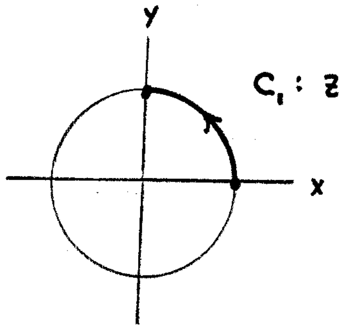
$$\int_C z^{\frac{1}{2}}\,dz$$

MAKES NO SENSE SINCE $z^{\frac{1}{2}}$ IS NOT A FUNCTION. ONE MUST CHOOSE A BRANCH.

IN ORDER TO HAVE A PROBLEM TO SOLVE LET'S CHOOSE THE PRINCIPAL BRANCH :

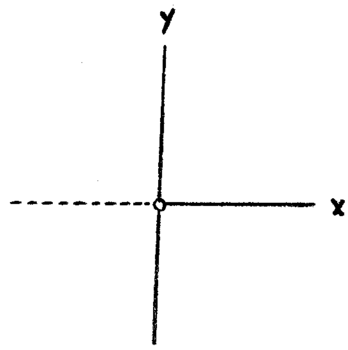
$$f(z) = \sqrt{r}\, e^{i\theta/2} \quad , \quad r > 0 \, , \; -\pi < \theta \leq \pi$$

THERE ARE NO SUBTLTIES INVOLVED IN INTEGRATING $f(z)$ OVER



$$C_1 : z(t) = e^{it} , \quad 0 \leq t \leq \frac{\pi}{2}$$

BECAUSE $f(z)$ IS CONTINUOUS ON A DOMAIN CONTAINING C_1, E.G.,



$$\int_{C_1} f(z)\,dz = \int_0^{\pi/2} f(z(t))\,z'(t)\,dt = \int_0^{\pi/2} e^{it/2}\,(i e^{it})\,dt$$

$$= i \int_0^{\pi/2} e^{\frac{3t}{2} i}\,dt = \frac{2}{3}\, e^{\frac{3t}{2} i} \Big|_0^{\pi/2} = \frac{2}{3}\left(e^{\frac{3\pi}{4} i} - 1\right)$$

$$= \frac{2}{3}\left[-\left(1 + \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}\, i \right]$$

SUPPOSE, HOWEVER, THAT WE WISH TO INTEGRATE $f(z)$ OVER THE ENTIRE UNIT CIRCLE

$$C_2 : z(t) = e^{it} , \quad -\pi \le t \le \pi .$$

$f(z)$ IS NOT CONTINUOUS ON ANY DOMAIN CONTAINING C_2. NEVERTHELESS, FOR $-\pi < t \le \pi$,

$$f(z(t)) = e^{it/2} = \cos\frac{t}{2} + i\sin\frac{t}{2}$$

AND $\cos\frac{t}{2} + i\sin\frac{t}{2}$ CERTAINLY IS CONTINUOUS ON $-\pi \le t \le \pi$.

> $f(z(t))$ IS NOT DEFINED AT $t = -\pi$, BUT EVERYWHERE ELSE ON $-\pi \le t \le \pi$ IT IS DEFINED AND AGREES WITH $\cos\frac{t}{2} + i\sin\frac{t}{2}$. SINCE A DEFINITE INTEGRAL IS UNAFFECTED BY THE VALUE OF ITS INTEGRAND AT ONE POINT WE MAY DEFINE $f(z(-\pi)) = \cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2}) = -i$ AND INTEGRATE THE RESULTING CONTINUOUS FUNCTION.

$$\int_{C_2} f(z)\,dz = \int_{-\pi}^{\pi} f(z(t))z'(t)\,dt = \int_{-\pi}^{\pi} e^{it/2}(ie^{it})\,dt$$

$$= i\int_{-\pi}^{\pi} e^{\frac{3t}{2}i}\,dt = \frac{2}{3}e^{\frac{3t}{2}i}\Big|_{-\pi}^{\pi}$$

$$= \frac{2}{3}\left[e^{\frac{3\pi}{2}i} - e^{-\frac{3\pi}{2}i}\right] = \frac{2}{3}[-i-i] = -\frac{4}{3}i$$

EXERCISES :

COMPUTE EACH OF THE FOLLOWING INTEGRALS OVER THE INDICATED CONTOURS.

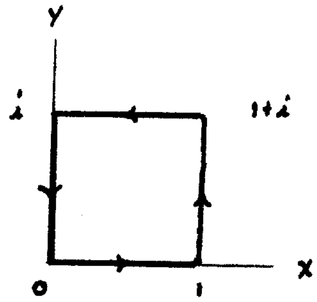
61. $\int_C \frac{z+2}{z}\,dz$, WHERE C IS THE TOP HALF OF THE CIRCLE OF RADIUS 2 ABOUT $z = 0$ FROM $z = 2$ TO $z = -2$.

ANS. $-4 + 2\pi i$

62. $\int_C e^z\,dz$, WHERE C IS THE STRAIGHT LINE SEGMENT FROM $z = \pi i$ TO $z = 1$.

ANS. $1 + e$

63. $\int_C (3z+1)\,dz$, WHERE C IS



ANS. 0

64. $\int_C 1\,dz$, WHERE C IS AN ARBITRARY CURVE FROM z_1 TO z_2.

ANS. $z_2 - z_1$

65. $\int_C z\,dz$, WHERE C IS AN ARBITRARY CURVE FROM z_1 TO z_2.

ANS. $\frac{1}{2}(z_2^2 - z_1^2)$

66. $\int_C \frac{1}{z-z_0}\,dz$, WHERE C IS THE CIRCLE $|z - z_0| = R$, TRAVERSED ONCE COUNTERCLOCKWISE

ANS. $2\pi i$

67. $\int_C f(z)\,dz$, WHERE $f(z)$ IS THE PRINCIPAL BRANCH OF z^{-1+i}

AND C IS THE UNIT CIRCLE TRAVERSED ONCE COUNTERCLOCKWISE.

ANS. $(2\sinh\pi)i$

ANTIDERIVATIVES :

LET $f(z)$ BE CONTINUOUS ON A DOMAIN D. IF THERE IS AN ANALYTIC FUNCTION $F(z)$ ON D FOR WHICH

$$F'(z) = f(z)$$

THEN $F(z)$ IS AN <u>ANTIDERIVATIVE</u> FOR $f(z)$.

> <u>NOTE</u> : ANY CONTINUOUS REAL FUNCTION $f(x)$ HAS AN ANTIDERIVATIVE, BUT THIS IS FAR FROM TRUE FOR COMPLEX FUNCTIONS. INDEED, WE WILL EVENTUALLY SEE THAT ANY $f(z)$ THAT HAS AN ANTIDERIVATIVE MUST ITSELF BE ANALYTIC.

LET $C : z(t) = x(t) + iy(t)$, $a \le t \le b$, BE AN ARBITRARY CURVE IN D AND DEFINE

$$w(t) = F(z(t))$$

WE WILL SHOW THAT

$$w'(t) = F'(z(t))\,z'(t)$$

PROOF : $w(t) = u(x(t), y(t)) + i v(x(t), y(t)) \Rightarrow$

$$w'(t) = \left(u_x \frac{dx}{dt} + u_y \frac{dy}{dt}\right) + i\left(v_x \frac{dx}{dt} + v_y \frac{dy}{dt}\right)$$

$$= (u_x x' - v_x y') + i(v_x x' + u_x y')$$

(CAUCHY-RIEMANN EQUATIONS)

AND

$$F'(z(t))\, z'(t) = (u_x(x(t), y(t)) + i v_x(x(t), y(t)))(x' + i y')$$

$$= (u_x x' - v_x y') + i(v_x x' + u_x y')$$

$$= w'(t)$$

THUS,

$$\int_C f(z)dz = \int_C F'(z)dz = \int_a^b F'(z(t))z'(t)dt$$

$$= \int_a^b w'(t)dt = w(t)\Big|_a^b = w(b) - w(a)$$

$$= F(z(b)) - F(z(a))$$

$$= F(z)\Big|_{z(a)}^{z(b)}$$

<u>CONCLUSION</u> : IF $f(z)$ IS CONTINUOUS ON A DOMAIN D AND HAS AN ANTIDERIVATIVE $F(z)$ ON D, THEN FOR <u>ANY</u> SMOOTH CURVE C FROM z_0 TO z_1 IN D,

$$\int_C f(z)dz = F(z)\Big|_{z_0}^{z_1}$$

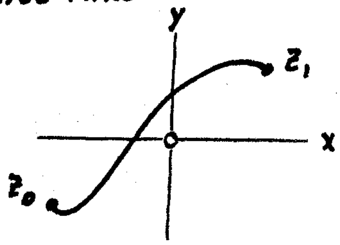
IN PARTICULAR, IF C IS CLOSED,

$$\oint_C f(z)dz = 0.$$

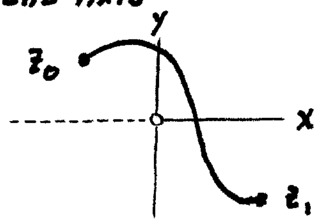
EXAMPLES :

1. $\oint_C (2z^2-5z)dz = 0$ FOR ANY CLOSED CURVE C.

2. $\int_C \frac{1}{z^2}\,dz = -\frac{1}{z}\Big|_{z_0}^{z_1}$ FOR ANY CURVE THAT DOES NOT PASS THROUGH THE ORIGIN



3. $\int_C \frac{1}{z}\,dz = \log z\Big|_{z_0}^{z_1}$ FOR ANY CURVE THAT DOES NOT PASS THROUGH THE NON-POSITIVE REAL AXIS



NOTE : IF C DOES PASS THROUGH THE NON-POSITIVE REAL AXIS, CHOOSE A DIFFERENT BRANCH OF $\log z$.

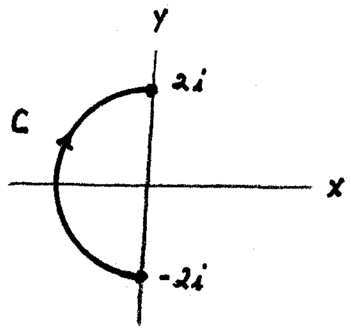
SINCE CONTOUR INTEGRALS OF FUNCTIONS WITH ANTIDERIVATIVES ARE INDEPENDENT OF PATH (I.E., DEPENDENT ONLY ON THE ENDPOINTS z_0 AND z_1) ONE OFTEN SEES THEM WRITTEN

$$\int_{z_0}^{z_1} f(z)\,dz \, .$$

EXERCISES :

68. COMPUTE $\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz$. ANS. $2 \cosh 1$

69. COMPUTE $\int_C \frac{1}{z}\, dz$, WHERE C IS THE LEFT HALF OF THE CIRCLE OF RADIUS 2 ABOUT $z=0$ FROM $-2i$ TO $2i$.



ANS. $-\pi i$

THERE ARE SOME REMARKABLE CONNECTIONS BETWEEN CONTOUR INTEGRATION AND SERIES EXPANSIONS THAT WE WILL LOOK INTO IN THE NEXT SECTION.

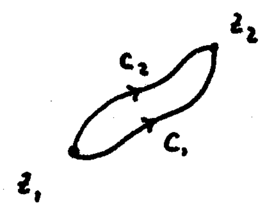
SOLUTIONS TO THE EXERCISES:

59. $z(t) = e^{z_0 t} = e^{x_0 t + y_0 t i} = e^{x_0 t}\cos(y_0 t) + i e^{x_0 t}\sin(y_0 t)$

$$z'(t) = [-y_0 e^{x_0 t}\sin(y_0 t) + x_0 e^{x_0 t}\cos(y_0 t)]$$
$$+ i[y_0 e^{x_0 t}\cos(y_0 t) + x_0 e^{x_0 t}\sin(y_0 t)]$$
$$= [x_0(e^{x_0 t}\cos(y_0 t)) - y_0(e^{x_0 t}\sin(y_0 t))]$$
$$+ i[x_0(e^{x_0 t}\sin(y_0 t)) + y_0(e^{x_0 t}\cos(y_0 t))]$$
$$= (x_0 + y_0 i)(e^{x_0}\cos(y_0 t) + i e^{x_0}\sin(y_0 t))$$
$$= z_0 e^{z_0 t}$$

60. THE CAUCHY-GOURSAT THEOREM (ONCE WE HAVE PROVED IT) WILL IMPLY THAT IF $f(z)$ IS ENTIRE, z_1 AND z_2 ARE ANY TWO POINTS IN $\mathbb{C}$, AND C_1 AND C_2 ARE ANY TWO CURVES FROM z_1 TO z_2, THEN

$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ BECAUSE



$C = C_1 + (-C_2)$ IS A CLOSED CURVE SO

$$\int_C f(z)dz = 0$$

$$\int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = 0$$

$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$$

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

61. $\displaystyle\int_C \frac{z+2}{z}\,dz$ $\qquad$ $C:\ z(t) = 2e^{it},\ 0 \le t \le \pi$

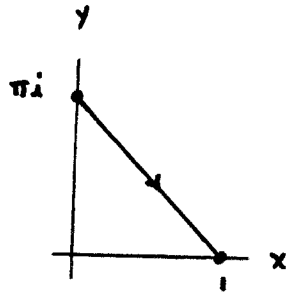
$z'(t) = 2ie^{it}$

$$= \int_0^{\pi} \frac{z(t)+2}{z(t)}\, z'(t)\,dt = \int_0^{\pi} \frac{2e^{it}+2}{2e^{it}}\,(2ie^{it})\,dt$$

$$= \int_0^{\pi} i(2e^{it}+2)\,dt = 2e^{it} + 2it \Big|_0^{\pi} = 2e^{i\pi} + 2\pi i - 2e^{0}$$

$$= -4 + 2\pi i$$

62. $\displaystyle\int_C e^{z}\,dz$ $\qquad$ $C:$



$y - \pi = -\pi(x-0)$

$y = -\pi x + \pi$

$z(t) = t - \pi(t-1)i,\ 0 \le t \le 1$

$z'(t) = 1 - \pi i$

$$= \int_0^1 e^{z(t)} z'(t)\,dt = \int_0^1 e^{t-\pi(t-1)i}(1-\pi i)\,dt$$

$$= \int_0^1 e^{(1-\pi i)t+\pi i}(1-\pi i)\,dt$$

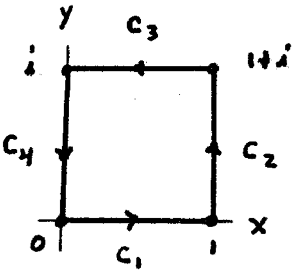
$$= e^{\pi i}\int_0^1 e^{(1-\pi i)t}(1-\pi i)\,dt$$

$$= (-1)\, e^{(1-\pi i)t}\Big|_0^1 = -\left[e^{1-\pi i} - e^{0}\right]$$

$$= -\left[e^{1}e^{-\pi i} - 1\right]$$

$$= 1+e$$

63. $\displaystyle\int_C (3z+1)\,dz$ $\qquad$ C :



$c_1 : \; z_1(t) = t \;, \quad 0 \le t \le 1$

$c_2 : \; z_2(t) = 1+ti \;, \quad 0 \le t \le 1$

$c_3 : \; z_3(t) = (1-t)+i \;, \quad 0 \le t \le 1$

$c_4 : \; z_4(t) = (1-t)i \;, \quad 0 \le t \le 1$

$$= \int_0^1 (3t+1)(1)\,dt + \int_0^1 (3(1+ti)+1)(i)\,dt + \int_0^1 (3((1-t)+i)+1)(-1)\,dt$$

$$+ \int_0^1 (3(1-t)i+1)(-i)\,dt$$

$$= \frac{3}{2}t^2\Big|_0^1 + t\Big|_0^1 + \int_0^1 (4i-3t)\,dt + \int_0^1 (-4+3t-3i)\,dt$$

$$+ \int_0^1 (3-3t-i)\,dt$$

$$= \frac{3}{2} + 1 + 4ti\Big|_0^1 - \frac{3}{2}t^2\Big|_0^1 - 4t\Big|_0^1 + \frac{3}{2}t^2\Big|_0^1 - 3ti\Big|_0^1$$

$$+ 3t\Big|_0^1 - \frac{3}{2}t^2\Big|_0^1 - it\Big|_0^1$$

$$= \frac{3}{2} + 1 + 4i - \frac{3}{2} - 4 + \frac{3}{2} - 3i + 3 - \frac{3}{2} - i$$

$= 0$ $\quad$ (AS THE CAUCHY-GOURSAT THEOREM SAYS IT SHOULD BE)

64. $\displaystyle\int_C 1\,dz$ $\qquad$ $C : z(t)\,,\; a \le t \le b$

$z(a) = z_1 \;,\; z(b) = z_2$

$$= \int_a^b 1\, z'(t)\,dt = \int_a^b z'(t)\,dt = z(t)\Big|_a^b = z(b) - z(a) = z_2 - z_1$$

65. $\int_C z\,dz$ $\qquad C: z(t),\ a \le t \le b$

$z(a) = z_1 \quad , \quad z(b) = z_2$

$$= \int_a^b z(t) z'(t)\,dt = \int_a^b [\tfrac{1}{2}(z(t))^2]'\,dt$$

$$= \tfrac{1}{2}(z(t))^2 \Big|_a^b = \tfrac{1}{2}\left((z(b))^2 - (z(a))^2\right)$$

$$= \tfrac{1}{2}(z_2^2 - z_1^2)$$

66. $\int_C \frac{1}{z - z_0}\,dz$ $\qquad C: z(t) = z_0 + Re^{it} \quad , \quad 0 \le t \le 2\pi$

$z'(t) = Rie^{it}$

$$= \int_0^{2\pi} \frac{1}{z(t) - z_0} z'(t) = \int_0^{2\pi} \frac{1}{Re^{it}} (Rie^{it})\,dt$$

$$= i \int_0^{2\pi} dt = it \Big|_0^{2\pi} = 2\pi i$$

67. $\int_C f(z)\,dz$ $\qquad f(z) =$ PRINCIPAL BRANCH OF z^{-1+i}

$= e^{(-1+i)\log z}$

$$C: z(t) = e^{it} \quad , \quad -\pi < t \le \pi$$

$$f(z(t)) = e^{(-1+i)\log(e^{it})} = e^{(-1+i)(it)} = e^{-t-ti}$$

$$= e^{-t}e^{-ti} = e^{-t}\cos t - ie^{-t}\sin t$$

<u>NOTE</u> : CONTINUOUS ON $[-\pi, \pi]$

THUS,

$$\int_C f(z)\,dz = \int_{-\pi}^{\pi} f(z(t))\,z'(t)\,dt$$

$$= \int_{-\pi}^{\pi} e^{-t}e^{-ti}(ie^{ti})\,dt = i\int_{-\pi}^{\pi} e^{-t}\,dt = -ie^{-t}\Big|_{-\pi}^{\pi}$$

$$= -i\,(e^{-\pi} - e^{-(-\pi)}) = (e^{\pi} - e^{-\pi})\,i$$

$$= (2\sinh\pi)\,i$$

68. $$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)dz = 2\sin\left(\frac{z}{2}\right)\Big|_0^{\pi+2i} = 2\sin\left(\frac{\pi}{2}+i\right)$$

$$= 2\left(\sin\frac{\pi}{2}\cosh 1 + i\cos\frac{\pi}{2}\sinh 1\right)$$

$$= 2\cosh 1$$

69. $$\int_C \frac{1}{z}\,dz \qquad C:\ z(t) = 2e^{(2\pi-t)i},\quad \frac{\pi}{2} \le t \le \frac{3\pi}{2}$$

THE BRANCH

$$\log z = \ln r + i\theta \quad,\ r>0,\ 0<\theta<2\pi$$

OF THE LOGARITHM IS ANALYTIC ON $\mathbb{C}$ MINUS THE NON-NEGATIVE REAL AXIS (WHICH CONTAINS C) AND ITS DERIVATIVE THERE IS $\frac{1}{z}$ SO

$$\int_C \frac{1}{z}\,dz = \log z\Big|_{-2i}^{2i} = \log z\Big|_{2e^{\frac{3\pi}{2}i}}^{2e^{\frac{\pi}{2}i}}$$

$$= \left(\ln 2 + \frac{\pi}{2}i\right) - \left(\ln 2 + \frac{3\pi}{2}i\right)$$

$$= -\pi i$$