# DEFINITE INTEGRALS BY SUBSTITUTION

SUBSTITUTION FOR DEFINITE INTEGRALS, E.G.,

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{(3+\sin x)^2}\,dx$$

TWO OPTIONS :

A. INDEFINITE INTEGRAL FIRST :

$$\int \frac{\cos x}{(3+\sin x)^2}\,dx = \int (3+\sin x)^{-2}\cos x\,dx = \int u^{-2}\,du$$

$u = 3+\sin x$

$du = \cos x\,dx$

$$= \frac{u^{-2+1}}{-2+1} + C$$

$$= -\frac{1}{u} + C$$

$$= -\frac{1}{3+\sin x} + C$$

SELECT AN ANTIDERIVATIVE AND SUBSTITUTE LIMITS :

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{(3+\sin x)^2}\,dx = -\frac{1}{3+\sin x}\Bigg|_0^{\frac{\pi}{2}}$$

$$= \left(-\frac{1}{3+1}\right) - \left(-\frac{1}{3+0}\right) = -\frac{1}{4} + \frac{1}{3}$$

$$= \frac{1}{12}$$

B. CHANGE $x$-LIMITS TO $u$-LIMITS WITH THE SUBSTITUTION :

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{(3+\sin x)^2}\,dx = \int_0^{\frac{\pi}{2}} (3+\sin x)^{-2}\cos x\,dx = \int_3^4 u^{-2}\,du$$

$u = 3 + \sin x$

$du = \cos x\,dx$

$x = 0 \Rightarrow u = 3 + \sin 0 = 3$

$x = \frac{\pi}{2} \Rightarrow u = 3 + \sin\frac{\pi}{2} = 4$

$$= -\frac{1}{u}\Big|_3^4$$

$$= \left(-\frac{1}{4}\right) - \left(-\frac{1}{3}\right)$$

$$= \frac{1}{12}$$

<u>MORE</u> <u>EXAMPLES</u> : OPTION (B) IS GENERALLY FASTER SO WE'LL USE THIS.

1. $$\int_0^{\frac{\pi}{8}} \sin^2 2x \cos 2x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{8}} \sin^2 2x\,(2\cos 2x\,dx) =$$

$u = \sin 2x$

$du = 2\cos 2x\,dx$

$x = 0 \Rightarrow u = \sin 2(0) = \sin 0 = 0$

$x = \frac{\pi}{8} \Rightarrow u = \sin 2(\frac{\pi}{8}) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$$\frac{1}{2}\int_0^{\sqrt{2}/2} u^2\,du =$$

$$\frac{1}{2}\,\frac{1}{3}u^3\Big|_0^{\sqrt{2}/2} =$$

$$\frac{1}{6}\left[\left(\frac{\sqrt{2}}{2}\right)^3 - 0^3\right] =$$

$$\frac{1}{6}\left[\frac{\sqrt{2}}{4}\right] = \frac{\sqrt{2}}{24}$$

2. $$\int_1^{\sqrt{2}} \frac{x}{4+x^4}\,dx = \int_1^{\sqrt{2}} \frac{x}{4+(x^2)^2}\,dx = \int_1^{\sqrt{2}} \frac{x}{4(1+(\frac{x^2}{2})^2)}\,dx =$$

$$\frac{1}{4}\int_1^{\sqrt{2}} \frac{1}{1+(\frac{x^2}{2})^2}\,x\,dx = \frac{1}{4}\int_{\frac{1}{2}}^1 \frac{1}{1+u^2}\,du = \frac{1}{4}\arctan u\,\Big|_{\frac{1}{2}}^1$$

$$= \frac{1}{4}\left[\arctan 1 - \arctan\frac{1}{2}\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{4} - \arctan\frac{1}{2}\right]$$

$u = \frac{x^2}{2}$

$du = x\,dx$

$x = 1 \Rightarrow u = \frac{1^2}{2} = \frac{1}{2}$

$x = \sqrt{2} \Rightarrow u = \frac{(\sqrt{2})^2}{2} = 1$

3. $$\int_4^7 \frac{dx}{x^2-6x+9} = \int_4^7 \frac{1}{(x-3)^2}\,dx = \int_1^4 u^{-2}\,du = -\frac{1}{u}\Big|_1^4$$

$u = x-3$

$du = dx$

$x=4 \Rightarrow u = 4-3 = 1$

$x=7 \Rightarrow u = 7-3 = 4$

$$= \left(-\frac{1}{4}\right) - \left(-\frac{1}{1}\right)$$

$$= \frac{3}{4}$$

4. $$\int_1^{\sqrt{e}} \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int_1^{\sqrt{e}} \frac{1}{\sqrt{1-(\ln x)^2}}\,\frac{1}{x}\,dx = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}}\,du =$$

$$\arcsin u \Big|_0^{\frac{1}{2}} =$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$x=1 \Rightarrow u = \ln 1 = 0$

$x=\sqrt{e} \Rightarrow u = \ln\sqrt{e} = \frac{1}{2}$

$$\arcsin\frac{1}{2} - \arcsin 0 = \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

5. $$\int_1^9 \frac{dx}{\sqrt{x}(1+x)} = \int_1^9 \frac{1}{1+x}\,\frac{1}{\sqrt{x}}\,dx = \int_1^9 \frac{1}{1+(\sqrt{x})^2}\,\frac{1}{\sqrt{x}}\,dx =$$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$x = 1 \Rightarrow u = \sqrt{1} = 1$

$x = 9 \Rightarrow u = \sqrt{9} = 3$

$$2\int_1^9 \frac{1}{1+(\sqrt{x})^2}\,\frac{1}{2\sqrt{x}}\,dx =$$

$$2\int_1^3 \frac{1}{1+u^2}\,du = 2\arctan u\Big|_1^3 = 2(\arctan 3 - \arctan 1)$$

$$= 2\left(\arctan 3 - \frac{\pi}{4}\right)$$

6. $\int_0^3 \frac{y\,dy}{\sqrt{y+1}} = \int_1^4 \frac{u-1}{\sqrt{u}}\,du = \int_1^4 \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right)du$

$u = y+1 \Rightarrow y = u-1$

$du = dy$

$y = 0 \Rightarrow u = 0+1 = 1$

$y = 3 \Rightarrow u = 3+1 = 4$

$= \int_1^4 (u^{\frac{1}{2}} - u^{-\frac{1}{2}})\,du$

$= \frac{2}{3}u^{\frac{3}{2}}\Big|_1^4 - 2u^{\frac{1}{2}}\Big|_1^4$

$= \frac{2}{3}(4^{\frac{3}{2}} - 1^{\frac{3}{2}}) - 2(4^{\frac{1}{2}} - 1^{\frac{1}{2}})$

$= \frac{2}{3}(8-1) - 2(2-1)$

$= \frac{14}{3} - 2 = \frac{8}{3}$

7. $\int_0^1 \sqrt{e^x}\,dx$

$= \int_0^1 (e^x)^{\frac{1}{2}}dx = \int_0^1 e^{\frac{x}{2}}dx = 2\int_0^1 e^{\frac{x}{2}}\left(\tfrac{1}{2}dx\right) = 2\int_0^{\frac{1}{2}} e^u\,du$

$u = \frac{x}{2}$

$du = \frac{1}{2}dx$

$x = 0 \Rightarrow u = \frac{0}{2} = 0$

$x = 1 \Rightarrow u = \frac{1}{2}$

$= 2e^u\Big|_0^{\frac{1}{2}}$

$= 2(e^{\frac{1}{2}} - e^0)$

$= 2(\sqrt{e} - 1)$