CONVERGENCE TESTS

BASIC QUESTION FOR NEXT THREE SECTIONS :

GIVEN AN INFINITE SERIES $\sum_{k=0}^{\infty} a_k$, HOW DO YOU DECIDE WHETHER OR NOT IT CONVERGES ?

FOR THE TIME BEING WE WILL NOT WORRY SO MUCH ABOUT WHAT IT CONVERGES TO (IN THE CASE IN WHICH IT ACTUALLY DOES CONVERGE).

FIRST STEP : BE ABLE TO RECOGNIZE A SERIES THAT HAS NO CHANCE AT ALL OF CONVERGING.

E.G., $\sum_{k=1}^{\infty} \frac{k}{k+1}$ CAN'T POSSIBLY CONVERGE BECAUSE $\frac{k}{k+1} \to 1$ SO IT'S LIKE TRYING TO ADD UP MORE AND MORE 1s. IT HAS TO BLOW UP.

MORE PRECISELY : IF $\sum_{k=0}^{\infty} a_k$ CONVERGES, THEN $\lim_{k\to\infty} a_k = 0$

(I.E., IF $\lim_{k\to\infty} a_k \neq 0$, THEN $\sum_{k=0}^{\infty} a_k$ CANNOT CONVERGE).

HERE'S THE REASON :

$$S_k = a_0 + a_1 + \dots + a_{k-1} + a_k$$
$$= S_{k-1} + a_k$$

SO

$$a_k = S_k - S_{k-1}$$

IF $\sum_{k=0}^{\infty} a_k = S$, THEN

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} (S_k - S_{k-1}) = S - S = 0.$$

<u>EXAMPLES</u> :

1. $\sum_{k=1}^{\infty} k \sin\left(\frac{1}{k}\right)$ DIVERGES BECAUSE $\lim_{k\to\infty} k \sin\left(\frac{1}{k}\right) = \lim_{k\to\infty} \frac{\sin\left(\frac{1}{k}\right)}{\frac{1}{k}} = 1$

2. $\sum_{k=0}^{\infty} \cos k\pi$

   $= \sum_{k=0}^{\infty} (-1)^k$ DIVERGES BECAUSE $\lim_{k\to\infty} (-1)^k$ DOES NOT EXIST

   (SO, IN PARTICULAR, IT IS NOT ZERO)

<u>NOTE</u> : IT IS <u>NOT</u> TRUE THAT IF $\lim_{k\to\infty} a_k = 0$, THEN $\sum_{k=0}^{\infty} a_k$ MUST CONVERGE, E.G., WE WILL SEE SOON THAT $\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES.

RECALL THAT A SEQUENCE THAT IS INCREASING AND BOUNDED FROM ABOVE MUST CONVERGE.

NOW NOTICE THAT, FOR AN INFINITE SERIES $\sum_{k=0}^{\infty} a_k$ OF <u>NON-NEGATIVE TERMS</u> ($a_k \geq 0$), THE CORRESPONDING SEQUENCE OF PARTIAL SUMS

$$
\begin{aligned}
S_0 &= a_0 \\
S_1 &= a_0 + a_1 \\
S_2 &= a_0 + a_1 + a_2 \\
&\vdots
\end{aligned}
$$

IS CLEARLY INCREASING. THUS,

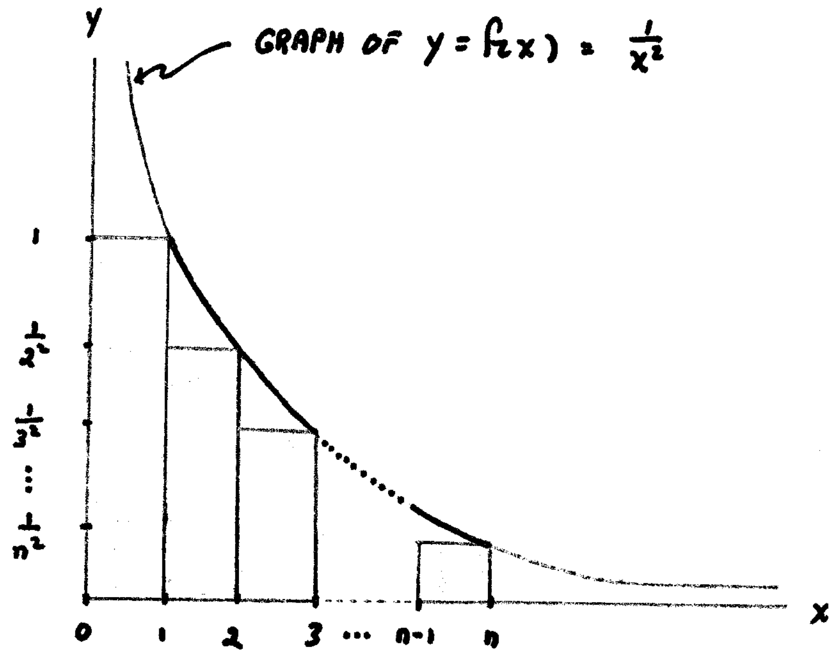
> A SERIES $\sum_{k=0}^{\infty} a_k$ OF NON-NEGATIVE TERMS FOR WHICH THE PARTIAL SUMS
>
> $$S_n = a_0 + a_1 + \cdots + a_n$$
>
> ARE ALL BOUNDED FROM ABOVE BY SOME CONSTANT M ($S_n \leq M$ FOR ALL n) MUST CONVERGE.

FINDING SUCH AN UPPER BOUND M IS OFTEN DIFFICULT, BUT WE WILL EVENTUALLY SEE SEVERAL DIFFERENT METHODS OF PRODUCING IT.

HERE'S AN EXAMPLE : $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$

$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

NOW LOOK AT THE FOLLOWING PICTURE :



S_n IS THE SUM OF THESE RECTANGULAR AREAS SO

$$S_n < 1 + \int_1^n \frac{1}{x^2}\,dx$$

$$< 1 + \int_1^{\infty} \frac{1}{x^2}\,dx = 1 + 1 = 2$$

[<u>RECALL</u> : $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{\ell\to\infty} \int_1^{\ell} x^{-2}\,dx = \lim_{\ell\to\infty} -\frac{1}{x}\Big|_1^{\ell} = \lim_{\ell\to\infty} \left(-\frac{1}{\ell} + 1\right) = 1$]

THUS, $S_n < 2$ FOR EVERY n AND WE CONCLUDE THAT $\sum_{k=1}^{\infty} \frac{1}{k^2}$

MUST CONVERGE TO SOMETHING ≤ 2.

<u>NOTE</u> : FINDING OUT WHAT THE SERIES CONVERGES TO THIS TIME IS QUITE A PROJECT. IT CAN BE SHOWN (BUT WE WILL NOT SHOW) THAT

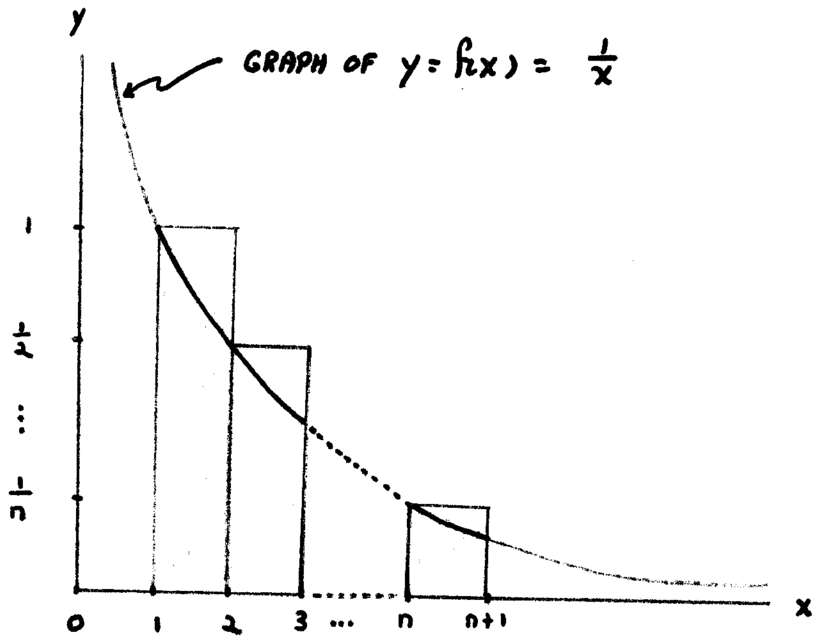
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

(EXACTLY WHAT YOU EXPECTED, RIGHT ?)

THE SAME TECHNIQUE CAN BE USED TO GO THE OTHER WAY AND SHOW THAT A SERIES OF POSITIVE TERMS MUST DIVERGE, E.G.,

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

(CALLED THE <u>HARMONIC</u> <u>SERIES</u>)



$$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

$$> \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$$

BUT $\ln(n+1) \longrightarrow \infty$ AS $n \to \infty$ AND THEREFORE SO DOES S_n, I.E.,

THE HARMONIC SERIES $\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES.

MORAL TO THE STORY HERE IS THAT

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ CONVERGES BECAUSE $\int_1^{\infty} \frac{1}{x^2}\,dx$ CONVERGES

AND

$\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES BECAUSE $\int_1^{\infty} \frac{1}{x}\,dx$ DIVERGES.

<u>THEOREM</u>: LET $\sum_{k=1}^{\infty} a_k$ BE A SERIES OF POSITIVE TERMS AND LET $f(x)$ BE THE FUNCTION FOR WHICH $f(k) = a_k$. IF $f(x)$ IS DECREASING AND CONTINUOUS ON $[1, \infty)$, THEN

$$\sum_{k=1}^{\infty} a_k \quad \text{AND} \quad \int_1^{\infty} f(x)\,dx$$

EITHER BOTH CONVERGE OR BOTH DIVERGE.

NOTES :

1. THIS IS CALLED THE INTEGRAL TEST.

2. THE PROOF IS EXACTLY THE SAME AS OUR EXAMPLES $\sum_{k=1}^{\infty} \frac{1}{k^2}$ AND $\sum_{k=1}^{\infty} \frac{1}{k}$.

3. THE " 1 " CAN BE REPLACED BY ANY POSITIVE INTEGER.

4. THE THEOREM DOES NOT SAY THAT $\sum_{k=1}^{\infty} a_k$ AND $\int_1^{\infty} f(x)\,dx$ ARE THE SAME (WHEN THEY CONVERGE), E.G., $\sum_{k=1}^{\infty} \frac{1}{k^2}$ IS $\frac{\pi^2}{6}$, BUT $\int_1^{\infty} \frac{1}{x^2}\,dx$ IS 1.

EXAMPLES : LET p BE SOME FIXED POSITIVE NUMBER. THEN

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

IS CALLED THE p-SERIES, E.G.,

$$p = 1 \quad : \quad \sum_{k=1}^{\infty} \frac{1}{k}$$

$$p = 2 \quad : \quad \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$p = \tfrac{1}{2} \quad : \quad \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

WE CLAIM THAT

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ CONVERGES IF AND ONLY IF } p > 1$$

TO SEE THIS LET $f(x) = \frac{1}{x^p}$. THIS IS DECREASING AND CONTINUOUS ON $[1, \infty)$. WE COMPUTE $\int_1^{\infty} \frac{1}{x^p}\,dx$.

SINCE WE ALREADY KNOW THAT THE INTEGRAL DIVERGES WHEN $p = 1$ WE WILL ASSUME $p \neq 1$:

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{\ell \to \infty} \int_1^{\ell} x^{-p}\,dx = \lim_{\ell \to \infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^{\ell}$$

$$= \lim_{\ell \to \infty} \left(\frac{\ell^{-p+1}}{-p+1} - \frac{1}{-p+1} \right)$$

AND THIS CAN COVERGE ONLY IF $-p+1 < 0$, I.E., IF $p > 1$.

E.G., $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ DIVERGES $(p = \frac{1}{2})$, BUT $\sum_{k=1}^{\infty} \frac{1}{k^4}$ CONVERGES $(p = 4)$.

IF YOU REALLY MUST KNOW,

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

MORE EXAMPLES :

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+9k^2}$: $f(x) = \dfrac{1}{1+9x^2}$ IS DECREASING AND CONTINUOUS ON $[1,\infty)$ AND

$$\int_1^{\infty} \frac{1}{1+9x^2}\,dx = \lim_{\ell\to\infty} \int_1^{\ell} \frac{1}{1+9x^2}\,dx$$

$$= \lim_{\ell\to\infty} \frac{1}{3} \int_1^{\ell} \frac{1}{1+(3x)^2}\,(3\,dx)$$

$$= \frac{1}{3} \lim_{\ell\to\infty} \text{ARCTAN}(3x)\Big|_1^{\ell}$$

$$= \frac{1}{3} \lim_{\ell\to\infty} [\,\text{ARCTAN}(3\ell) - \text{ARTAN}\,3\,]$$

$$= \frac{1}{3} [\,\frac{\pi}{2} - \text{ARCTAN}\,3\,]$$

WHICH IS FINITE SO $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+9k^2}$ CONVERGES.

2. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln k}$: $f(x) = \dfrac{1}{x\ln x}$ IS DECREASING AND CONTINUOUS ON $[2,\infty)$ AND

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{\ell\to\infty} \int_2^{\ell} \frac{1}{x\ln x}\,dx \qquad (u = \ln x,\ \text{ETC.})$$

$$= \lim_{\ell\to\infty} \ln(\ln x)\Big|_2^{\ell} = \lim_{\ell\to\infty} (\ln(\ln\ell) - \ln(\ln 2))$$

WHICH DIVERGES AND THEREFORE SO DOES $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln k}$.

WE CONCLUDE THIS SECTION WITH A FEW GENERAL REMARKS.

1. THE CONVERGENCE OR DIVERGENCE OF A SERIES DOES NOT DEPEND ON THE FIRST FEW TERMS, E.G.,

$$\sum_{k=1}^{\infty} a_k \quad \text{CONVERGES IF AND ONLY IF} \quad \sum_{k=213}^{\infty} a_k \quad \text{CONVERGES}$$

(THEY DON'T CONVERGE TO THE SAME THING, OF COURSE).

2. IF $\sum_{k=1}^{\infty} a_k$ CONVERGES, THEN SO DOES $\sum_{k=1}^{\infty} c\,a_k$ FOR ANY CONSTANT c AND

$$\sum_{k=1}^{\infty} c\,a_k = c \sum_{k=1}^{\infty} a_k \,,$$

E.G., $\sum_{k=1}^{\infty} \frac{7}{k^2}$ CONVERGES TO $7 \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{7\pi^2}{6}$.

3. IF $\sum_{k=1}^{\infty} a_k$ AND $\sum_{k=1}^{\infty} b_k$ BOTH CONVERGE, THEN SO DO $\sum_{k=1}^{\infty} (a_k + b_k)$ AND $\sum_{k=1}^{\infty} (a_k - b_k)$ AND

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

$$\sum_{k=1}^{\infty} (a_k - b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$$

FOR EXAMPLE,

$$\sum_{k=0}^{\infty}\left(\frac{3}{4^k}-\frac{7}{5^k}\right)$$

SINCE $\displaystyle\sum_{k=0}^{\infty}\frac{3}{4^k} = \sum_{k=0}^{\infty}3\left(\frac{1}{4}\right)^k = \frac{3}{1-\frac{1}{4}} = 4$ AND

$\displaystyle\sum_{k=0}^{\infty}\frac{7}{5^k} = \sum_{k=0}^{\infty}7\left(\frac{1}{5}\right)^k = \frac{7}{1-\frac{1}{5}} = \frac{35}{4}$ BOTH CONVERGE,

$$\sum_{k=0}^{\infty}\left(\frac{3}{4^k}-\frac{7}{5^k}\right) = 4-\frac{35}{4} = -\frac{19}{4}$$