CONVERGENCE OF TAYLOR SERIES

GIVEN $f(x)$ AND x_0 CONSTRUCT THE TAYLOR SERIES FOR $f(x)$ ABOUT x_0 :

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{n!}(x-x_0)^n$$

QUESTIONS :

1. FOR WHICH VALUES OF x DOES THE SERIES CONVERGE, I.E., WHAT IS THE INTERVAL OF CONVERGENCE ?

   RATIO TEST FOR ABSOLUTE CONVERGENCE
   THEN TEST THE ENDPOINTS SEPARATELY.

2. FOR WHICH VALUES OF x IN THE INTERVAL OF CONVERGENCE DOES THE SERIES CONVERGE TO $f(x)$?

   FIND THE VALUES OF x FOR WHICH
   $\lim_{n\to\infty} |R_n(x)| = 0.$

3. FOR THOSE x FOR WHICH THE SERIES DOES CONVERGE TO $f(x)$, WHAT SIZE PARTIAL SUM SHOULD BE USED TO APPROXIMATE $f(x)$ TO SOME DESIRED ACCURACY ?

   FIND n FOR WHICH $|R_n(x)|$ IS SUFFICIENTLY SMALL.

EXAMPLES :

1. $f(x) = \sin x$

   $x_0 = 0$

WE HAVE ALREADY COMPUTED THE MACLAURIN SERIES :

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

INTERVAL OF CONVERGENCE :

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{(-1)^k x^{2k+1}} \right| =$$

$$\lim_{k\to\infty} \left| \frac{(-1)\, x^2}{(2k+3)(2k+2)} \right| = x^2 \lim_{k\to\infty} \frac{1}{(2k+3)(2k+2)} = 0 \text{ FOR ALL } x$$

SO THE SERIES CONVERGES FOR ALL x. INTERVAL OF CONVERGENCE IS

$$(-\infty, \infty).$$

FOR ANY x

$$0 \le |R_n(x)| \le \frac{M}{(n+1)!} |x|^{n+1} \le \frac{1}{(n+1)!} |x|^{n+1}$$

SINCE $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ FOR ALL x, THE SERIES CONVERGES TO $\sin x$ FOR ALL x. THUS, WE ARE JUSTIFIED IN WRITING

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

NOW LET'S SEE WHAT SIZE PARTIAL SUM IS REQUIRED TO APPROXIMATE SIN 3° TO FIVE DECIMAL PLACE ACCURACY, I.E., WITH AN ERROR < 0.000005.

$$|R_n(x)| \le \frac{M}{(n+1)!} |x|^{n+1} \le \frac{1}{(n+1)!} |x|^{n+1}$$

SO WE WANT n SO THAT

$$\frac{1}{(n+1)!} \left(\frac{\pi}{60}\right)^{n+1} < 0.000005$$

<u>NOTE</u>: WE <u>MUST</u> WORK WITH RADIANS (WHY?)

SINCE $\pi < 4$, $\frac{\pi}{60} < \frac{4}{60} = \frac{1}{15}$ SO IT WILL SUFFICE TO FIND n SO THAT

$$\frac{1}{(n+1)!\, 15^{n+1}} < 0.000005$$

$$(n+1)!\, 15^{n+1} > 200{,}000$$

$n = 2$: $\quad 3!\, 15^3 = 20{,}250$

$n = 3$: $\quad 4!\, 15^4 = 1{,}215{,}000$ SO THIS WILL DO

$P_3(x) = x - \frac{1}{3!} x^3$ SO

$$\sin\left(\frac{\pi}{60}\right) \approx \frac{\pi}{60} - \frac{1}{6}\left(\frac{\pi}{60}\right)^3 \approx 0.05234$$

2. $f(x) = \ln(1+x)$

$x_0 = 0$

$$f^{(0)}(x) = \ln(1+x) \qquad f^{(0)}(0) = 0$$

$$f^{(1)}(x) = (1+x)^{-1} \qquad f^{(1)}(0) = 1$$

$$f^{(2)}(x) = (-1)(1+x)^{-2} \qquad f^{(2)}(0) = -1$$

$$f^{(3)}(x) = (-1)(-2)(1+x)^{-3} \qquad f^{(3)}(0) = 2!$$

$$f^{(4)}(x) = (-1)(-2)(-3)(1+x)^{-4} \qquad f^{(4)}(0) = -3!$$

$$\vdots \qquad \vdots$$

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!(1+x)^{-k} \quad \text{FOR } k \geq 1 \qquad f^{(k)}(0) = (-1)^{k-1}(k-1)! \quad \text{FOR } k \geq 1$$

MACLAURIN SERIES

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k!} x^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$$

INTERVAL OF CONVERGENCE :

$$\lim_{k\to\infty} \left| \frac{(-1)^k x^{k+1}}{k+1} \cdot \frac{k}{(-1)^{k-1} x^k} \right| = \lim_{k\to\infty} \left| \frac{(-1)\, x \cdot k}{k+1} \right|$$

$$= |x| \lim_{k\to\infty} \frac{k}{k+1}$$

$$= |x|$$

SO THE SERIES CONVERGES ABSOLUTELY WHEN

$$|x| < 1$$

$$-1 < x < 1$$

ENDPOINTS:

$$x = -1 \quad : \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (-1)^k = -\sum_{k=1}^{\infty} \frac{1}{k}$$

WHICH DIVERGES $(p = 1)$

$$x = 1 \quad : \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \cdot 1^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

WHICH CONVERGES (ALTERNATING SERIES TEST)

SERIES CONVERGES FOR

$$-1 < x \leq 1$$

FOR THE REMAINDER, NOTE THAT THE PATTERN IN THE DERIVATIVES COMPUTED ABOVE SHOWS THAT

$$f^{(n+1)}(x) = \frac{(-1)^n \, n!}{(1+x)^{n+1}} \qquad \text{FOR } n \geq 0$$

$$\left| f^{(n+1)}(x) \right| = \frac{n!}{(1+x)^{n+1}} \qquad \text{FOR } n \geq 0$$

AND THIS IS <u>NOT</u> <u>BOUNDED</u> ON $-1 < x \leq 1$ (IT BLOWS UP AS $x \to -1$). ON $0 \leq x \leq 1$ IT IS BOUNDED FROM ABOVE BY

$$M = n!$$

THUS, ON $[0,1]$,

$$|R_n(x)| \le \frac{M}{(n+1)!}|x|^{n+1} = \frac{n!}{(n+1)!}x^{n+1}$$

$$|R_n(x)| \le \frac{x^{n+1}}{n+1}$$

SINCE $\frac{x^{n+1}}{n+1} \longrightarrow 0$ FOR EACH x IN $[0,1]$ THE SERIES CONVERGES TO $\ln(1+x)$ FOR EACH SUCH x :

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} x^k \quad , \quad 0 \le x \le 1$$

IN PARTICULAR,

$$\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

HERE THE CONVERGENCE IS QUITE SLOW, HOWEVER. TO OBTAIN FIVE DECIMAL PLACE ACCURACY IN AN APPROXIMATION TO $\ln 2$ WE WOULD NEED A PARTIAL SUM OF SIZE n WHERE

$$|R_n(1)| < 0.000005$$

SO WE TAKE

$$\frac{1}{n+1} < 0.000005$$

$$n+1 > 200{,}000$$

$$n > 199{,}999$$

WHICH IS NOT SO HOT.