COUNTABLE AND UNCOUNTABLE SETS

LET A AND B BE TWO NONEMPTY SETS. A FUNCTION

$$\varphi : A \to B$$

FROM A TO B IS SAID TO BE <u>INJECTIVE</u> (<u>ONE-TO-ONE</u>) IF $a_1, a_2 \in A$ AND $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ (DISTINCT POINTS IN A MAP TO DISTINCT POINTS IN B); EQUIVALENTLY, φ IS INJECTIVE IF AND ONLY IF

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2 .$$

IN THIS CASE, φ IS CALLED AN <u>INJECTION</u>.

THE MAP $\varphi : A \to B$ IS <u>SURJECTIVE</u> (<u>ONTO</u>) IF, FOR EVERY $b \in B$, THERE IS AN $a \in A$ FOR WHICH $\varphi(a) = b$ (EVERYTHING IN B IS THE IMAGE OF SOMETHING IN A). φ IS THEN SAID TO BE A <u>SURJECTION</u>.

$\varphi : A \to B$ IS <u>BIJECTIVE</u> (OR A <u>BIJECTION</u>) IF IT IS BOTH INJECTIVE AND SURJECTIVE.

TWO SETS A AND B ARE SAID TO HAVE THE <u>SAME CARDINALITY</u> (WRITTEN $A \sim B$) IF THERE IS A BIJECTIVE MAP φ OF A ONTO B. A IS <u>FINITE</u> IF IT IS EITHER EMPTY OR HAS THE SAME CARDINALITY AS $\{1, \dots, n\}$ FOR SOME $n \in \mathbb{N}$ (THEN n IS CALLED THE <u>CARDINALITY</u> OF A AND WE WRITE $|A| = n$; IF $A = \emptyset$ WE DEFINE $|\emptyset| = 0$). IF A IS NOT FINITE IT IS SAID TO BE <u>INFINITE</u>.

EXAMPLES :

1. DENOTE BY $2\mathbb{N}$ THE SET OF ALL EVEN NATURAL NUMBERS, I.E.,

$$2\mathbb{N} = \{2n : n \in \mathbb{N}\}.$$

DEFINE $\varphi : \mathbb{N} \to 2\mathbb{N}$ BY

$$\varphi(n) = 2n.$$

THEN φ IS INJECTIVE SINCE

$$\varphi(n_1) = \varphi(n_2) \Rightarrow 2n_1 = 2n_2 \Rightarrow n_1 = n_2$$

AND SURJECTIVE SINCE ANY $2k \in 2\mathbb{N}$ IS THE IMAGE OF k UNDER φ, I.E.,

$$\varphi(k) = 2k.$$

THUS, $\mathbb{N}$ AND $2\mathbb{N}$ HAVE THE SAME CARDINALITY

$$2\mathbb{N} \sim \mathbb{N}$$

(EVEN THOUGH $2\mathbb{N}$ IS A PROPER SUBSET OF $\mathbb{N}$).

2. WE SHOW THAT $\mathbb{N}$ AND $\mathbb{Z}$ HAVE THE SAME CARDINALITY. WE DEFINE A MAP $\varphi : \mathbb{N} \to \mathbb{Z}$ BASED ON THE FOLLOWING DIAGRAM :

$$\begin{array}{ccccccc}
\mathbb{N} : & 1 & 2 & 3 & 4 & 5 & \cdots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\mathbb{Z} : & 0 & -1 & 1 & -2 & 2 & \cdots
\end{array}$$

SPECIFICALLY, WE DEFINE

$$\varphi(n) = \begin{cases} -\dfrac{n}{2}, & \text{IF } n \text{ IS EVEN} \\[2mm] \dfrac{n-1}{2}, & \text{IF } n \text{ IS ODD} \end{cases}$$

WE SHOW THAT φ IS A BIJECTION.

INJECTIVE : SUPPOSE $\varphi(n_1) = \varphi(n_2)$. IF THIS IS NEGATIVE, THEN n_1 AND n_2 MUST BE EVEN AND

$$-\frac{n_1}{2} = -\frac{n_2}{2}$$

SO

$$n_1 = n_2 .$$

IF $\varphi(n_1) = \varphi(n_2)$ IS NON-NEGATIVE, THEN n_1 AND n_2 MUST BE ODD AND

$$\frac{n_1 - 1}{2} = \frac{n_2 - 1}{2}$$

SO

$$n_1 = n_2 .$$

THUS, $\varphi(n_1) = \varphi(n_2) \Rightarrow n_1 = n_2$.

SURJECTIVE : LET $m \in \mathbb{Z}$. THEN

$$m = 0 \Rightarrow m = \frac{1-1}{2} = \varphi(1)$$

$$m < 0 \Rightarrow m = -\frac{-2m}{2} = \varphi(-2m)$$

$$m > 0 \Rightarrow m = \frac{(2m+1)-1}{2} = \varphi(2m+1)$$

THUS, EVERY $m \in \mathbb{Z}$ IS IN THE IMAGE OF φ.

THUS,

$$\mathbb{Z} \sim \mathbb{N} .$$

A SET A WITH THE SAME CARDINALITY AS $\mathbb{N}$ (E.G., $2\mathbb{N}$ AND $\mathbb{Z}$) IS SAID TO BE <u>COUNTABLY INFINITE</u>; IF A IS EITHER FINITE OR COUNTABLY INFINITE, THEN IT IS SAID TO BE <u>COUNTABLE</u>.

NOTE : INTUITIVELY, ONE THINKS OF A COUNTABLY INFINITE SET AS ONE WHOSE ELEMENTS CAN BE "LISTED IN A SEQUENCE" :

SUPPOSE $\varphi : \mathbb{N} \to A$ IS A BIJECTION.

DEFINE

$$a_1 = \varphi(1),\ a_2 = \varphi(2), \ldots, a_n = \varphi(n), \ldots$$

THEN

$$A = \{ a_1, a_2, \ldots, a_n, \ldots \}.$$

A SET THAT IS NOT COUNTABLE IS SAID TO BE UNCOUNTABLE. WE WILL SEE MORE INTERESTING EXAMPLES SHORTLY, BUT HERE IS ONE.

FOR ANY SET A WE DEFINE THE POWER SET OF A, DENOTED 2^A, TO BE THE SET OF ALL SUBSETS OF A, E.G., IF

$$A = \{1, 2, 3\}$$

THEN

$$2^A = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$$

(NOTE THAT, IN THIS EXAMPLE ANYWAY, $|2^A| = 2^{|A|}$).

LEMMA : THERE IS NO SURJECTIVE MAP OF A ONTO 2^A.

NOTE : IN PARTICULAR, THERE IS NO BIJECTION OF $\mathbb{N}$ ONTO $2^{\mathbb{N}}$. SINCE $2^{\mathbb{N}}$ IS CERTAINLY NOT FINITE, IT IS NOT COUNTABLE, I.E., $2^{\mathbb{N}}$ IS UNCOUNTABLE.

<u>PROOF OF THE LEMMA</u>: LET $\varphi : A \to 2^A$ BE ANY MAP.
CONSIDER THE SET

$$B = \{ b \in A : b \notin \varphi(b) \}.$$

THIS IS A SUBSET OF A, I.E., AN ELEMENT OF 2^A. WE CLAIM THAT IT CANNOT BE IN THE IMAGE OF φ (SO THAT φ IS NOT SURJECTIVE).

SUPPOSE, TO THE CONTRARY, THAT $B = \varphi(a)$ FOR SOME $a \in A$. NOTE THAT a CANNOT BE IN THE SET B SINCE

$$a \in B \Rightarrow a \notin \varphi(a) = B.$$

THUS, $a \notin B$. BUT THEN, BY DEFINITION OF B, $a \in \varphi(a) = B$ AND THIS IS A CONTRADICTION. $\square$

TO FIND SOME INTERESTING EXAMPLES IN $\mathbb{R}$ WE WILL NEED TO OBTAIN SOME GENERAL TOOLS, WHICH WE NOW PROCEED TO DO.

<u>LEMMA</u>: LET A BE ANY INFINITE SET. THEN THERE EXISTS AN INJECTIVE MAP FROM $\mathbb{N}$ INTO A.

<u>PROOF</u>: WE MUST CONSTRUCT SOME MAP $\varphi : \mathbb{N} \to A$ SUCH THAT $n_1 \neq n_2 \Rightarrow \varphi(n_1) \neq \varphi(n_2)$. THE CONSTRUCTION WILL BE <u>INDUCTIVE</u>.

> <u>NOTE</u>: WE HAVE PROVED MANY THINGS BY INDUCTION, BUT THIS IS THE FIRST TIME WE HAVE CONSTRUCTED SOMETHING BY INDUCTION SO WE WILL EXPLAIN WHAT

THIS MEANS. BASICALLY, WE WILL DEFINE $\varphi(1)$ AND THEN, ASSUMING THAT, FOR SOME $k \geq 2$, $\varphi(1), \ldots, \varphi(k-1)$ HAVE ALL BEEN DEFINED AND HAVE THE PROPERTY THAT $\varphi(i) \neq \varphi(j)$ FOR $i \neq j$, $i, j = 1, \ldots, k-1$, WE WILL DEFINE $\varphi(k)$ IN SUCH A WAY THAT $\varphi(i) \neq \varphi(j)$ FOR $i \neq j$, $i, j = 1, \ldots, k$. THIS PROVES THE EXISTENCE OF A $\varphi(n) \in A$ $\forall n \in \mathbb{N}$ S.T. $\varphi(i) \neq \varphi(j)$ FOR $i \neq j$, $i, j = 1, \ldots, n$ AND SO WE HAVE A MAP $\varphi : \mathbb{N} \to A$. φ MUST BE INJECTIVE SINCE, IF $n_1 \neq n_2$, THEN, WITHOUT LOSS OF GENERALITY, $n_1 < n_2$ AND, BY THE CONSTRUCTION OF $\varphi(n_2)$, $\varphi(n_1) \neq \varphi(n_2)$.

NOW FOR THE CONSTRUCTION. A IS INFINITE SO $A \neq \emptyset$ AND SO WE CAN SELECT SOME $a_1 \in A$ AND DEFINE

$$\varphi(1) = a_1 .$$

JUST TO GET AN IDEA OF WHAT THE INDUCTION WILL LOOK LIKE LET'S ALSO CONSTRUCT $\varphi(2)$.

NOTE THAT $A - \{a_1\} \neq \emptyset$ SINCE OTHERWISE $\varphi : \{1\} \to A$, $\varphi(1) = a_1$, WOULD BE A BIJECTION AND WE WOULD HAVE $|A| = 1$. THUS, WE CAN CHOOSE $a_2 \in A - \{a_1\}$ AND DEFINE

$$\varphi(2) = a_2 .$$

IN PARTICULAR, $\varphi(1) \neq \varphi(2)$.

NOW SUPPOSE WE HAVE DEFINED $\varphi(1), \dots, \varphi(k-1)$ IN A SUCH THAT $\varphi(i) \neq \varphi(j)$ FOR $i \neq j$, $i, j = 1, \dots, k-1$. THEN

$A - \{\varphi(1), \dots, \varphi(k-1)\} \neq \emptyset$ SINCE OTHERWISE $\varphi : \{1, \dots, k-1\} \to A$ WOULD BE A BIJECTION AND WE WOULD HAVE $|A| = k-1$. THUS, WE CAN CHOOSE AN $a_k \in A - \{\varphi(1), \dots, \varphi(k-1)\}$ AND DEFINE

$$\varphi(k) = a_k .$$

THEN $\varphi(k) \neq \varphi(j)$, $j = 1, \dots, k-1$, AND THE INDUCTION HYPOTHESIS GIVES $\varphi(i) \neq \varphi(j)$ FOR $i \neq j$, $i, j = 1, \dots, k$ AS REQUIRED. THUS, $\varphi(n)$ IS DEFINED FOR EVERY $n \in \mathbb{N}$ AND $\varphi : \mathbb{N} \to A$ IS INJECTIVE.

$\square$

EXERCISE 1: PROVE THAT ANY INFINITE SET CONTAINS A COUNTABLY INFINITE SUBSET.

THEOREM: EVERY SUBSET OF A COUNTABLE SET IS COUNTABLE.

PROOF: SUPPOSE A IS COUNTABLE. IF $A = \emptyset$, THEN THE ONLY SUBSET OF A IS $\emptyset$, WHICH IS FINITE AND THEREFORE COUNTABLE. THUS, WE ASSUME $A \neq \emptyset$.

EXERCISE 2: SHOW THAT IF A IS FINITE, THEN EVERY SUBSET OF A IS FINITE (AND THEREFORE COUNTABLE).

THUS, WE ASSUME A IS COUNTABLY INFINITE.

LET B BE A SUBSET OF A. IF B IS FINITE, THEN IT IS COUNTABLE SO WE MAY ASSUME B IS INFINITE.

TO SHOW THAT B IS COUNTABLY INFINITE WE BEGIN BY CHOOSING A BIJECTION

$$\varphi : \mathbb{N} \to A$$

OF $\mathbb{N}$ ONTO A. NEXT WE DEFINE A MAP

$$\psi_B : \mathbb{N} \to \mathbb{N}$$

AS FOLLOWS : EVERY ELEMENT OF $B \subseteq A$ IS THE IMAGE UNDER φ OF SOME NATURAL NUMBER SO THE SET OF ALL $k \in \mathbb{N}$ FOR WHICH $\varphi(k) \in B$ IS NONEMPTY. BY THE WELL-ORDERING PRINCIPLE WE MAY THEREFORE DEFINE

$$\psi_B(1) = \text{THE LEAST } n \in \mathbb{N} \text{ FOR WHICH } \varphi(n) \in B$$

(SO, IN PARTICULAR, $\varphi(\psi_B(1)) \in B$).

NOW SUPPOSE $k \geq 2$ AND $\psi_B(1), \ldots, \psi_B(k-1)$ HAVE BEEN DEFINED SO THAT $\psi_B(1) < \cdots < \psi_B(k-1)$ AND $\{\varphi(\psi_B(1)), \ldots, \varphi(\psi_B(k-1))\} \subseteq B$. SINCE B IS INFINITE, $B - \{\varphi(\psi_B(1)), \ldots, \varphi(\psi_B(k-1))\} \neq \emptyset$ SO WE CAN DEFINE

$$\psi_B(k) = \text{THE LEAST } n \in \mathbb{N} \text{ SUCH THAT}$$
$$n > \psi_B(k-1) \text{ AND } \varphi_B(n) \in B.$$

(SO, IN PARTICULAR, $\varphi(\psi_B(k)) \in B$ AND $\psi_B(k) > \psi_B(k-1)$).

THUS, $\psi_B(1) < \cdots < \psi_B(k)$ AND $\{\varphi(\psi_B(1)), \ldots, \varphi(\psi_B(k))\} \subseteq B$.
THIS COMPLETES THE INDUCTIVE CONSTRUCTION OF A MAP

$$\psi_B : \mathbb{N} \to \mathbb{N}$$

THAT SATISFIES

$$\psi_B(1) < \psi_B(2) < \cdots$$

(SO ψ_B IS INJECTIVE) AND $\varphi(\psi_B(n)) \in B \;\; \forall\, n \in \mathbb{N}$.

NOW CONSIDER THE COMPOSITION

$$\varphi \circ \psi_B : \mathbb{N} \to B.$$

IT IS INJECTIVE BECAUSE BOTH ψ_B AND φ ARE INJECTIVE.
TO SEE THAT IT IS SURJECTIVE, LET $b \in B$. SINCE
$\varphi : \mathbb{N} \to A$ IS A BIJECTION AND $B \subseteq A$, $\exists!\, k \in \mathbb{N}$ S.T.
$\varphi(k) = b$. WE MUST SHOW THAT $k = \psi_B(n)$ FOR SOME $n \in \mathbb{N}$.
SUPPOSE NOT, I.E., SUPPOSE $k \neq \psi_B(n)$ FOR $n = 1, 2, \cdots$
THEN, SINCE $\psi_B(1) < \psi_B(2) < \cdots$, $\{n \in \mathbb{N} : k < \psi_B(n)\}$ IS
NONEMPTY AND WE MAY SET

$$n_0 = \text{THE LEAST } n \in \mathbb{N} \text{ S.T. } k < \psi_B(n)$$

THEN $n_0 \neq 1$ SINCE $\psi_B(1)$ IS THE LEAST POSITIVE INTEGER FOR
WHICH $\varphi(\psi_B(1)) \in B$ AND $\varphi(k) \in B$ SO $k < \psi_B(1)$ IS IMPOSSIBLE.
THUS, $n_0 - 1 \in \mathbb{N}$ SO $\psi_B(n_0 - 1) \leq k < \psi_B(n_0)$. BUT, BY OUR
ASSUMPTION, $\psi_B(n_0 - 1) \neq k$ SO $\psi_B(n_0 - 1) < k < \psi_B(n_0)$. BUT, BY
DEFINITION, $\psi_B(n_0)$ IS THE LEAST POSITIVE INTEGER GREATER

THAN $\psi_B(n_0-1)$ WHOSE IMAGE UNDER φ IS IN B SO THIS CONTRADICTS THE FACT THAT $\varphi(k) \in B$. THUS, k MUST BE $\psi_B(n)$ FOR SOME $n \in \mathbb{N}$ AND $\varphi \circ \psi_B : \mathbb{N} \to B$ IS SURJECTIVE AS WELL AS INJECTIVE. B IS THEREFORE COUNTABLY INFINITE. $\square$

> EXERCISE 3 : SHOW THAT IF THERE IS AN INJECTIVE MAP OF A INTO $\mathbb{N}$, THEN A IS COUNTABLE.

THEOREM : IF A AND B ARE COUNTABLE, THEN THE CARTESIAN PRODUCT

$$A \times B = \{(a,b) : a \in A, b \in B\}$$

IS ALSO COUNTABLE.

PROOF :

> EXERCISE 4 : SHOW THAT IT IS ENOUGH TO PROVE THE THEOREM WHEN A AND B ARE COUNTABLY INFINITE.

CHOOSING BIJECTIONS OF $\mathbb{N}$ ONTO A AND B WE CAN WRITE (AS ON PAGE 4),

$$A = \{a_1, a_2, \cdots\}$$

AND

$$B = \{b_1, b_2, \cdots\}.$$

DEFINE

$$\varphi : A \times B \longrightarrow \mathbb{N}$$

BY

$$\varphi(a_n, b_m) = 2^n(2m-1).$$

WE CLAIM THAT φ IS INJECTIVE (SO $A \times B$ IS COUNTABLE BY EXERCISE 3). TO SEE THIS NOTE THAT

$$\varphi(a_{n_1}, b_{m_1}) = \varphi(a_{n_2}, b_{m_2}) \Rightarrow$$

$$2^{n_1}(2m_1 - 1) = 2^{n_2}(2m_2 - 1).$$

SUPPOSE, WITHOUT LOSS OF GENERALITY, THAT $n_1 \geq n_2$. THEN

$$2^{n_1 - n_2}(2m_1 - 1) = 2m_2 - 1.$$

BUT $2m_2 - 1$ IS ODD SO WE MUST HAVE $n_1 - n_2 = 0$, I.E. $n_1 = n_2$. THUS,

$$2m_1 - 1 = 2m_2 - 1$$

SO $m_1 = m_2$. THUS,

$$(a_{n_1}, b_{m_1}) = (a_{n_2}, b_{m_2})$$

SO φ IS INJECTIVE. $\square$

COROLLARY : THE SET $\mathbb{Q}$ OF RATIONAL NUMBERS IS COUNTABLY INFINITE.

PROOF : EACH RATIONAL NUMBER CAN BE UNIQUELY WRITTEN AS $\frac{m}{n}$, WHERE m AND n HAVE NO COMMON FACTORS GREATER THAN 1 AND $n > 0$. DEFINE

$$\varphi : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$$

BY

$$\varphi\left(\frac{m}{n}\right) = (m, n).$$

φ IS INJECTIVE BECAUSE

$$\varphi\left(\frac{m_1}{n_1}\right) = \varphi\left(\frac{m_2}{n_2}\right) \Rightarrow (m_1, n_1) = (m_2, n_2)$$

$$\Rightarrow m_1 = m_2 \text{ AND } n_1 = n_2$$

$$\Rightarrow \frac{m_1}{n_1} = \frac{m_2}{n_2}$$

$\mathbb{Z} \times \mathbb{N}$ IS COUNTABLY INFINITE BY THE PREVIOUS THEOREM SO $\varphi(\mathbb{Q}) \subseteq \mathbb{Z} \times \mathbb{N}$ IS COUNTABLE (AND NOT FINITE). THUS, WE CAN CHOOSE A BIJECTION

$$\psi : \varphi(\mathbb{Q}) \to \mathbb{N}.$$

SINCE $\varphi : \mathbb{Q} \to \varphi(\mathbb{Q})$ IS A BIJECTION SO IS

$$\psi \circ \varphi : \mathbb{Q} \to \mathbb{N}$$

SO $\mathbb{Q} \sim \mathbb{N}$. $\square$

<u>EXERCISE 5</u>: SHOW THAT A COUNTABLE UNION OF COUNTABLE SETS IS COUNTABLE, I.E., IF A_n IS A COUNTABLE SET FOR EACH $n = 1, 2, \ldots$, THEN $\bigcup_{n=1}^{\infty} A_n$ IS COUNTABLE.

NEXT WE WILL SHOW THAT THE SET $\mathbb{R}$ OF REAL NUMBERS IS UNCOUNTABLE. IN FACT, WE SHOW THAT ANY SUBSET OF $\mathbb{R}$ THAT CONTAINS AN INTERVAL (a, b), $a < b$, IS UNCOUNTABLE.

<u>EXERCISE 6</u>: FIND A BIJECTION OF $(0,1)$ ONTO (a, b) AND CONCLUDE THAT $(a,b) \sim (0,1)$.

THUS, IF WE CAN SHOW THAT $(0,1)$ IS UNCOUNTABLE IT WILL FOLLOW THAT EVERY (a,b) IS UNCOUNTABLE. MOREOVER, SINCE A SUBSET OF A COUNTABLE SET IS COUNTABLE, ANY SUBSET OF $\mathbb{R}$ THAT CONTAINS AN (a,b) MUST ALSO BE UNCOUNTABLE; IN PARTICULAR, $\mathbb{R}$ ITSELF IS UNCOUNTABLE.

A SUBSET OF $\mathbb{R}$ NEED NOT CONTAIN AN INTERVAL IN ORDER TO BE UNCOUNTABLE. INDEED, SINCE $\mathbb{Q}$ IS COUNTABLE AND THE UNION OF TWO COUNTABLE SETS IS COUNTABLE, THE SET $\mathbb{R}-\mathbb{Q}$ OF IRRATIONAL NUMBERS MUST BE UNCOUNTABLE (ASSUMING, OF COURSE, THAT WE ACTUALLY PROVE THAT $(0,1)$, AND THEREFORE $\mathbb{R}$, IS UNCOUNTABLE).

THUS, OUR TASK IS TO SHOW THAT $(0,1)$ IS UNCOUNTABLE.

THE USUAL WAY TO DO THIS IS TO USE AN INGENIOUS "DIAGONAL ARGUMENT" DUE TO CANTOR. THIS USES THE FACT THAT THE REAL NUMBERS α IN $(0,1)$ HAVE DECIMAL REPRESENTATIONS

$$\alpha = 0.a_1 a_2 a_3 a_4 \cdots$$

WHERE $a_i \in \{0,1,2,\ldots,9\}$ FOR EACH $i \in \mathbb{N}$. HOWEVER, THE EXISTENCE OF THESE DECIMAL REPRESENTATIONS IS NOT OBVIOUS AND PROVING THAT THEY EXIST IS A BIT TEDIOUS (ALTHOUGH EVERYONE SHOULD GO THROUGH IT ONCE IN THEIR LIVES). WE PREFER TO SPEND THE TIME DEVELOPING ANOTHER ARGUMENT THAT IS A BIT MORE "TOPOLOGICAL" IN NATURE AND HAS IMPORTANT GENERALIZATIONS. WE BEGIN WITH

<u>THE CANTOR INTERSECTION THEOREM</u> : SUPPOSE THAT, FOR EACH $n \in \mathbb{N}$, $[a_n, b_n]$ IS A CLOSED, BOUNDED INTERVAL IN $\mathbb{R}$ AND THAT

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_n, b_n] \supseteq [a_{n+1}, b_{n+1}] \supseteq \cdots .$$

THEN

$$\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset .$$

<u>PROOF</u> : FIRST NOTICE THAT, FOR ALL $n, m \in \mathbb{N}$,

$$a_n \leq b_m .$$

INDEED, IF WE LET $k = \max(n, m)$, THEN $[a_k, b_k] \subseteq [a_n, b_n]$ SO $a_n \leq a_k$ AND $[a_k, b_k] \subseteq [a_m, b_m]$ SO $b_k \leq b_m$. THUS,

$$a_n \leq a_k \leq b_k \leq b_m .$$

THUS, $\{a_n : n \in \mathbb{N}\}$ IS BOUNDED ABOVE BY ANY b_m. LET

$$c = \sup \{a_n : n \in \mathbb{N}\} .$$

THEN

$$a_n \leq c \qquad \forall\, n \in \mathbb{N}$$

(c IS AN UPPER BOUND) AND

$$c \leq b_n \qquad \forall\, n \in \mathbb{N}$$

(c IS THE LEAST UPPER BOUND AND EACH b_n IS AN UPPER BOUND).

THUS, $c \in [a_n, b_n]$ $\forall\, n \in \mathbb{N}$, I.E., $c \in \bigcap_{n=1}^{\infty} [a_n, b_n]$. $\square$

<u>COROLLARY</u> : THE INTERVAL $(0, 1)$ IN $\mathbb{R}$ IS UNCOUNTABLE.

PROOF: WE WILL SHOW THAT NO COUNTABLE SET $\{x_1, x_2, \dots\}$ OF REAL NUMBERS IN $(0,1)$ CAN CONTAIN ALL OF THE REAL NUMBERS IN $(0,1)$. FOR THIS WE WILL REPEATED USE THE FOLLOWING OBSERVATION:

> GIVEN A NONEMPTY OPEN INTERVAL (a,b) IN $\mathbb{R}$ AND ANY $x \in \mathbb{R}$, $\exists\ c < d$ SUCH THAT $[c,d] \subseteq (a,b)$ AND $x \notin [c,d]$.

(IF THIS ISN'T COMPLETELY OBVIOUS TO YOU THEN WRITE OUT A PROOF.)

THUS, WE CAN CHOOSE $a_1 < b_1$ SUCH THAT $[a_1, b_1] \subseteq (0,1)$ AND $x_1 \notin [a_1, b_1]$. NOW ASSUME (FOR AN INDUCTIVE CONSTRUCTION) THAT $k \geq 1$ AND THAT WE HAVE CHOSEN $a_1, b_1, \dots, a_k, b_k$ SUCH THAT $a_i < b_i$ FOR $i = 1, \dots, k$, $[a_{i+1}, b_{i+1}] \subseteq (a_i, b_i)$ FOR $i = 1, \dots, k-1$, AND $x_i \notin [a_i, b_i]$ FOR $i = 1, \dots, k$. THE OBSERVATION ABOVE THEN ALLOWS US TO CHOOSE $a_{k+1} < b_{k+1}$ SUCH THAT $[a_{k+1}, b_{k+1}] \subseteq (a_k, b_k)$ AND $x_{k+1} \notin [a_{k+1}, b_{k+1}]$. THUS, WE HAVE INDUCTIVELY DEFINED, FOR EACH $n \in \mathbb{N}$, A CLOSED INTERVAL $[a_n, b_n]$ SUCH THAT $x_n \notin [a_n, b_n]$ AND $[a_{n+1}, b_{n+1}] \subseteq (a_n, b_n) \subseteq [a_n, b_n]$. BY THE PREVIOUS THEOREM, $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$. IF $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$, THEN $x \neq x_n\ \forall n \in \mathbb{N}$ SINCE $x_n \notin [a_n, b_n]$. HOWEVER, x IS IN $(0,1)$ SINCE $(0,1) \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$. THUS, $\{x_1, x_2, \dots\}$ CANNOT EXHAUST ALL OF $(0,1)$ AND THE PROOF IS COMPLETE. $\square$

ADDITIONAL PROBLEMS :

7. SHOW THAT THE SET OF ALL FUNCTIONS FROM $\mathbb{N}$ TO $\{0,1\}$ IS UNCOUNTABLE.

8. SHOW THAT "SAME CARDINALITY" $\sim$ IS AN EQUIVALENCE RELATION, I.E., IF A, B AND C ARE SETS, THEN

   (i) $A \sim A$

   (ii) $A \sim B \Rightarrow B \sim A$

   (iii) $A \sim B$ AND $B \sim C \Rightarrow A \sim C$.

9. SHOW THAT THE SET OF ALL FINITE SUBSETS OF $\mathbb{N}$ IS COUNTABLE.

10. LET D BE THE SET OF POINTS IN THE PLANE SATISFYING $x^2 + y^2 < 1$. SHOW THAT THE SET OF POINTS IN D WITH BOTH COORDINATES RATIONAL IS COUNTABLE.

11. SHOW THAT $[2,3]$ AND $(2,3) \cup \{4\}$ HAVE THE SAME CARDINALITY. HINT: THINK ABOUT $[2,3] \cap \mathbb{Q}$ AND $((2,3) \cup \{4\}) \cap \mathbb{Q}$.

12. SHOW THAT ANY SET OF PAIRWISE DISJOINT, NONEMPTY OPEN INTERVALS IN $\mathbb{R}$ IS COUNTABLE.

    LET A BE AN INFINITE SET AND B A COUNTABLY INFINITE SET WITH $A \cap B = \phi$. SHOW THAT $A \cup B$ HAS THE SAME CARDINALITY AS A.

    HINT: SHOW THAT THERE IS A BIJECTION φ OF B ONTO A SUBSET $\varphi(B)$ OF A AND THEN FIND A BIJECTION OF $\varphi(B)$ ONTO A PROPER SUBSET OF $\varphi(B)$.

SOLUTIONS TO THE EXERCISES :

1. A AN INFINITE SET. THEN THERE IS AN INJECTIVE MAP φ OF $\mathbb{N}$ INTO A. THEN $\varphi : \mathbb{N} \to \varphi(\mathbb{N})$ IS A BIJECTION SO $\varphi(\mathbb{N}) \subseteq A$ IS COUNTABLE.

2. A A FINITE SET. THEN FOR SOME $n \in \mathbb{N}$ THERE IS A BIJECTION $\varphi : \{1, \dots, n\} \to A$ OF $\{1, \dots, n\}$ ONTO A. LET B BE A SUBSET OF A. IF $B = \emptyset$, THEN IT IS FINITE SO SUPPOSE $B \neq \emptyset$. THEN $\varphi^{-1}(B)$ IS A NONEMPTY SUBSET OF $\{1, \dots, n\}$, SAY,

$$\varphi^{-1}(B) = \{ i_1, \dots, i_k \}$$

THEN

$$B = \{ \varphi(i_1), \dots, \varphi(i_k) \}.$$

NOW,

$$\psi : \{1, \dots, k\} \to \varphi^{-1}(B)$$

$$\psi(j) = i_j$$

IS A BIJECTION AND SO IS

$$\varphi : \varphi^{-1}(B) \to B$$

SO

$$\varphi \circ \psi : \{1, \dots, k\} \to B$$

IS A BIJECTION AND THEREFORE $|B| = k$.

3. LET $\varphi : A \to \mathbb{N}$ BE INJECTIVE. THEN $\varphi(A)$ IS COUNTABLE (BEING A SUBSET OF A COUNTABLE SET). SO THERE IS A BIJECTION ψ OF $\varphi(A)$ ONTO EITHER $\mathbb{N}$ OR $\{1, \dots, n\}$ FOR SOME $n \in \mathbb{N}$. THEN $\psi \circ \varphi$ IS A BIJECTION OF A ONTO EITHER $\mathbb{N}$ OR $\{1, \dots, n\}$ SO A IS COUNTABLE.

4. SUPPOSE WE HAVE SHOWN THAT $A \times B$ IS COUNTABLE WHENEVER A AND B ARE COUNTABLY INFINITE. IF EITHER A' OR B' IS FINITE WE CAN CONSIDER $A = A' \cup \mathbb{N}$ AND $B = B' \cup \mathbb{N}$. THEN A AND B ARE COUNTABLY INFINITE SO $A \times B$ IS COUNTABLE. BUT $A' \times B' \subseteq A \times B$ SO $A' \times B'$ IS ALSO COUNTABLE.

5. SUPPOSE A_n IS COUNTABLE FOR EACH $n = 1, 2, \ldots$ WE SHOW THAT $\bigcup_{n=1}^{\infty} A_n$ IS COUNTABLE. AS IN EXERCISE 4 IT IS ENOUGH TO PROVE THIS WHEN EACH A_n IS COUNTABLY INFINITE (ANY FINITE A_n CAN BE ENLARGED TO A COUNTABLY INFINITE SET, $\bigcup_{n=1}^{\infty} A_n$ IS CONTAINED IN THE UNION OF THE ENLARGED SETS, AND A SUBSET OF A COUNTABLE SET IS COUNTABLE).

THUS, LET $A_n = \{a_{n1}, a_{n2}, \ldots, a_{nm}, \ldots\}$ FOR $n = 1, 2, \ldots$

THEN

$$\bigcup_{n=1}^{\infty} A_n = \{a_{nm} : n, m \in \mathbb{N}\}$$

THE MAP

$$\varphi : \bigcup_{n=1}^{\infty} A_n \to \mathbb{N} \times \mathbb{N}$$

$$\varphi(a_{nm}) = (n, m)$$

IS INJECTIVE SINCE

$$a_{n_1 m_1} \neq a_{n_2 m_2} \Longrightarrow (n_1, m_1) \neq (n_2, m_2)$$
$$\Longrightarrow \varphi(a_{n_1 m_1}) \neq \varphi(a_{n_2 m_2})$$

IF $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ IS A BIJECTION, THEN $\psi \circ \varphi : \bigcup_{n=1}^{\infty} A_n \to \mathbb{N}$ IS INJECTIVE SO $\bigcup_{n=1}^{\infty} A_n$ IS COUNTABLE.

6. $\varphi : (0,1) \to (a,b)$ DEFINED BY

$$\varphi(x) = (b-a)x + a$$

IS INJECTIVE

$$\varphi(x_1) = \varphi(x_2) \Rightarrow (b-a)x_1 + a = (b-a)x_2 + a$$
$$\Rightarrow x_1 = x_2$$

AND SURJECTIVE

$$y \in (a,b) \Rightarrow \frac{y-a}{b-a} \in (0,1) \text{ AND}$$
$$\varphi\left(\frac{y-a}{b-a}\right) = y$$

AND SO IS A BIJECTION.