DIFFERENTIAL TOPOLOGY AND PHYSICS

THE WITTEN CONJECTURE

GREG NABER

DONALDSON INVARIANTS $\hookrightarrow$ SEIBERG-WITTEN INVARIANTS

$SU(2)$ $U(1)$

$$*F_\omega = -F_\omega \qquad \begin{cases} \not{D}_A \psi = 0 \\ F_A^+ = \sigma^+((\psi \otimes \psi^*)_0) \end{cases}$$

$$\mathcal{D}_M(x) = EXP\left(Q_M(x,x)/2\right) \sum_{\mathcal{L} \in \Lambda} 2^{m(M)} SW_0(M,\mathcal{L})\, EXP\left(c_1(L^0(\mathcal{L}))(x)\right)$$

ADDENDA (http://www.pages.drexel.edu/~gln22/) :

M = COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH 4-MANIFOLD

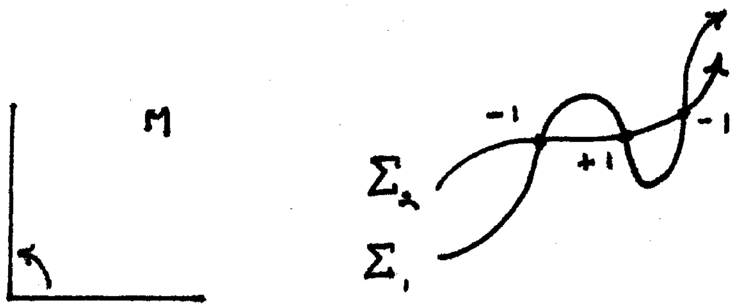
+ ASSUMPTIONS ON $b_2^+(M)$ AS THE NEED ARISES

REGARDING $b_2^+(M)$:

<u>INTERSECTION</u> <u>FORM</u> (ASSUMING $H_2(M;\mathbb{Z}) \not\equiv 0$) :

$$Q_M : H_2(M;\mathbb{Z}) \oplus H_2(M;\mathbb{Z}) \rightarrow \mathbb{Z}$$

A UNIMODULAR, SYMMETRIC, $\mathbb{Z}$-VALUED, BILINEAR FORM



$Q_M(x_1, x_2)$ = SUM OF SIGNED INTERSECTION POINTS

$b_2^+(M)$ = MAXIMAL DIMENSION OF A SUBSPACE OF $H_2(M;\mathbb{Z})$ ON WHICH Q_M IS POSITIVE DEFINITE

DONALDSON'S 1983 THEOREM (FIELDS MEDAL):

$$b_2^+(M) = 0 \quad \Rightarrow \quad Q_M = -id$$

SKETCH OF THE PROOF:

$$SU(2) \hookrightarrow P_1 \xrightarrow{\pi_1} M \qquad \text{(CHERN CLASS 1)}$$

CHOOSE RIEMANNIAN METRIC g ON M. TOGETHER WITH THE ORIENTATION THIS GIVES A HODGE STAR $*$.

A CONNECTION ω ON P_1 IS g-ASD IF ITS CURVATURE

$$F_\omega \in \Omega^2(M, ad P_1)$$

SATISFIES

$$*F_\omega = -F_\omega .$$

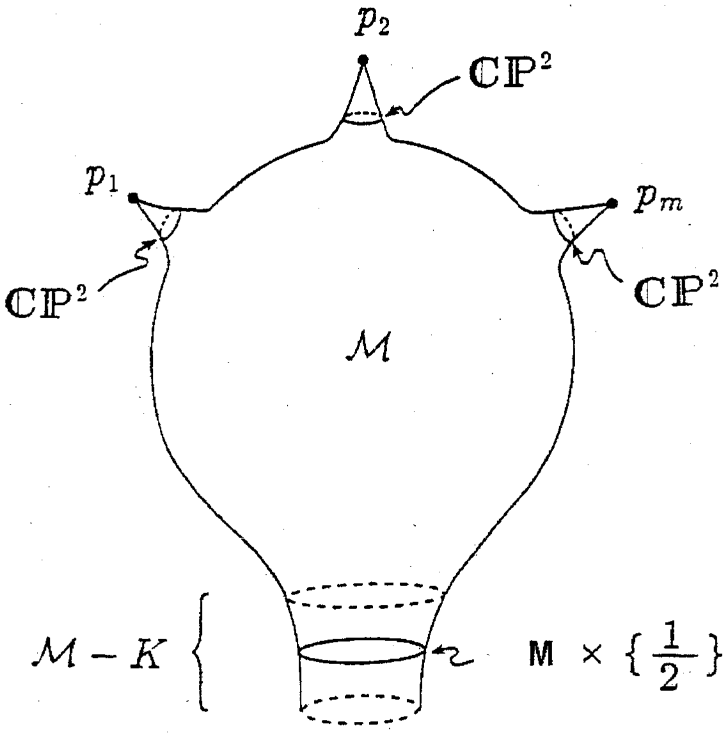
NOTE: $b_2^+(M) = 0 \Rightarrow$ SUCH THINGS EXIST (TAUBES)

$\mathcal{G}(P_1)$ = GAUGE GROUP = DIFFEOMORPHISMS $f : P_1 \to P_1$
SATISFYING $f(p \cdot g) = f(p) \cdot g$
AND $\pi_1 \circ f = \pi_1$

ω, ω' ARE GAUGE EQUIVALENT IF $\omega' = f^*\omega$ FOR SOME $f \in \mathcal{G}(P_1)$

$\mathcal{M} = \mathcal{M}(P_1, g)$ = MODULI SPACE OF GAUGE EQUIVALENCE CLASSES OF g-ASD CONNECTIONS ON P_1

FOR "GENERIC" g, $\mathcal{M}$ LOOKS LIKE THIS :

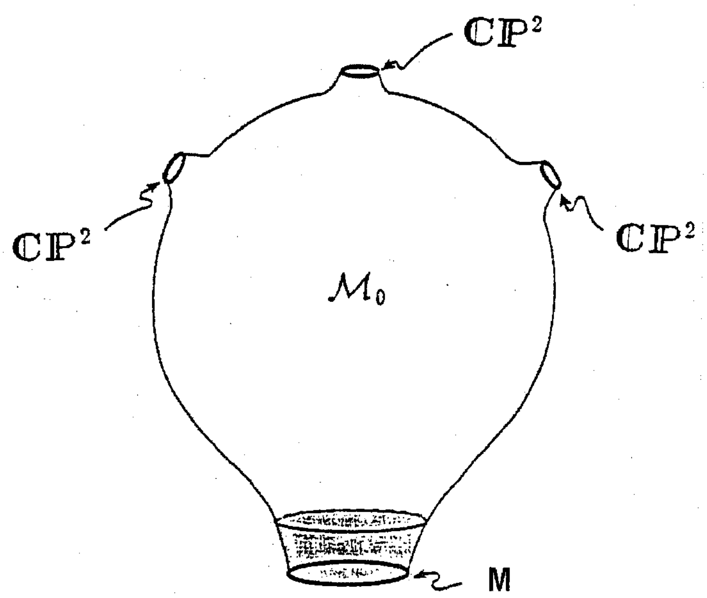


1. $\exists$ POINTS $\{p_1, \ldots, p_m\}$ SUCH THAT $\mathcal{M} - \{p_1, \ldots, p_m\}$ IS A SMOOTH, ORIENTED, 5-MANIFOLD.

2. EACH p_i HAS A NEIGHBORHOOD IN $\mathcal{M}$ HOMEOMORPHIC TO A CONE OVER $\mathbb{CP}^2$.

3. $\exists$ COMPACT $K \subseteq \mathcal{M}$ SUCH THAT $\mathcal{M} - K$ IS A SUBMANIFOLD OF $\mathcal{M} - \{p_1, \ldots, p_m\}$ DIFFEOMORPHIC TO $M \times (0, 1)$.

NOW CUT OFF THE TOP HALF OF EACH CONE AND THE BOTTOM HALF OF THE CYLINDER :



M IS COBORDANT TO A DISJOINT UNION OF $\mathbb{CP}^2$S AND THE SIGNATURE OF THE INTERSECTION FORM IS A COBORDISM INVARIANT ETC. $\square$

<u>DONALDSON</u> <u>POLYNOMIAL</u> <u>INVARIANTS</u> : $\gamma_d(M)$, $d = 0, 1, 2, \ldots$

$$\gamma_0(M) \in \mathbb{Z}$$

$$\gamma_d(M) : H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}[\tfrac{1}{2}] , \quad d = 1, 2, \ldots$$

CONSTRUCTION :

$$SU(2) \hookrightarrow P_k \xrightarrow{\pi_k} M \qquad (\text{CHERN CLASS } k > 0)$$

CHOOSE A RIEMANNIAN METRIC g ON M

$\mathcal{M}(P_k, g)$ = MODULI SPACE OF GAUGE EQUIVALENCE CLASSES OF g-ASD CONNECTIONS ON P_k

ASSUME

$$b_2^+(M) > 1 \quad \text{AND ODD}$$

FOR A "GENERIC" g, $\mathcal{M}(P_k, g)$ IS EITHER EMPTY OR A SMOOTH, ORIENTED MANIFOLD OF DIMENSION

$$8k - 3(1 + b_2^+(M)).$$

CASE 1 : $\quad 8k - 3(1+b_2^+(M)) = 0$

$$\gamma_0(M) = \begin{cases} \displaystyle\sum_{[\omega] \in \mathcal{M}(P_k, g)} (-1)^{[\omega]} & , \ \mathcal{M}(P_k, g) \neq \phi \\ 0 & , \ \mathcal{M}(P_k, g) = \phi \end{cases}$$

CASE 2 : $\quad 8k - 3(1+b_2^+(M)) = 2d_k > 0$

DONALDSON μ - MAP :

$$\mu : H_2(M; \mathbb{Z}) \longrightarrow H^2(\mathcal{M}(P_k, g); \mathbb{Q})$$

"NAIVE" DEFINITION OF $\gamma_{d_k}(M) : H_2(M;\mathbb{Z}) \to \mathbb{Z}[\tfrac{1}{2}]$:

$$x \in H_2(M;\mathbb{Z}) \Rightarrow \mu(x) \wedge \overset{d_k}{\cdots} \wedge \mu(x) \in H^{2d_k}(\mathcal{M}(P_k, g); \mathbb{Q})$$

$$\gamma_{d_k}(M) = \int_{\mathcal{M}(P_k,g)} \mu(x) \wedge \overset{d_k}{\cdots} \wedge \mu(x)$$

ORIENTATION PRESERVING DIFFEOMORPHISM INVARIANTS OF M.

<u>DONALDSON</u> (FORMAL POWER) <u>SERIES</u> :

$$\mathcal{D}_M(x) = \sum_{d=0}^{\infty} \frac{\gamma_d(M)(x)}{d!}$$

<u>EXAMPLE</u> : $M = K3$

$$\mathcal{D}_{K3}(x) = \sum_{d=0}^{\infty} \frac{(Q_{K3}(x,x)/2)^d}{d!} = \mathrm{EXP}\left(Q_{K3}(x,x)/2\right)$$

<u>A BREAKTHROUGH</u> (SPRING, 1994):

<u>KRONHEIMER - MROWKA</u> <u>STRUCTURE THEOREM</u> : IF M IS OF "D-SIMPLE TYPE", THEN $\exists\ K_1, \dots, K_s \in H^2(M;\mathbb{Z})$ (<u>D-BASIC CLASSES</u>) AND RATIONAL NUMBERS $a_1, \dots, a_s$ (<u>COEFFICIENTS</u>) SUCH THAT

$$\mathcal{D}_M(x) = \mathrm{EXP}(Q_M(x,x)/2) \sum_{r=1}^{s} a_r\, \mathrm{EXP}(K_r(x))$$

RENDERED MOOT (FALL, 1994):

TO TELL THIS STORY WE RETURN TO 1988.

DONALDSON INVARIANTS AS INTERSECTION NUMBERS :

$$\begin{tikzcd}
\hat{\mathcal{A}} \arrow[d] & \hat{\mathcal{A}} \times_{\hat{\mathcal{G}}} \Omega^2_+(M, \mathrm{ad}P) \arrow[d] & {[\omega, F^+_\omega]} \\
\hat{\mathcal{A}}/\hat{\mathcal{G}} & \hat{\mathcal{A}}/\hat{\mathcal{G}} & {[\omega]} \arrow[u]
\end{tikzcd}$$

MODULI SPACE OF ALL (IRREDUCIBLE) CONNECTIONS	ASSOCIATED VECTOR BUNDLE WITH FIBER $\Omega^2_+(M, \mathrm{ad}P)$	SECTION s

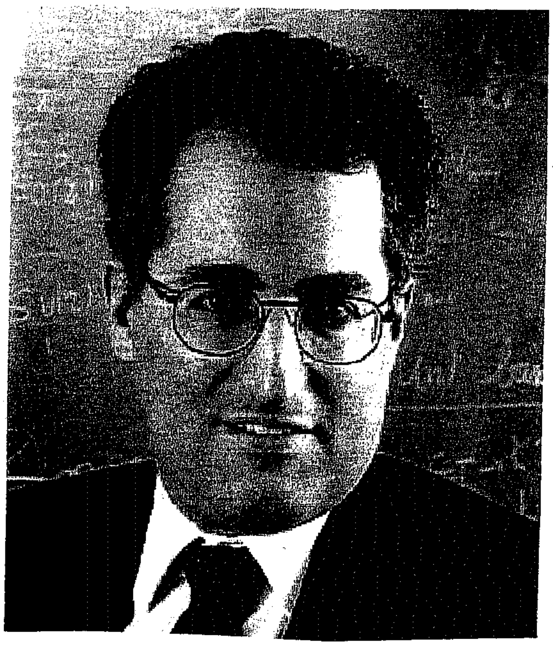
$$\mathcal{M} = s \cap s_0$$

IS THERE A "GAUSS-BONNET-CHERN" TYPE INTEGRAL REPRESENTATION OF THE DONALDSON INVARIANTS ?

> " THIS IS SUCH STUFF AS QUANTUM FIELD THEORY IS MADE OF. "
>
> \- NIGEL HITCHIN

... SO

> " WHO YA GONNA CALL ? "
>
> \- GHOSTBUSTERS



WITTEN'S 1988 TOPOLOGICAL QUANTUM FIELD THEORY (TQFT):

M AS ABOVE WITH RIEMANNIAN METRIC g

$$SU(2) \hookrightarrow P \xrightarrow{\pi} M$$

FIELD CONTENT:

GAUGE FIELD (CONNECTION) ω + "MATTER FIELDS"

BOSONIC

$\phi \in \Omega^0(M, \mathrm{ad}\,P)$

$\lambda \in \Omega^0(M, \mathrm{ad}\,P)$

FERMIONIC

$\eta \in \Omega^0(M, \mathrm{ad}\,P)$

$\psi \in \Omega^1(M, \mathrm{ad}\,P)$

$\chi \in \Omega^2_+(M, \mathrm{ad}\,P)$

$$\Phi = (\omega, \phi, \lambda, \eta, \psi, \chi)$$

+ DONALDSON-WITTEN ACTION FUNCTIONAL

$$S_{DW}[\Phi] = \int_M \mathrm{Tr}\ \Big\{ \tfrac{1}{4} F_\omega \wedge {}^*F_\omega + \tfrac{1}{4} F_\omega \wedge F_\omega - \tfrac{1}{2} \zeta \wedge [\zeta, \phi] +$$
$$i\, d^\omega \psi \wedge \zeta - 2i [\psi, {}^*\psi]\lambda + i\,\phi\, d^\omega {}^* d^\omega \lambda - \psi \wedge {}^* d^\omega \eta \Big\}$$

PARTITION FUNCTION :

$$Z_{DW} = \int \mathrm{EXP}\left(- S_{DW}[\Phi]/e^2\right) \mathcal{D}\Phi$$

WHERE e IS A COUPLING CONSTANT AND $\mathcal{D}\Phi$ IS A (NONEXISTENT) MEASURE ON THE SPACE OF FIELDS.

PHYSICS (WITTEN) : TWISTED VERSION OF "$N=2$ SUPERSYMMETRIC YANG-MILLS THEORY"

GEOMETRY (ATIYAH-JEFFREY) : EULER CLASS OF $\hat{\mathcal{A}} \times_{\hat{\mathcal{G}}} \Omega^2_+(M, \mathrm{ad}\,P)$

WITTEN "PROVES" Z_{DW} INDEPENDENT OF g AND e, COMPUTES IT IN THE "WEAK COUPLING LIMIT" $e \to 0$ WHERE THE INTEGRAL LOCALIZES TO THE ASD MODULI SPACE GIVING, WHEN $8k - 3(1 + b_2^+(M)) = 0$,

$$\boxed{Z_{DW} = \gamma_0(M)}$$

REMARKABLE, BUT ONLY THE BEGINNING OF THE STORY !

"DUALITY" IN WITTEN'S TQFT (1988-1994)

$e \to 0$	$e \to \infty$
WEAK COUPLING	STRONG COUPLING
PERTURBATIVE	NON-PERTURBATIVE
COMPUTABLE	INTRACTIBLE (1988)
$\boxed{\text{DONALDSON INVARIANTS}}$	$\boxed{\quad ? \quad}$

SEIBERG - WITTEN (1994) : EXACT SOLUTIONS IN STRONG COUPLING

$\boxed{\text{SEIBERG-WITTEN INVARIANTS}}$

SW - EQUATIONS

+

SW - GAUGE GROUP

$\downarrow$

SW - MODULI SPACE

$\downarrow$

SW - INVARIANTS

M AS BEFORE + RIEMANNIAN METRIC g + "SPIN^c STRUCTURE" $\mathcal{L}$

$$
\begin{tikzcd}[column sep=large]
 & & U(1)/\pm 1 = U(1) \\
\mathrm{Spin}^c(4) = SU(2)\times SU(2)\times U(1)/\pm 1 \arrow[d, "\text{DOUBLE COVER}"] \arrow[urr, "\delta"] \arrow[rr, "\Delta_\pm"] \arrow[drr, "\pi"'] & & SU(2)\times U(1)/\pm 1 = U(2) \\
SO(4)\times U(1) & & SU(2)\times SU(2)/\pm 1 = SO(4)
\end{tikzcd}
$$

$$
\mathcal{L}:\quad
\begin{tikzcd}
\mathrm{Spin}^c(4) \arrow[r, hook] & S^c(M) \arrow[d] \arrow[dr] & \\
 & & M \\
SO(4) \arrow[r, hook] & F_{SO}(M) \arrow[ur] &
\end{tikzcd}
$$

ASSOCIATED BUNDLES :

$L(\mathcal{L}) = S^c(M) \times_\delta \mathbb{C}$

$L^0(\mathcal{L})$ = CORRESPONDING PRINCIPAL $U(1)$-BUNDLE

$\mathbb{S}^{\pm}(\mathcal{L}) = S^c(M) \times_{\Delta_\pm} \mathbb{C}^2$ ($\pm$ SPINOR BUNDLES)

DATA FOR SW-EQUATIONS :

A = CONNECTION ON $L^0(\mathcal{L})$ ($U(1)$-GAUGE FIELD)

ψ = SECTION OF $\mathbb{S}^+(\mathcal{L})$ (POSITIVE SPINOR FIELD)

<u>SW</u>-<u>EQUATIONS</u> :

$$
\begin{cases}
\not{D}_A \psi = 0 \\
F_A^+ = \sigma^+((\psi\otimes\psi^*)_0)
\end{cases}
$$

SW - GAUGE GROUP :

$$\mathcal{G}(\mathcal{L}) = \text{AUTOMORPHISMS OF } SPIN^c(4) \hookrightarrow S^c(M) \rightarrow M$$
$$\text{THAT COVER THE IDENTITY ON } F_{SO}(M)$$

" GENERICALLY " THE CORRESPONDING MODULI SPACE OF SOLUTIONS IS EITHER EMPTY OR A SMOOTH, ORIENTED MANIFOLD OF DIMENSION

$$\frac{1}{4}\left(c_1(L^0(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M) \right)$$

AND IS ALWAYS COMPACT.

FOR A GIVEN (GENERIC) g, LET Λ BE THE SET OF ALL (EQUIVALENCE CLASSES OF) SPIN^c STRUCTURES $\mathcal{L}$ FOR WHICH

$$c_1(L^0(\mathcal{L}))^2 - 2\chi(M) - 3\sigma(M) = 0$$

MODULI SPACE IS EITHER EMPTY (IN WHICH CASE WE TAKE $SW_0(M,\mathcal{L}) = 0$) OR A FINITE SET OF ISOLATED POINTS, EACH EQUIPPED WITH A SIGN ± 1, IN WHICH CASE WE TAKE

$$SW_0(M,\mathcal{L}) = \text{SUM OF THESE SIGNS}.$$

THIS IS THE 0-DIMENSIONAL SEIBERG-WITTEN INVARIANT.

WHEN THE MODULI SPACE IS NOT 0-DIMENSIONAL ONE CAN DEFINE SEIBERG-WITTEN INVARIANTS BY INTEGRATING OVER IT, BUT IT IS CONJECTURED THAT NONZERO INVARIANTS ARISE ONLY FROM 0-DIMENSIONAL MODULI SPACES (MANIFOLDS FOR WHICH THIS IS TRUE ARE SAID TO BE OF <u>SW</u>-<u>SIMPLE</u> <u>TYPE</u>).

ELEMENTS OF $H^2(M;\mathbb{Z})$ THAT ARE $c_1(L^0(\mathcal{L}))$ FOR SOME $\mathcal{L} \in \Lambda$ ARE CALLED <u>SW</u>-<u>BASIC</u> <u>CLASSES</u>.

<u>THE</u> <u>WITTEN</u> <u>CONJECTURE</u>: LET M BE A COMPACT, SIMPLY CONNECTED, ORIENTED, SMOOTH 4-MANIFOLD WITH $b_2^+(M) > 1$ AND ODD. THEN

1. M IS OF D-SIMPLE TYPE IF AND ONLY IF IT IS OF SW-SIMPLE TYPE.
2. D-BASIC CLASSES COINCIDE WITH SW-BASIC CLASSES AND

$$\mathcal{D}_M(x) = \operatorname{EXP}(Q_M(x,x)/2) \sum_{r=1}^{s} a_r \operatorname{EXP}(K_r(x))$$

$$= \operatorname{EXP}(Q_M(x,x)/2) \sum_{\mathcal{L} \in \Lambda} 2^{m(M)} SW_0(M,\mathcal{L}) \operatorname{EXP}\big(c_1(L^0(\mathcal{L}))(x)\big)$$

WHERE

$$m(M) = 2 + \tfrac{1}{4}(7\chi(M) + 11\sigma(M))$$

ATTITUDES ONE MIGHT ADOPT TOWARD THE CONJECTURE :

1. IT SHOULD BE RIGOROUSLY PROVED.

   - PIDSTRIGATCH AND TYURIN

   - FEEHAN AND LENESS

   (12 YEARS)

2. RIGOROUSLY TRUE OR NOT THE SW-INVARIANTS PROVIDE A MUCH MORE TRACTABLE TOOL FOR THE STUDY OF 4-MANIFOLDS SO IT MAKES GOOD PRACTICAL SENSE TO ABANDON THE ASD EQUATIONS IN FAVOR OF THE SW-EQUATIONS.

   - ESSENTIALLY EVERYONE ELSE

3. IF PHYSICS IS TRULY CAPABLE OF CASTING SUCH A PENETRATING LIGHT UPON MATHEMATICS AT THE DEEPEST LEVELS, THEN MATHEMATICIANS WILL WANT TO TAKE HEED AND TURN THEIR ATTENTION ONCE AGAIN TO THEIR HISTORICAL ROOTS IN PHYSICS.

   - ATIYAH