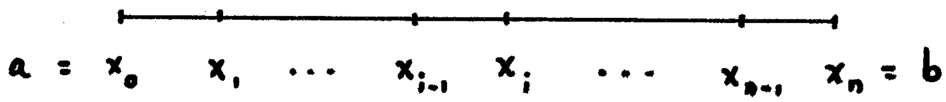
DEFINITE INTEGRALS

RECALL : RIEMANN SUM PROCEDURE

LET $f(x)$ BE CONTINUOUS ON $[a,b]$.

1. SUBDIVIDE $[a,b]$ INTO n SUBINTERVALS WITH ENDPOINTS

$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_{n-1} < x_n = b$$



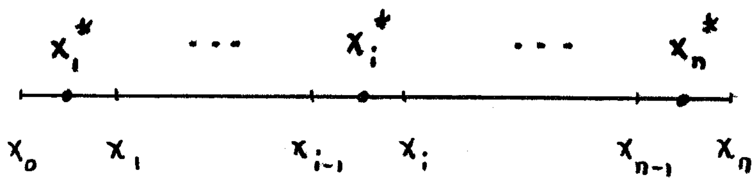
FOR EACH $i = 1, \ldots, n$ LET

$$\Delta x_i = x_i - x_{i-1}$$

AND LET

$$\Delta x_{max} = \text{LARGEST } \Delta x_i$$

2. INSIDE EACH $[x_{i-1}, x_i]$ SELECT A POINT x_i^*



EVALUATE

$$f(x_1^*), \ \ldots, \ f(x_i^*), \ \ldots, \ f(x_n^*)$$

AND COMPUTE

$$f(x_1^*)\Delta x_1, \ \ldots, \ f(x_i^*)\Delta x_i, \ \ldots, \ f(x_n^*)\Delta x_n$$

3. FORM THE RIEMANN SUM

$$f(x_1^*)\Delta x_1 + \cdots + f(x_n^*)\Delta x_n = \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

4. REPEAT # 1-3 OVER AND OVER WITH SMALLER AND SMALLER Δx_{max} AND TAKE THE LIMIT

$$\lim_{\Delta x_{max} \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

IF $f(x) \geq 0$ ON $[a,b]$, THIS LIMIT CAN BE THOUGHT OF AS THE AREA UNDER THE GRAPH OF f AND ABOVE $[a,b]$.

OTHERWISE, IT CAN BE THOUGHT OF AS THE NET SIGNED AREA BETWEEN THE GRAPH OF f AND $[a,b]$.

HOWEVER, THIS LIMIT CAN ALSO BE THOUGHT OF IN MANY OTHER WAYS AS WELL.

E.G., WE WILL SEE LATER THAT IF THE INTERVAL $[a,b]$ IS ACTUALLY A METAL WIRE WHOSE DENSITY AT ANY x IN $[a,b]$ IS $f(x)$, THEN $\lim_{\Delta x_{max} \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ IS THE MASS OF THE WIRE.

THE POINT IS THAT THIS LIMIT, BECAUSE IT MEANS MANY DIFFERENT THINGS IN DIFFERENT CONTEXTS, DESERVES A NAME AND A SYMBOL.

$\lim_{\Delta x_{max} \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$, WHEN THE LIMIT EXISTS, IS CALLED THE <u>DEFINITE INTEGRAL OF</u> $f(x)$ <u>OVER</u> $[a,b]$ AND IS DENOTED

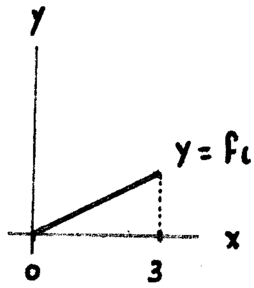
$$\int_a^b f(x)\,dx$$

a IS CALLED THE <u>LOWER</u>, AND b THE <u>UPPER LIMIT OF INTEGRATION</u>.

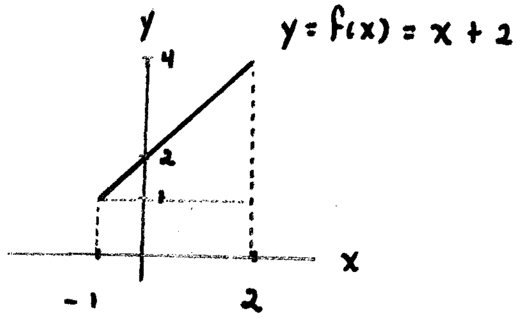
BE CAREFUL NOT TO CONFUSE $\int_a^b f(x)\,dx$ AND $\int f(x)\,dx$. THEY ARE RELATED IN A WAY WE WILL DESCRIBE NEXT TIME, BUT ARE ENTIRELY DIFFERENT TYPES OF THINGS. THE FIRST IS A <u>NUMBER</u>. THE SECOND IS A <u>COLLECTION OF FUNCTIONS</u>.

WE WILL NEED METHODS FOR EVALUATING THE NUMBERS $\int_a^b f(x)\,dx$ OTHER THAN COMPUTING THE LIMIT THAT DEFINES THEM.

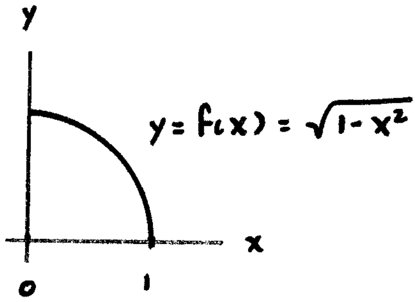
THESE METHODS GENERALLY INVOLVE ANTIDIFFERENTIATION, BUT SOME DEFINITE INTEGRALS CAN BE EVALUATED BY THINKING OF THEM AS AREAS, E.G.,



$$\int_0^3 \tfrac{1}{2}x\,dx = \tfrac{1}{2}(3)(\tfrac{1}{2}\cdot 3) = \frac{9}{4}$$



$$\int_{-1}^{2} (x+2)\,dx = \tfrac{1}{2}(3)(3) + (3)(1) = \frac{9}{2} + 3 = \frac{15}{2}$$



$$\int_0^1 \sqrt{1-x^2}\,dx = \tfrac{1}{4}(\pi\cdot 1^2) = \frac{\pi}{4}$$

A FEW DEFINITIONS AND PROPERTIES :

<u>DEFINITION</u> 1 : $\displaystyle\int_a^a f(x)\,dx = 0$

<u>DEFINITION</u> 2 : $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

E.G., $\displaystyle\int_1^0 \sqrt{1-x^2}\,dx = -\frac{\pi}{4}$

PROPERTY 1 : $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(LARGER SUMS ALSO)

PROPERTY 2 : $\displaystyle\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

(LARGER DIFFERENCES ALSO)

PROPERTY 3 : FOR ANY CONSTANT c,

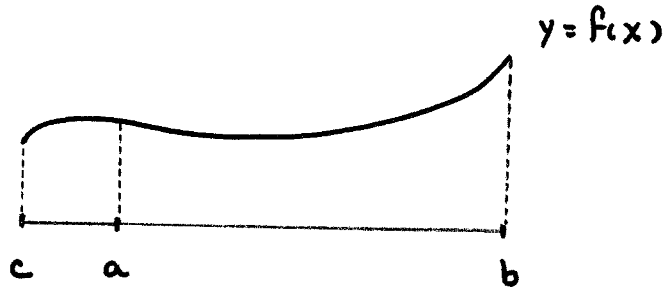
$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$$

E.G., $\displaystyle\int_0^1 7\sqrt{1-x^2}\,dx = 7\int_0^1 \sqrt{1-x^2}\,dx = 7\left(\frac{\pi}{4}\right) = \frac{7\pi}{4}$

PROPERTY 4 : $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

FOR ANY CHOICE OF C.

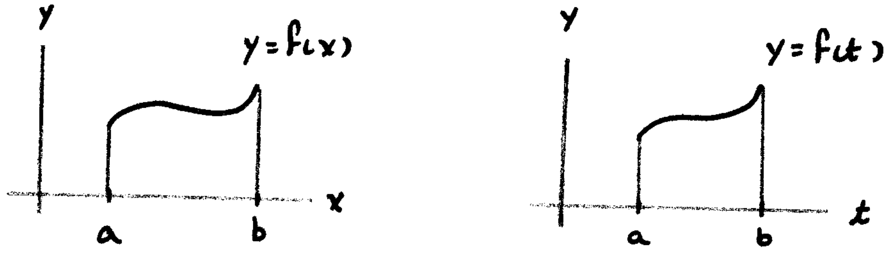
NOTE : PROPERTY 4 IS VALID EVEN IF C IS NOT IN $[a,b]$, E.G.,



$$\int_a^b = \int_c^b - \int_c^a = \int_c^b + \int_a^c$$

$$= \int_a^c + \int_c^b$$

NOTICE THAT, FOR A DEFINITE INTEGRAL, THE NAME OF THE VARIABLE IS IRRELEVANT:



$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

FOR THIS REASON THE VARIABLE IN A DEFINITE INTEGRAL IS OFTEN REFERRED TO AS A <u>DUMMY</u> <u>VARIABLE</u>.

OF COURSE, THIS IS <u>NOT</u> TRUE FOR AN INDEFINITE INTEGRAL, E.G.,

$$\int x^3\,dx = \frac{1}{4}x^4 + C$$

BUT

$$\int t^3\,dt = \frac{1}{4}t^4 + C$$

OR, EVEN BETTER YET,

$$\int 1\,dx = x + C$$

BUT

$$\int 1\,dt = t + C$$