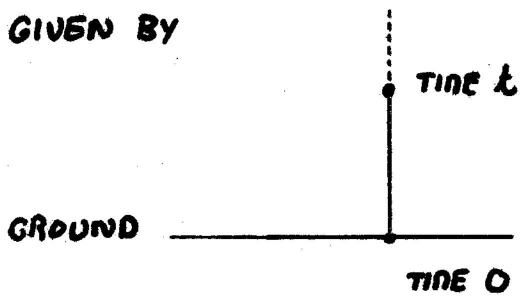
DERIVATIVES, INTEGRALS AND THE FUNDAMENTAL THEOREM OF CALCULUS

A QUICK AERIAL SURVEY OF THE TERRAIN

HERE'S A FACT FROM PHYSICS (WE WILL SEE WHY IT IS TRUE SHORTLY).

A BALL THROWN VERTICALLY UPWARD FROM THE GROUND AT AN INITIAL SPEED OF 64 FT/SEC WILL, t SEC LATER, HAVE A HEIGHT s (IN FEET) GIVEN BY

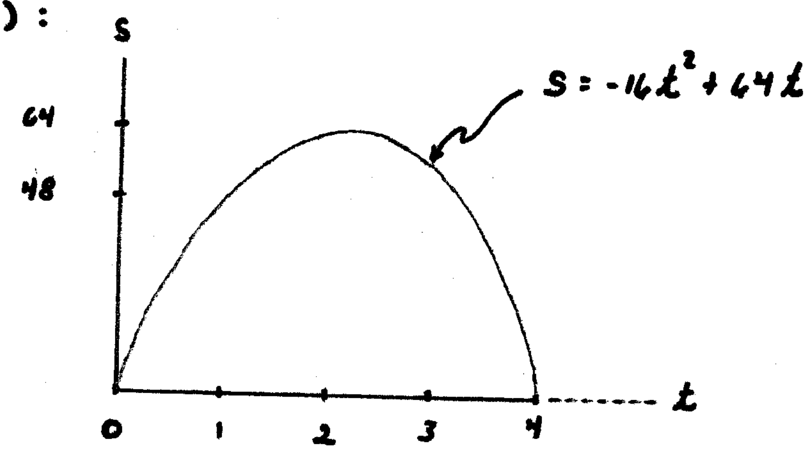


$$\boxed{s = s(t) = -16t^2 + 64t}$$

(AT LEAST UNTIL IT HITS THE GROUND AGAIN).

E.G.,
$$s(1) = -16\cdot 1^2 + 64\cdot 1 = 48$$
$$s(2) = -16\cdot 2^2 + 64\cdot 2 = 64$$
$$s(3) = -16\cdot 3^2 + 64\cdot 3 = 48$$
$$s(4) = -16\cdot 4^2 + 64\cdot 4 = 0$$

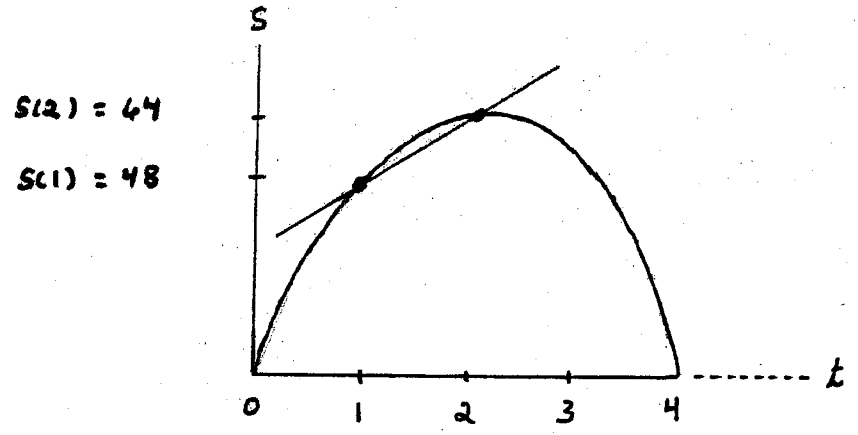
GRAPH OF $s(t)$:



NOTE : THE <u>AVERAGE</u> <u>VELOCITY</u> OF THE BALL OVER THE TIME INTERVAL FROM $t = 1$ TO $t = 2$ IS

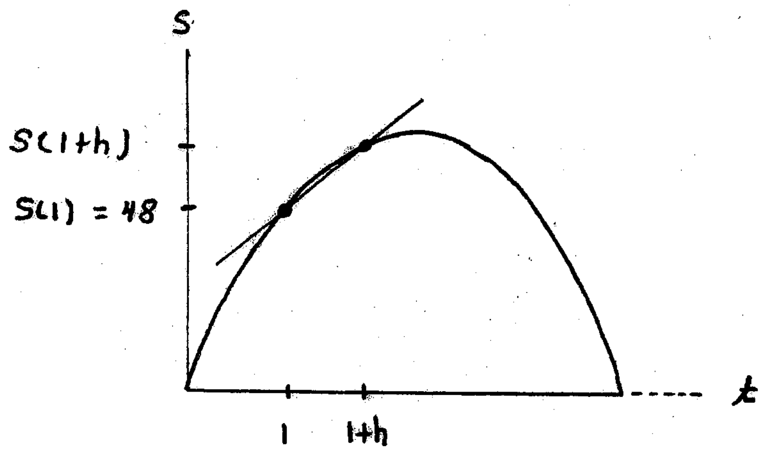
$$\frac{\text{DISPLACEMENT}}{\text{LENGTH OF TIME INTERVAL}} = \frac{S(2) - S(1)}{2 - 1} = \frac{64 - 48}{1} = 16 \text{ FT/SEC}$$

WHICH IS ALSO THE <u>SLOPE</u> OF THE STRAIGHT LINE JOINING THE POINTS $(1, S(1)) = (1, 48)$ AND $(2, S(2)) = (2, 64)$ ON THE GRAPH OF $S(t)$.



SIMILARLY, THE <u>AVERAGE</u> <u>VELOCITY</u> OF THE BALL BETWEEN $t = 2$ AND $t = 3$ IS -16 FT/SEC AND THIS IS THE <u>SLOPE</u> OF THE LINE JOINING $(2, 64)$ AND $(3, 48)$.

NEXT, LET'S COMPUTE THE <u>AVERAGE</u> <u>VELOCITY</u> OF THE BALL OVER SOME " SMALL " TIME INTERVAL FROM $t = 1$ TO $t = 1 + h$ (h IS JUST SOME "SMALL", BUT NONZERO NUMBER).



$$\frac{S(1+h)-S(1)}{h} = \frac{-16(1+h)^2+64(1+h) - 48}{h}$$

$$= \frac{-16(1+2h+h^2)+64+64h-48}{h}$$

$$= \frac{-16-32h-16h^2+64+64h-48}{h}$$

$$= \frac{-16h^2+32h}{h} = \boxed{-16h+32}$$

= SLOPE OF THE LINE JOINING THE POINTS $(1, S(1))$ AND $(1+h, S(1+h))$ ON THE GRAPH OF $S(t)$.

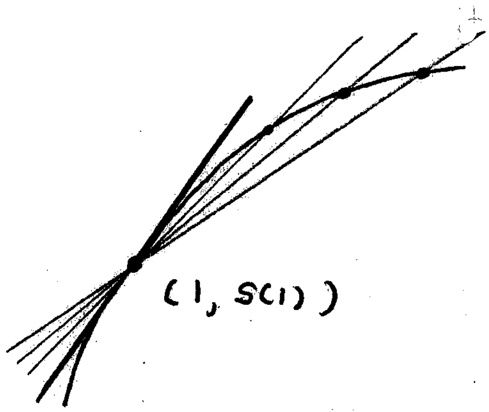
THIS IS TRUE FOR ANY NONZERO h. IF h IS VERY SMALL, THE BALL HAS VERY LITTLE TIME TO SLOW DOWN OR SPEED UP AND THIS AVERAGE VELOCITY IS VERY CLOSE TO THE BALL'S VELOCITY "AT $t = 1$"

("INSTANTANEOUS VELOCITY")

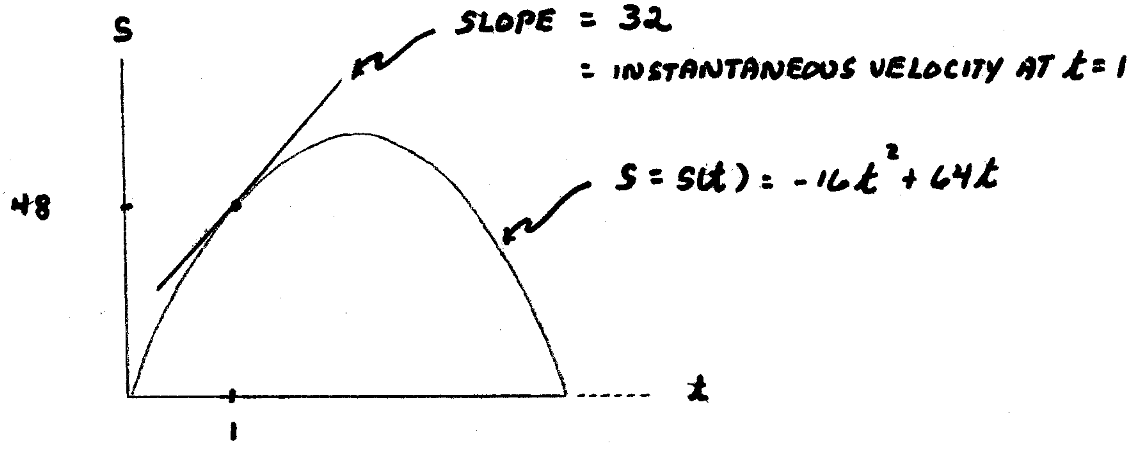
AS h IS CHOSEN CLOSER AND CLOSER TO 0, THE AVERAGE VELOCITY OVER THE TIME INTERVAL $[1, 1+h]$ GETS CLOSER AND CLOSER TO THE INSTANTANEOUS VELOCITY AT $t = 1$.

BUT IT'S CLEAR THAT AS h IS CHOSEN CLOSER AND CLOSER TO 0, $-16h + 32$ GETS CLOSER AND CLOSER TO 32 SO THE INSTANTANEOUS VELOCITY OF THE BALL AT $t = 1$ MUST BE 32 FT/SEC.

IN THE PICTURE, THE AVERAGE VELOCITY OVER $[1, 1+h]$ IS THE SLOPE OF THE LINE JOINING $(1, s(1))$ AND $(1+h, s(1+h))$ AND, AS h IS CHOSEN CLOSER AND CLOSER TO 0,



THESE GET CLOSER AND CLOSER TO THE SLOPE OF THE LINE THAT JUST "GRAZES" THE GRAPH OF $s(t)$ AT $(1, s(1))$, CALLED THE "TANGENT LINE" TO THE GRAPH AT $t = 1$.



IN THE JARGON OF CALCULUS WE SAY THAT THE INSTANTANEOUS VELOCITY AT $t = 1$ (SLOPE OF THE TANGENT LINE TO THE GRAPH OF $S(t)$ AT $t = 1$) IS "THE LIMIT, AS h GOES TO 0, OF THE AVERAGE VELOCITY OVER THE TIME INTERVAL $[1, 1+h]$ (SLOPE OF THE LINE JOINING $(1, S(1))$ AND $(1+h, S(1+h))$)"

SYMBOLICALLY, WE WILL WRITE THIS AS

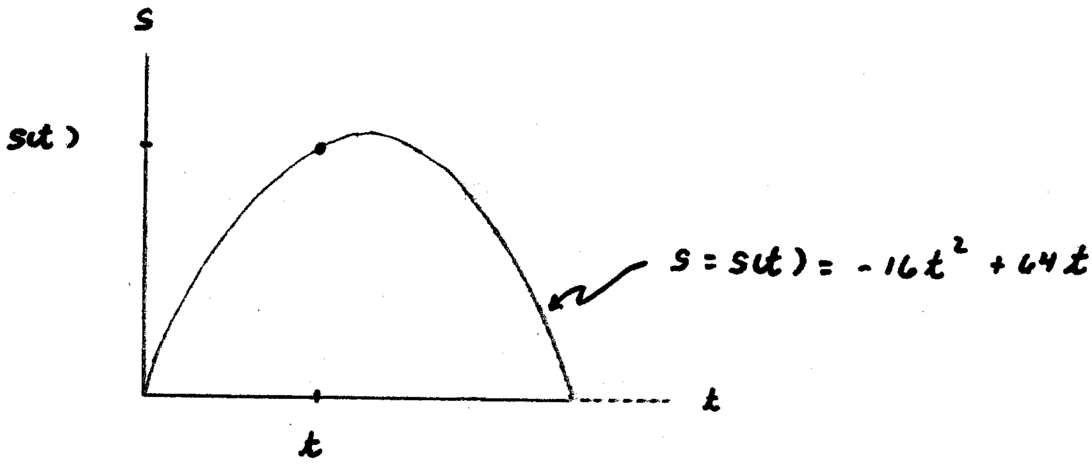
$$-16h + 32 \longrightarrow 32 \quad \text{AS} \quad h \to 0$$

OR

$$\lim_{h \to 0} (-16h + 32) = 32$$

<u>NOTE</u>: FOR THE TIME BEING WE WILL USE SYMBOLS LIKE THIS SORT OF CASUALLY, AS A CONVENIENT SHORTHAND. SOON WE WILL HAVE A MORE CAREFUL DISCUSSION OF SUCH "LIMITS".

NEXT, LET'S DO FOR AN ARBITRARY t WHAT WE JUST DID FOR $t = 1$.



INSTANTANEOUS VELOCITY AT TIME t = LIMIT, AS h GOES TO 0, OF THE AVERAGE VELOCITY OVER THE TIME INTERVAL $[t, t+h]$

$$= \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

= SLOPE OF THE TANGENT LINE TO THE GRAPH OF $s(t)$ AT t

$$\frac{s(t+h) - s(t)}{h} = \frac{[-16(t+h)^2 + 64(t+h)] - [-16t^2 + 64t]}{h}$$

$$= \frac{-16t^2 - 32th - 16h^2 + 64t + 64h + 16t^2 - 64t}{h}$$

$$= \frac{-32th - 16h^2 + 64h}{h}$$

$$= -32t - 16h + 64$$

AS $h \to 0$, $\quad -32t - 16h + 64 \longrightarrow -32t + 64$ SO

$$\lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = -32t + 64$$

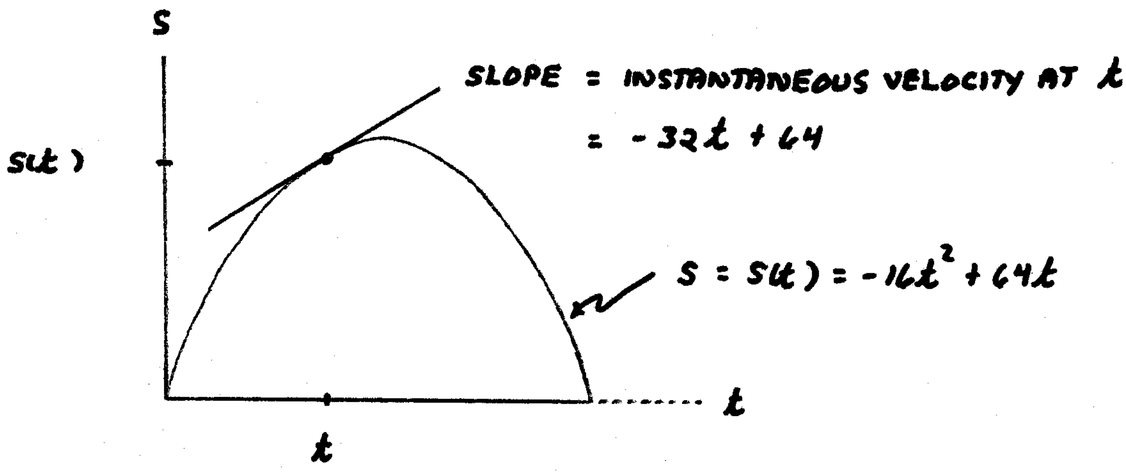
E.G., INSTANTANEOUS VELOCITY (SLOPE OF TANGENT LINE) AT

$t = 1$ IS $-32(1) + 64 = 32$ (AS BEFORE)

$t = \frac{1}{2}$ IS $-32(\frac{1}{2}) + 64 = 48$

$t = 3$ IS $-32(3) + 64 = -32$

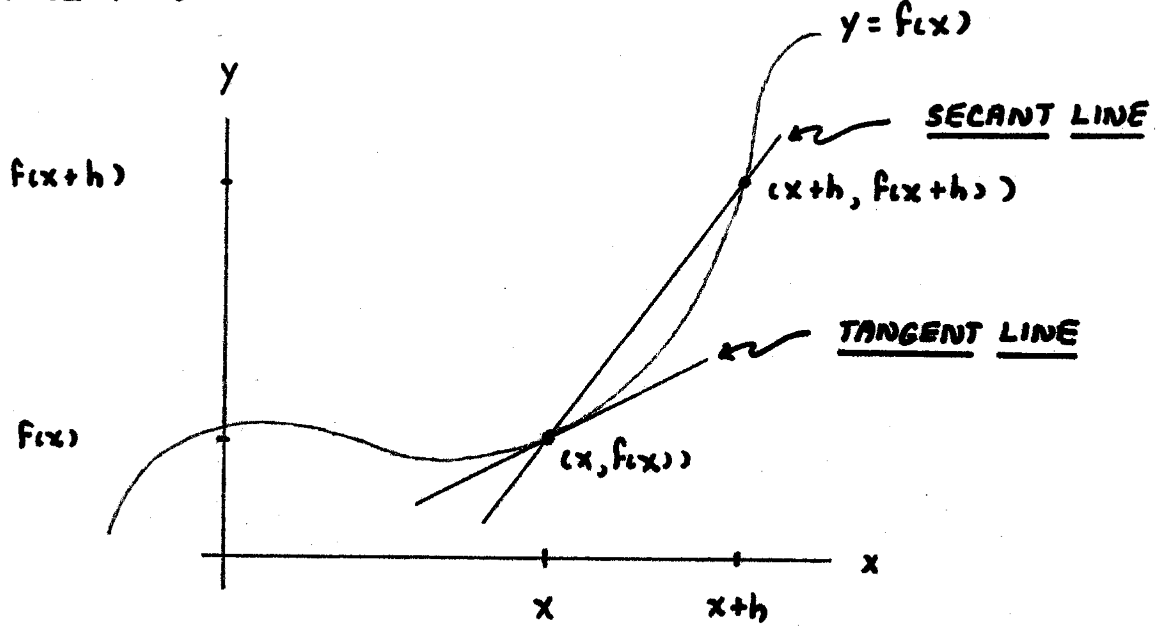
$t = 4$ IS $-32(4) + 64 = -64$



WE WILL COME BACK TO THIS EXAMPLE (AND TO MANY OTHER EXAMPLES OF "MOTION ALONG A STRAIGHT LINE") SOON. FIRST, WE <u>GENERALIZE</u>.

VELOCITY (EITHER AVERAGE OR INSTANTANEOUS) IS THE <u>RATE</u> <u>OF</u> <u>CHANGE</u> <u>OF</u> <u>POSITION</u>.

NOW WE'LL DO FOR AN ARBITRARY FUNCTION $y = f(x)$ AT A POINT x IN ITS DOMAIN WHAT WE JUST DID FOR THE POSITION FUNCTION OF THE BALL AT t



<u>AVERAGE RATE OF CHANGE OF $f(x)$ OVER THE INTERVAL</u> $[x, x+h]$

$$= \frac{f(x+h) - f(x)}{h}$$

= SLOPE OF THE SECANT LINE THROUGH THE POINTS $(x, f(x))$ AND $(x+h, f(x+h))$ ON THE GRAPH OF $f(x)$

<u>INSTANTANEOUS RATE OF CHANGE OF $f(x)$ AT</u> x

= WHAT THE AVERAGE RATE OF CHANGE OVER $[x, x+h]$ APPROACHES AS $h \to 0$

= SLOPE OF TANGENT LINE AT $(x, f(x))$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

LET'S DO AN <u>EXAMPLE</u> : LET $f(x) = x^3$ AND FIND

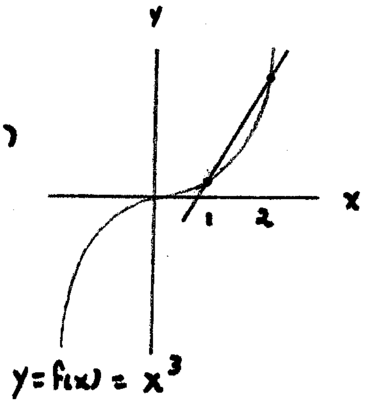
(a) THE AVERAGE RATE OF CHANGE OF $f(x)$ OVER THE INTERVAL $[1,2]$.

(b) THE AVERAGE RATE OF CHANGE OF $f(x)$ OVER THE INTERVAL $[1, 1+h]$.

(c) THE INSTANTANEOUS RATE OF CHANGE OF $f(x)$ AT $x = 1$.

(d) THE INSTANTANEOUS RATE OF CHANGE OF $f(x)$ AT AN ARBITRARY x.

(a)

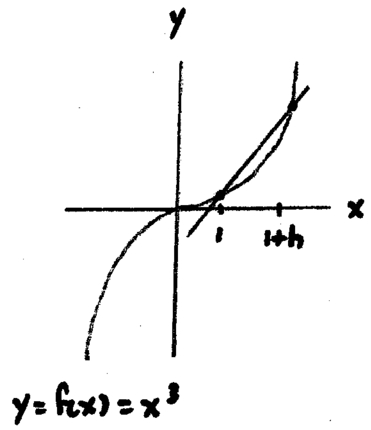


AVERAGE RATE OF CHANGE OF $f(x) = x^3$ OVER THE INTERVAL $[1,2]$

= SLOPE OF THE SECANT LINE JOINING $(1, f(1))$ AND $(2, f(2))$

$$= \frac{f(2) - f(1)}{2-1} = \frac{2^3 - 1^3}{1} = 7$$

(b)



AVERAGE RATE OF CHANGE OF $f(x) = x^3$ OVER THE INTERVAL $[1, 1+h]$

= SLOPE OF THE SECANT LINE JOINING $(1, f(1))$ AND $(1+h, f(1+h))$

$$= \frac{f(1+h) - f(1)}{h} = \frac{(1+h)^3 - 1^3}{h}$$

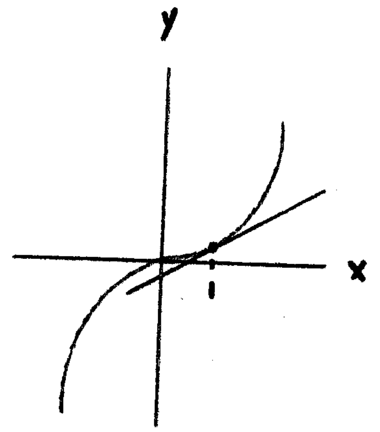
$$= \frac{1 + 3h + 3h^2 + h^3 - 1}{h} = \frac{3h + 3h^2 + h^3}{h}$$

$$= 3 + 3h + h^2$$

E.G., THE AVERAGE RATE OF CHANGE OF $f(x) = x^3$ OVER THE INTERVAL $[1, 1.1]$ $(h = 0.1)$ IS

$$3 + 3(0.1) + (0.1)^2 = 3.31$$

(c)



$y = f(x) = x^3$

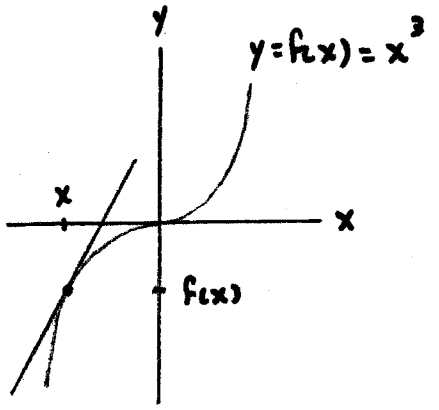
INSTANTANEOUS RATE OF CHANGE OF $f(x) = x^3$ AT $x = 1$

= SLOPE OF THE TANGENT LINE TO THE GRAPH OF $y = f(x) = x^3$ AT $(1, f(1))$

= WHAT $3 + 3h + h^2$ GETS CLOSE TO AS $h \to 0$

$$= \lim_{h \to 0} (3 + 3h + h^2) = 3$$

(d)



INSTANTANEOUS RATE OF CHANGE OF $f(x) = x^3$ AT x

= SLOPE OF THE TANGENT LINE TO THE GRAPH OF $y = f(x) = x^3$ AT THE POINT $(x, f(x))$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

BUT

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^3-x^3}{h}$$

$$= \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$

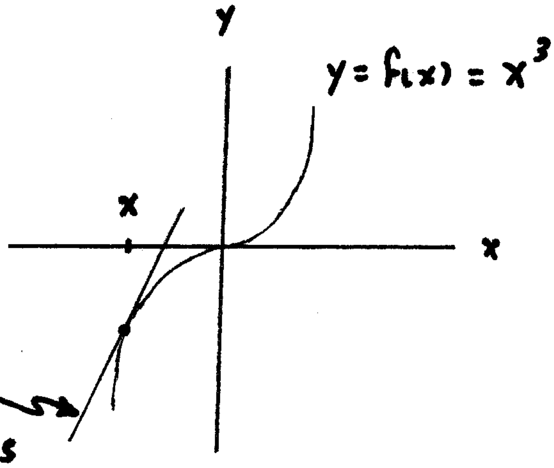
$$= \frac{3x^2h+3xh^2+h^3}{h}$$

$$= 3x^2+3xh+h^2$$

AS $h \to 0$ THIS APPROACHES $3x^2$ SO

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(3x^2+3xh+h^2)$$

$$= 3x^2$$



$3x^2$ = SLOPE

= INSTANTANEOUS
RATE OF CHANGE
OF $f(x) = x^3$
AT x

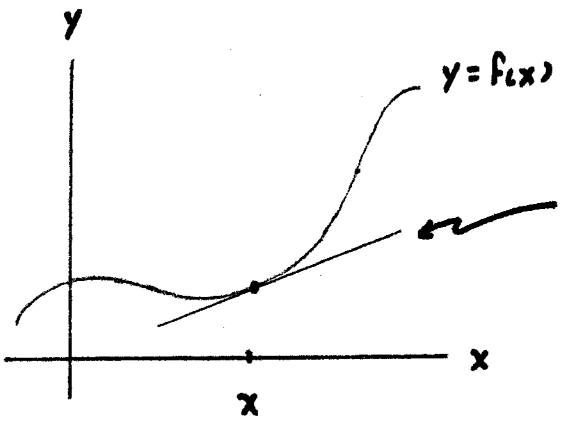
E.G., THE INSTANTANEOUS RATE OF CHANGE OF $f(x) = x^3$ AT $x = -\frac{1}{2}$ IS $3(-\frac{1}{2})^2 = \frac{3}{4}$.

ASSIGNMENT # 1 : READ SECTION 3.1 OF THE TEXT AND WORK THE FOLLOWING PROBLEMS IN EXERCISE SET 3.1 (PAGE 176) :

1, 3, 5, 7, 8, 9, 11, 13, 15, 17, 21, 23.

NOTE FOR FUTURE REFERENCE : I EXPECT YOU TO READ THE TEXT THOROUGHLY. IT CONTAINS A GREAT DEAL OF INFORMATION; I CANNOT COVER ALL OF IT IN LECTURE ; YOU ARE, HOWEVER, RESPONSIBLE FOR ALL OF IT.

THE FUNCTION $(3x^2)$ WHICH ASSIGNS TO EACH x THE INSTANTANEOUS RATE OF CHANGE OF $f(x) = x^3$ AT x IS CALLED THE " DERIVATIVE " OF $f(x)$. IN GENERAL,



SLOPE OF THE TANGENT LINE TO THE GRAPH OF $f(x)$ AT x

$=$ INSTANTANEOUS RATE OF CHANGE OF $f(x)$ AT x

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$=$ THE DERIVATIVE OF f AT x

$= f'(x)$

<u>NOTE</u> : OTHER SYMBOLS FOR THE DERIVATIVE OF $y = f(x)$, BESIDES $f'(x)$, INCLUDE y', $\frac{dy}{dx}$, $\frac{d}{dx}(f(x))$, E.G.,

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2$$

$$y = x^3 \Rightarrow y' = 3x^2$$

$$y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$$

$$\frac{d}{dx}(x^3) = 3x^2$$

OF COURSE, IF THE NAMES OF THE VARIABLES CHANGE, THE NOTATION CHANGES ACCORDINGLY, E.G.,

$$s(t) = -16t^2 + 64t \Rightarrow s'(t) = -32t + 64 \quad \text{(OR}$$

$$\frac{ds}{dt} = -32t + 64, \text{ ETC.)}$$

BEFORE WE ARE THROUGH WE WILL BE ABLE TO QUICKLY COMPUTE THE DERIVATIVE OF A HUGE NUMBER OF RATHER COMPLICATED FUNCTIONS. FOR THE TIME BEING WE WILL SETTLE FOR A FEW SIMPLE EXAMPLES.

<u>EXAMPLES</u> :

1. $f(x) = x^3 \Rightarrow f'(x) = 3x^2$ (SHOWN ON PAGES 10-11)

2. $s(t) = -16t^2 + 64t \Rightarrow s'(t) = -32t + 64$ (SHOWN ON PAGES 6-7)

3. THE DERIVATIVE OF A CONSTANT FUNCTION IS ZERO

($f(x) = c$ FOR EVERY x $\Rightarrow$ $f'(x) = 0$ FOR EVERY x)

<u>REASON</u> : THE GRAPH IS A HORIZONTAL STRAIGHT LINE. EACH OF ITS TANGENT LINES IS THE SAME HORIZONTAL STRAIGHT LINE AND SO ITS SLOPE IS 0.

4. $f(x) = x^n$ (FOR ANY POSITIVE INTEGER n)

$$\Rightarrow \quad f'(x) = n x^{n-1}$$

<u>REASON</u> : $$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^n - x^n}{h}$$

TO SIMPLIFY THIS NOTICE THAT

$$A^2 - B^2 = (A-B)(A+B)$$
$$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$$
$$A^4 - B^4 = (A-B)(A^3 + A^2B + AB^2 + B^3)$$
$$\vdots$$
$$A^n - B^n = (A-B)(A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1})$$

(IF YOU DON'T BELIEVE ME, JUST MULTIPLY OUT THE RIGHT-HAND SIDE AND WATCH EVERYTHING CANCEL)

WITH $A = x+h$ AND $B = x$ THE LAST ONE GIVES US

$$(x+h)^n - x^n = \big((x+h)-x\big)\big((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}\big)$$

$$= h\left((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}\right)$$

THUS,

$$\frac{(x+h)^n - x^n}{h} = (x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}$$

SO

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \left((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}\right)$$

$$= x^{n-1} + x^{n-2}x + \cdots + xx^{n-2} + x^{n-1}$$

$$= x^{n-1} + x^{n-1} + \cdots + x^{n-1} + x^{n-1}$$

$$= n\,x^{n-1}$$

E.G., $f(x) = x^7 \Rightarrow f'(x) = 7x^6$

$y = x^{12} \Rightarrow \frac{dy}{dx} = 12x^{11}$

$\frac{d}{dx}(x^1) = 1x^0 = 1$

<u>NOTE</u>: THE FORMULA $(x^n)' = nx^{n-1}$ IS ACTUALLY TRUE FOR ANY REAL NUMBER n, BUT THE PROOF IS MORE DIFFICULT.

THUS, COMPUTING THE DERIVATIVE ("DIFFERENTIATING") POWERS OF x (OR t, OR WHATEVER THE INDEPENDENT VARIABLE IS CALLED) IS EASY.

WITH A LITTLE MORE WORK (WHICH WE WILL GO THROUGH WHEN WE CONSIDER "DIFFERENTIATION TECHNIQUES" MORE CAREFULLY) ONE CAN SHOW THAT POLYNOMIALS

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

ARE JUST AS EASY:

$$f'(x) = a_n (n x^{n-1}) + a_{n-1}((n-1) x^{n-2}) + \cdots + a_2 (2x) + a_1 (1) + 0$$

$$= n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + 2a_2 x + a_1$$

<u>EXAMPLES</u>:

1. $f(x) = 3x^2 - 2x + 5$

   $f'(x) = 3(2x) - 2(1) + 0 = 6x - 2$

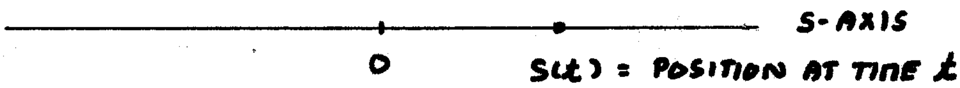
2. $s(t) = -16t^2 + 64t$

   $\dfrac{ds}{dt} = -16(2t) + 64(1) = -32t + 64$

3. $\dfrac{d}{dx}(5x^3 - 2x^2 + 7x - 13) = 5(3x^2) - 2(2x) + 7(1) - 0$

   $= 15x^2 - 4x + 7$

NOW NOTICE THAT IF WE CONSIDER AN OBJECT MOVING ALONG A STRAIGHT LINE (E.G., OUR BALL THROWN VERTICALLY, A ROCK FALLING VERTICALLY, A CAR MOVING ALONG A STRAIGHT HIGHWAY, A MASS ATTACHED TO A VIBRATING SPRING, ETC.) WE CAN CALL THE STRAIGHT LINE THE "S-AXIS" AND DESCRIBE THE MOTION BY THE "POSITION FUNCTION"



THEN THE DERIVATIVE OF $s(t)$ (RATE OF CHANGE OF POSITION) IS THE OBJECT'S "VELOCITY"

$$v(t) = \frac{ds}{dt}$$

THE DERIVATIVE OF $v(t)$ (RATE OF CHANGE OF VELOCITY) IS CALLED THE OBJECT'S "ACCELERATION"

$$a(t) = \frac{dv}{dt} .$$

$a(t)$ IS THE "DERIVATIVE OF THE DERIVATIVE" OF $s(t)$.

IN GENERAL, IF $y = f(x)$ IS A FUNCTION WITH DERIVATIVE

$$y' = f'(x) = \frac{dy}{dx}$$

THEN THE DERIVATIVE OF THIS DERIVATIVE IS CALLED THE

SECOND DERIVATIVE OF THE FUNCTION AND IS WRITTEN IN ONE OF THE FOLLOWING WAYS:

$$y'' = f''(x) = \frac{d^2y}{dx^2}$$

EXAMPLE: $y = f(x) = \frac{1}{3}x^7 - 2x^3 + 17x + 2$

$$f'(x) = \frac{1}{3}(7x^6) - 2(3x^2) + 17(1) + 0$$

$$= \frac{7}{3}x^6 - 6x^2 + 17$$

$$f''(x) = \frac{7}{3}(6x^5) - 6(2x) + 0$$

$$= 14x^5 - 12x$$

ASSIGNMENT #2: READ SECTION 3.2 THROUGH EXAMPLE 5 (PAGE 181) AND SECTION 3.3 OF THE TEXT AND WORK THE FOLLOWING PROBLEMS.

EXERCISE SET 3.2 (PAGE 187): # 1, 3, 5, 7, 9, 21, 37, 39

EXERCISE SET 3.3 (PAGE 196): # 1, 3, 5, 7, 13, 15, 17, 19, 21, 23, 29, 31, 33, 37(a), (b), (d), 43, 47, 49, 51, 54

STATUS REPORT: AT THIS POINT, IF WE ARE GIVEN THE POSITION FUNCTION $s(t)$ FOR AN OBJECT MOVING ALONG A STRAIGHT LINE, E.G.,

$$s(t) = -16t^2 + 64t$$

THEN WE CAN COMPUTE THE VELOCITY

$$v(t) = s'(t) = -16(2t) + 64(1)$$
$$= -32t + 64$$

AND ACCELERATION

$$a(t) = v'(t) = s''(t) = -32(1) + 0$$
$$= -32$$

BY FINDING DERIVATIVES.

BUT SUPPOSE THINGS ARE THE OTHER WAY AROUND. SUPPOSE, SAY, WE KNOW THE ACCELERATION $a(t) = s''(t)$. HOW DO WE "UNDO" THE DERIVATIVES TO FIND THE POSITION $s(t)$?

NOTE : THIS IS, BY FAR, THE MORE COMMON SITUATION. PHYSICS IS INTERESTED IN DESCRIBING THE MOTION (I.E., POSITION) OF AN OBJECT KNOWING THE FORCES THAT ARE ACTING ON IT. ACCORDING TO NEWTON'S SECOND LAW

$$F = ma$$

KNOWING THE FORCE F AMOUNTS TO KNOWING THE ACCELERATION a. THE PROBLEM THEN IS TO DIG $s(t)$ OUT OF $a(t)$.

GENERAL DEFINITION :

GIVEN A FUNCTION $f(x)$, AN <u>ANTIDERIVATIVE</u> FOR $f(x)$ IS ANOTHER FUNCTION $F(x)$ SATISFYING

$$F'(x) = f(x)$$

THUS, POSITION $s(t)$ IS AN ANTIDERIVATIVE FOR VELOCITY $v(t)$, AND VELOCITY $v(t)$ IS AN ANTIDERIVATIVE FOR ACCELERATION $a(t)$.

<u>EXAMPLES</u> : IF $f(x) = 3x^2$, THEN ALL OF THE FOLLOWING ARE ANTIDERIVATIVES FOR $f(x)$:

$$x^3, \quad x^3+5, \quad x^3 - \frac{\pi^2}{6}, \ \dots$$

$$x^3 + C \quad \text{FOR ANY CONSTANT } C$$

$$f(x) \xrightarrow{\text{DIFFERENTIATION}} f'(x)$$

$$f(x) \xrightarrow[\text{INDEFINITE INTEGRATION}]{\text{ANTIDIFFERENTIATION}} F(x) \text{ FOR WHICH } F'(x) = f(x)$$

INTEGRAL NOTATION :

$$\int f(x)\,dx \;=\; \underline{\text{INDEFINITE}}\ \underline{\text{INTEGRAL}}\ \text{OF } f(x)$$

$$= \text{COLLECTION OF ALL ANTIDERIVATIVES FOR } f(x)$$

$$= F(x) + C$$

WHERE $F(x)$ IS ANY FUNCTION SATISFYING $F'(x) = f(x)$ AND C IS AN ARBITRARY CONSTANT (THE <u>CONSTANT</u> <u>OF</u> <u>INTEGRATION</u>)

E.G.,

$$\int 3x^2\,dx = x^3 + C$$

THE SYMBOLS :

$$\underbrace{\int}_{\text{INTEGRAL SIGN}} \underbrace{f(x)}_{\text{INTEGRAND}} \overbrace{dx}^{\text{DIFFERENTIAL}}$$

FINDING ANTIDERIVATIVES (I.E., INTEGRATION) CAN BE A VERY TRICKY BUSINESS. FOR THE TIME BEING WE WILL NEED ONLY A FEW SIMPLE EXAMPLES.

EXAMPLES :

1. $(x^3)' = 3x^2$ SO

$$\int 3x^2 dx = x^3 + C$$

2. $(\frac{1}{3}x^3)' = \frac{1}{3}(3x^2) = x^2$ SO

$$\int x^2 dx = \frac{1}{3}x^3 + C$$

3. $(\frac{1}{7}x^7)' = \frac{1}{7}(7x^6) = x^6$ SO

$$\int x^6 dx = \frac{1}{7}x^7 + C$$

4. FOR ANY $n = 0, 1, 2, \ldots$,

$$\left(\frac{1}{n+1}x^{n+1}\right)' = \frac{1}{n+1}\left((n+1)x^n\right) = x^n$$

SO

$$\boxed{\int x^n dx = \frac{1}{n+1}x^{n+1} + C}$$

NOTE : THIS FORMULA IS ACTUALLY TRUE FOR ANY $n \neq -1$.

5. $\int x\,dx = \int x^1 dx = \frac{1}{1+1}x^{1+1} + C = \frac{1}{2}x^2 + C$

6. $\int 1\,dx = \int x^0 dx = \frac{1}{0+1}x^{0+1} + C = x + C$

IF THE NAMES OF THE VARIABLES CHANGE, THE NOTATION CHANGES ACCORDINGLY.

7. $\int t^{32}\,dt = \frac{1}{32+1} t^{32+1} + C = \frac{1}{33} t^{33} + C$

BECAUSE THE DERIVATIVE OF A POLYNOMIAL CAN BE COMPUTED " TERM-BY-TERM" , SO CAN ITS INTEGRAL.

8. $\int (5x^3 + x^2 + 3x + 1)\,dx = 5(\frac{1}{4}x^4) + \frac{1}{3}x^3 + 3(\frac{1}{2}x^2) + x + C$

$$= \frac{5}{4}x^4 + \frac{1}{3}x^3 + \frac{3}{2}x^2 + x + C$$

9. $\int (2t^2+1)(t^3+3)\,dt = \int (2t^5 + t^3 + 6t^2 + 3)\,dt$

$$= 2(\frac{1}{6}t^6) + \frac{1}{4}t^4 + 6(\frac{1}{3}t^3) + 3(t) + C$$

$$= \frac{1}{3}t^6 + \frac{1}{4}t^4 + 2t^3 + 3t + C$$

10. SUPPOSE AN OBJECT MOVES ALONG THE S-AXIS WITH VELOCITY $v(t) = 4t^3 - 3t^2$. IF THE OBJECT IS INITIALLY AT $S = 3$, WHAT IS ITS POSITION AT ANY TIME t ?

WE WANT $S(t)$. SINCE $v(t) = S'(t)$, $S(t)$ IS AN ANTIDERIVATIVE FOR $v(t)$.

HERE ARE ALL OF THE ANTIDERIVATIVES FOR $v(t)$:

$$\int v(t)\,dt = \int (4t^3 - 3t^2)\,dt = 4(\frac{1}{4}t^4) - 3(\frac{1}{3}t^3) + C$$

$$= t^4 - t^3 + C$$

S(t) MUST BE ONE OF THESE, I.E., FOR SOME CHOICE OF C,

$$s(t) = t^4 - t^3 + C$$

WHICH C ?

INITIALLY (I.E., AT $t=0$), S = 3 SO

$$3 = 0^4 - 0^3 + C$$

$$3 = C$$

AND THEREFORE

$$s(t) = t^4 - t^3 + C$$

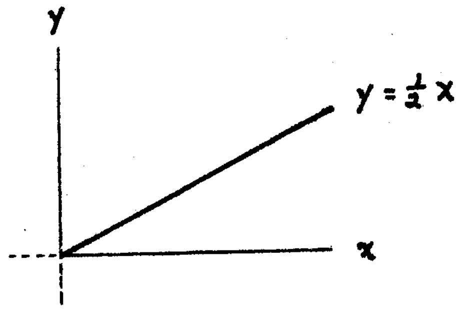
ASSIGNMENT #3: READ SECTION 6.2 (PAGES 355 - 356 AND EXAMPLE 5 ONLY) AND WORK THE FOLLOWING PROBLEMS.

EXERCISE SET 6.2 (PAGE 363): # 9(a), 15, 16, 18, 40, 47

EXERCISE SET 6.7 (PAGE 416): # 5(a), 7(a)

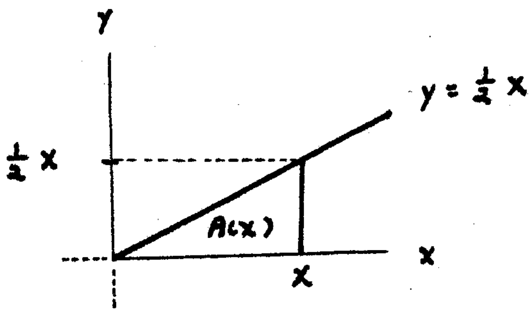
WE HAVE NOW ARRIVED AT A CRUCIAL POINT. WE KNOW A LITTLE ABOUT DERIVATIVES AND A LITTLE ABOUT ANTIDERIVATIVES (INDEFINITE INTEGRALS) AND A LITTLE ABOUT WHERE THEY CAME FROM. IN SOME SENSE, HOWEVER, THE REAL SIGNIFICANCE OF THESE THINGS LIES IN THEIR CONNECTION WITH SOMETHING TOTALLY UNEXPECTED THAT WE NOW INVESTIGATE.

THINK ABOUT THE FUNCTION $f(x) = \frac{1}{2}x$ ON THE INTERVAL $[0, \infty)$.



FOR ANY X IN $[0, \infty)$ LET

$A(x)$ = AREA UNDER THE GRAPH OF $f(x) = \frac{1}{2}x$ FROM 0 TO X



$$A(x) = \tfrac{1}{2}(\text{BASE})(\text{HEIGHT}) = \tfrac{1}{2}(x)(\tfrac{1}{2}x)$$

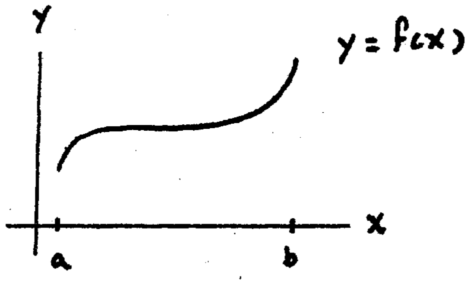
$$A(x) = \tfrac{1}{4}x^2$$

NOW, NOTICE SOMETHING PECULIAR :

$$A'(x) = (\tfrac{1}{4}x^2)' = \tfrac{1}{2}x = f(x)$$

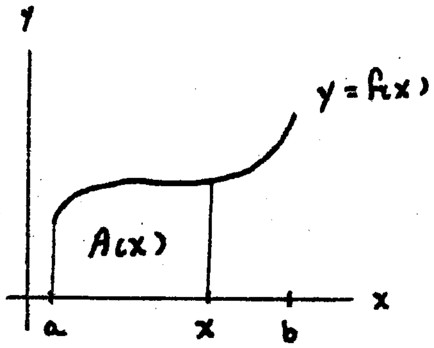
THE MOST "PECULIAR" THING ABOUT THIS IS THAT IT ISN'T REALLY PECULIAR AT ALL. IT HAPPENS ESSENTIALLY ALL THE TIME, AS I WILL NOW SHOW YOU :

LET $f(x)$ BE DEFINED AND ≥ 0 ON THE INTERVAL $[a,b]$.

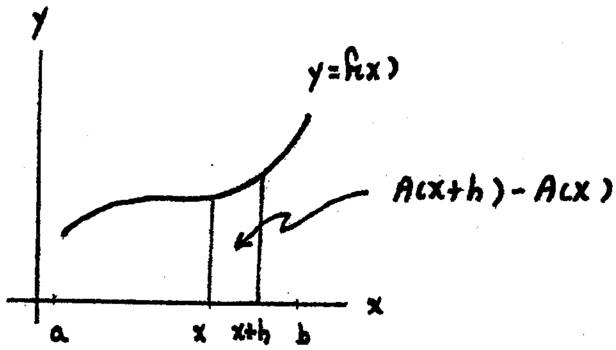


FOR ANY x IN $[a,b]$ LET

$A(x)$ = AREA UNDER THE GRAPH OF $f(x)$ FROM a TO x



NOW WE'LL "COMPUTE" THE DERIVATIVE $A'(x)$ GEOMETRICALLY.



$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

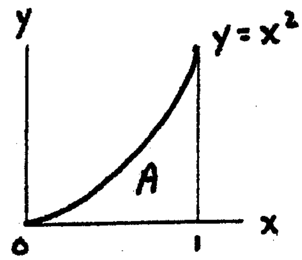
$A(x+h) - A(x) \approx f(x)h$ AND THE APPROXIMATION IMPROVES AS $h \to 0$

$$A'(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x)h}{h}$$
$$= \lim_{h\to 0} f(x)$$
$$= f(x)$$

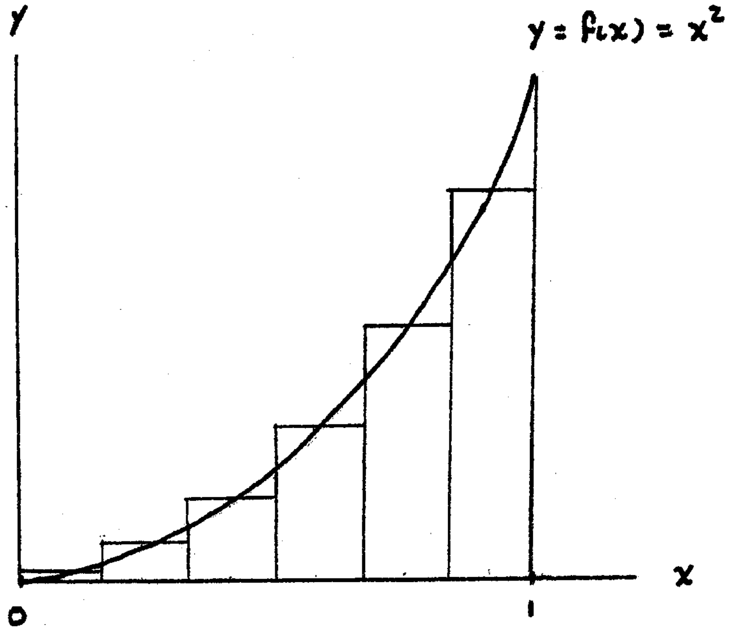
THE DERIVATIVE OF THE "AREA FUNCTION" OF $f(x)$ IS $f(x)$ ITSELF

BECAUSE OF THIS CLOSE CONNECTION BETWEEN AREAS AND DERIVATIVES WE WILL SPEND SOME TIME DESCRIBING A DIRECT APPROACH TO THE PROBLEM OF COMPUTING SUCH AREAS.

<u>ILLUSTRATION</u> : COMPUTE THE AREA UNDER THE GRAPH OF $f(x) = x^2$ FROM $x = 0$ TO $x = 1$



THE IDEA IS VERY SIMPLE : BEGIN BY APPROXIMATING THE REGION WHOSE AREA WE WANT BY RECTANGULAR STRIPS.



THE SUM OF THE RECTANGULAR AREAS IS AN APPROXIMATION TO A.

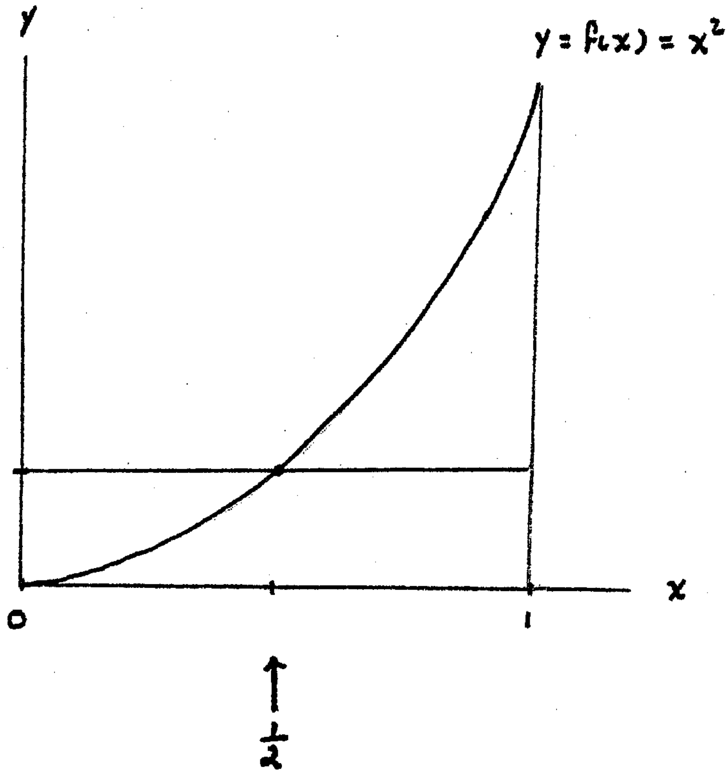
CHOOSING MORE AND MORE, SKINNIER AND SKINNIER RECTANGLES GIVES BETTER AND BETTER APPROXIMATIONS.

THE LIMIT OF THESE APPROXIMATIONS AS THE NUMBER OF RECTANGLES $\to \infty$ AND THEIR WIDTHS $\to 0$ GIVES PRECISELY THE AREA A WE WANT.

TO HAVE A SYSTEMATIC WAY OF BUILDING THESE RECTANGLES WE WILL (FOR THIS EXAMPLE)

1. CHOOSE ALL OF THEIR BASES TO HAVE THE SAME LENGTH.
2. CHOOSE THE HEIGHT OF EACH TO BE THE VALUE OF THE FUNCTION AT THE MIDPOINT OF THE BASE (OTHER COMMON CHOICES ARE THE VALUES OF THE FUNCTION AT THE LEFT OR RIGHT HAND ENDPOINTS OF THE BASE).

$\boxed{\text{1 RECTANGLE}}$

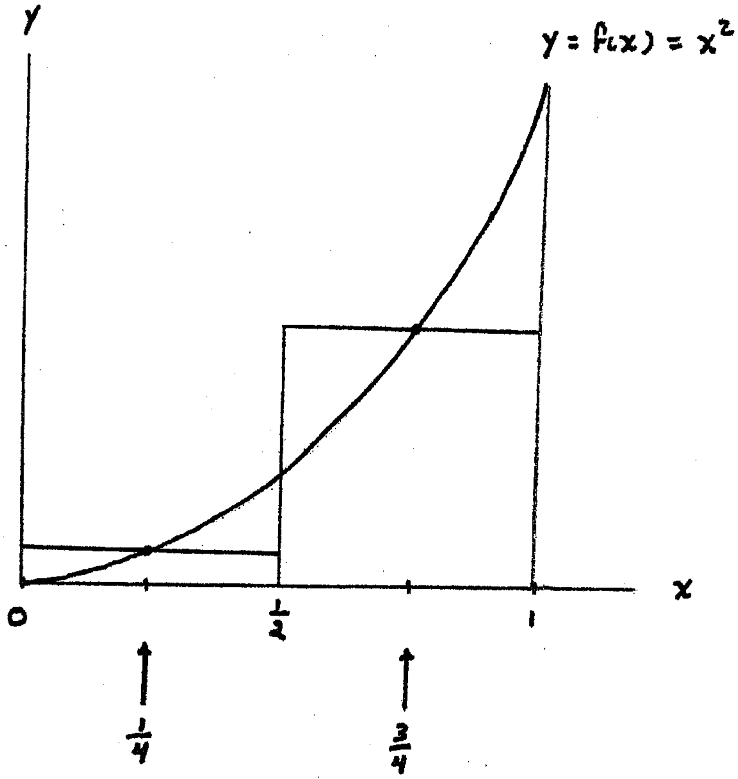


BASE : $[0,1]$

MIDPOINT : $\frac{1}{2}$

$$A \approx f(\tfrac{1}{2})\cdot 1 = (\tfrac{1}{2})^2\cdot 1 = \tfrac{1}{4}$$

$\boxed{\text{2 RECTANGLES}}$



BASES : $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$

MIDPOINTS : $\frac{1}{4}$, $\frac{3}{4}$

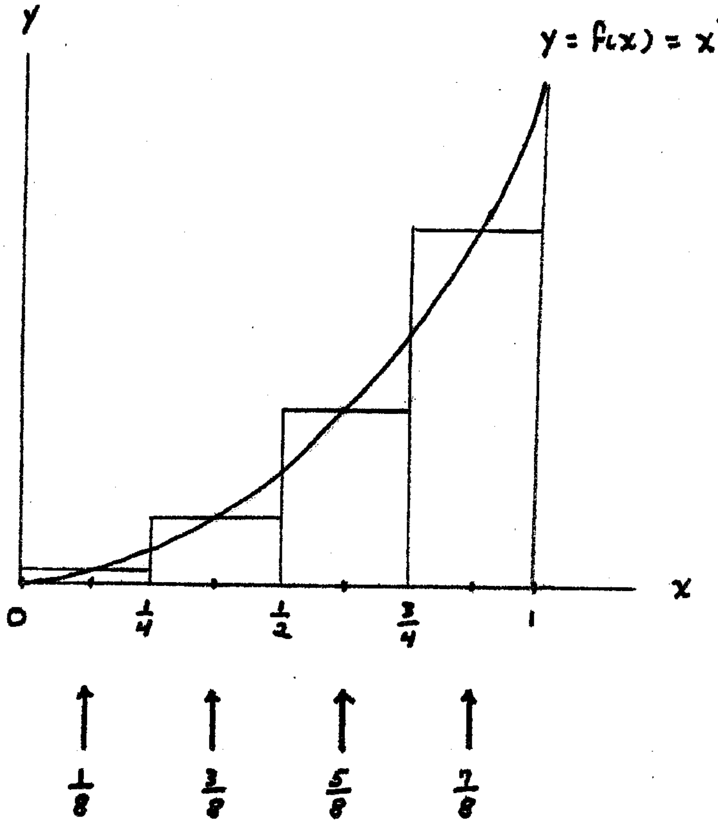
$$A \approx f(\tfrac{1}{4}) \cdot \tfrac{1}{2} + f(\tfrac{3}{4}) \cdot \tfrac{1}{2} = (\tfrac{1}{4})^2 \cdot \tfrac{1}{2} + (\tfrac{3}{4})^2 \cdot \tfrac{1}{2}$$

$$= (\tfrac{1}{16} + \tfrac{9}{16}) \cdot \tfrac{1}{2}$$

$$= \tfrac{5}{16}$$

$$= 0.3125$$

$\boxed{\text{4 RECTANGLES}}$



BASES : $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$, $[\frac{3}{4}, 1]$

MIDPOINTS : $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$

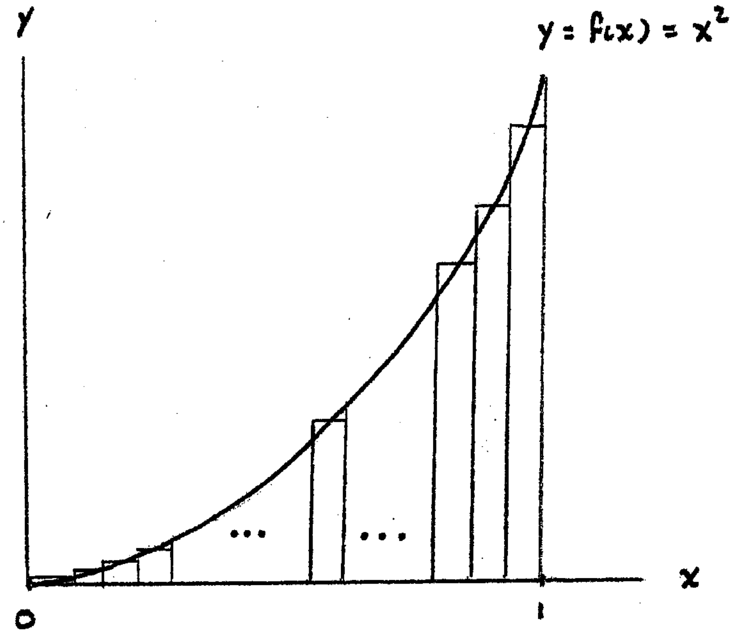
$$A \approx f(\tfrac{1}{8})\cdot\tfrac{1}{4} + f(\tfrac{3}{8})\cdot\tfrac{1}{4} + f(\tfrac{5}{8})\cdot\tfrac{1}{4} + f(\tfrac{7}{8})\cdot\tfrac{1}{4} =$$

$$(\tfrac{1}{8})^2\cdot\tfrac{1}{4} + (\tfrac{3}{8})^2\cdot\tfrac{1}{4} + (\tfrac{5}{8})^2\cdot\tfrac{1}{4} + (\tfrac{7}{8})^2\cdot\tfrac{1}{4} =$$

$$(\tfrac{1}{64} + \tfrac{9}{64} + \tfrac{25}{64} + \tfrac{49}{64})\cdot\tfrac{1}{4} = \tfrac{21}{64}$$

$$= 0.3281$$

$\boxed{n \text{ RECTANGLES}}$



BASES : $[0, \frac{1}{n}]$, $[\frac{1}{n}, \frac{2}{n}]$, ... , $[\frac{n-1}{n}, 1]$

MIDPOINTS : $\frac{1}{2n}$, $\frac{3}{2n}$, ... , $\frac{2n-1}{2n}$

$$A \approx f\left(\frac{1}{2n}\right)\cdot\frac{1}{n} + f\left(\frac{3}{2n}\right)\cdot\frac{1}{n} + \cdots + f\left(\frac{2n-1}{2n}\right)\cdot\frac{1}{n} =$$

$$\left(\frac{1}{2n}\right)^2\cdot\frac{1}{n} + \left(\frac{3}{2n}\right)^2\cdot\frac{1}{n} + \cdots + \left(\frac{2n-1}{2n}\right)^2\cdot\frac{1}{n} =$$

$$\frac{1^2}{4n^2}\cdot\frac{1}{n} + \frac{3^2}{4n^2}\cdot\frac{1}{n} + \cdots + \frac{(2n-1)^2}{4n^2}\cdot\frac{1}{n} =$$

$$\left(1^2 + 3^2 + \cdots + (2n-1)^2\right)\cdot\frac{1}{4n^3}$$

E.G., IF $n = 5$,

$$A \approx (1^2 + 3^2 + 5^2 + 7^2 + 9^2) \cdot \frac{1}{4 \cdot 5^3} = \frac{165}{500} = 0.3300$$

<u>FACT</u> (GENERALLY PROVED BY "MATHEMATICAL INDUCTION"):

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3}$$

E.G., IF $n = 3$,

$$1^2 + 3^2 + 5^2 = 1 + 9 + 25 = 35$$

$$\frac{4 \cdot 3^3 - 3}{3} = \frac{108 - 3}{3} = 35$$

THUS,

$$A \approx (1^2 + 3^2 + \cdots + (2n-1)^2) \frac{1}{4n^3} = \frac{4n^3 - n}{3} \frac{1}{4n^3}$$

$$= \frac{4n^3 - n}{12n^3}$$

$$= \frac{4n^3}{12n^3} - \frac{n}{12n^3}$$

$$= \frac{1}{3} - \frac{1}{12n^2}$$

THE APPROXIMATIONS BECOME BETTER AS n GETS LARGE (I.E., AS $n \to \infty$). BUT CLEARLY,

$$\frac{1}{3} - \frac{1}{12n^2} \longrightarrow \frac{1}{3} \quad \text{AS } n \to \infty$$

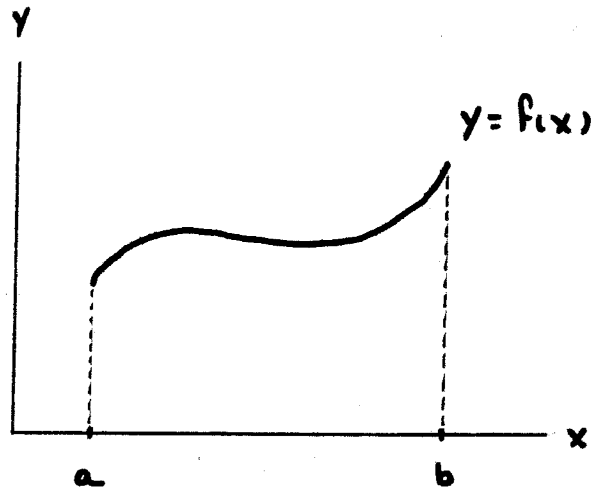
SO THE AREA UNDER THE GRAPH OF $y = f(x) = x^2$ FROM $x = 0$ TO $x = 1$ IS <u>PRECISELY</u> $\frac{1}{3}$.

SYMBOLICALLY, WE WOULD WRITE

$$A = \lim_{n \to \infty} \left(\frac{1}{3} - \frac{1}{12n^2} \right) = \frac{1}{3}$$

(WE WILL DISCUSS LIMITS LIKE THIS MORE PRECISELY IN JUST A LITTLE WHILE).

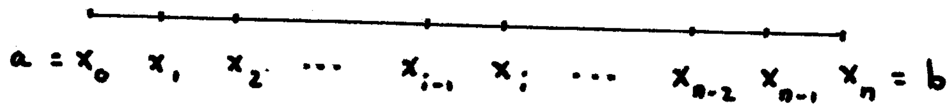
NOW LET'S DO THIS IN GENERAL. SUPPOSE $f(x) \geq 0$ ON THE INTERVAL $[a, b]$.



TO COMPUTE THE AREA UNDER THE GRAPH OF $f(x)$ AND ABOVE $[a, b]$ WE PROCEED AS FOLLOWS:

1. SUBDIVIDE THE INTERVAL $[a, b]$ INTO n SUBINTERVALS WITH ENDPOINTS

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} < x_n = b$$



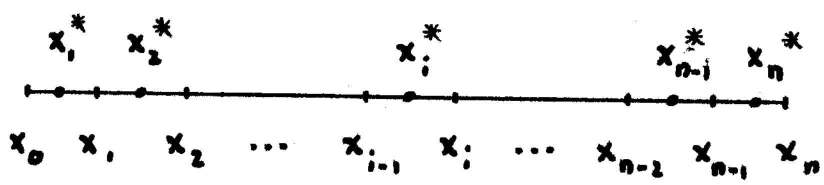
FOR EACH $i = 1, 2, \ldots, n-1, n$, LET

$$\Delta x_i = x_i - x_{i-1} = \text{LENGTH OF } [x_{i-1}, x_i]$$

<u>NOTE</u> : IF ALL OF THE SUBINTERVALS HAVE THE SAME LENGTH WE DENOTE ALL OF THE Δx_i SIMPLY Δx. OTHERWISE, THE LARGEST OF THE Δx_i WILL BE DENOTED

$$\Delta x_{max}$$

2. INSIDE EACH SUBINTERVAL $[x_{i-1}, x_i]$ SELECT A POINT x_i^*

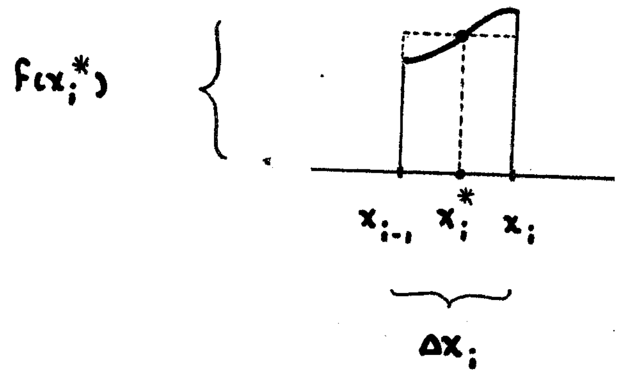


EVALUATE

$$f(x_1^*),\ f(x_2^*),\ \ldots,\ f(x_i^*),\ \ldots,\ f(x_{n-1}^*),\ f(x_n^*)$$

AND COMPUTE

$$f(x_1^*)\Delta x_1,\ f(x_2^*)\Delta x_2,\ \ldots,\ f(x_i^*)\Delta x_i,\ \ldots,\ f(x_{n-1}^*)\Delta x_{n-1},\ f(x_n^*)\Delta x_n$$

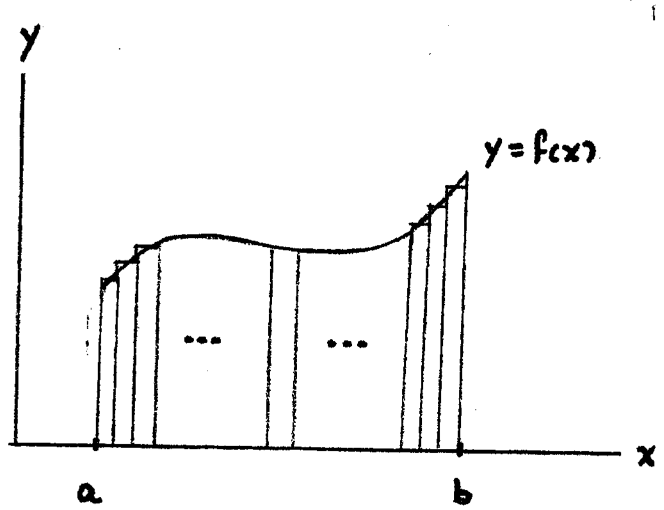


3. FORM THE <u>RIEMANN</u> <u>SUM</u> APPROXIMATION

$$f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_{n-1}^*)\Delta x_1 + f(x_n^*)\Delta x_n$$

$$= \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

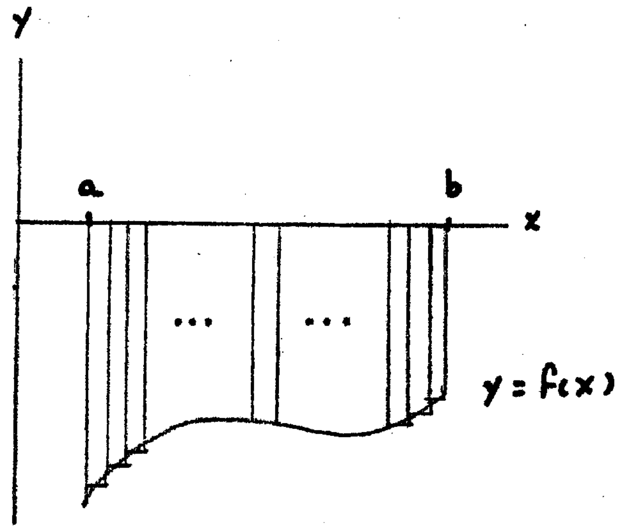
(Σ- NOTATION : THE SUM OF ALL THE $f(x_i^*)\Delta x_i$ FOR i TAKING VALUES FROM 1 TO n)



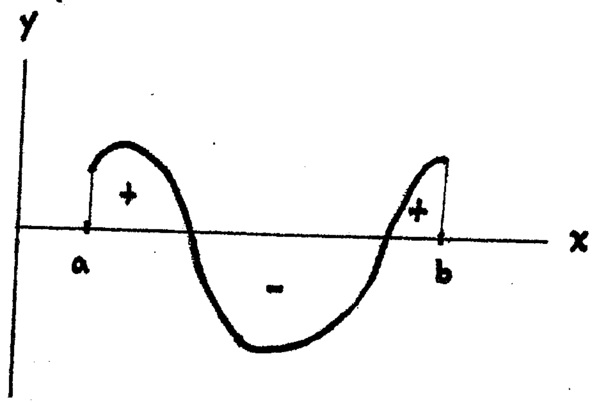
4. REPEAT STEPS # 1-3 OVER AND OVER WITH FINER AND FINER SUBDIVISIONS OF $[a,b]$, I.E., SMALLER AND SMALLER Δx_{max}, TO GENERATE BETTER AND BETTER APPROXIMATIONS AND THEN DETERMINE WHAT THESE APPROXIMATIONS ARE APPROACHING, I.E., "TAKE THE LIMIT"

$$\lim_{\Delta x_{max} \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

NOTICE THAT IF $f(x) \leq 0$ (RATHER THAN $f(x) \geq 0$) ON $[a,b]$, THEN THE RESULT OF THIS PROCEDURE WILL BE <u>MINUS</u> THE AREA BETWEEN THE GRAPH OF $f(x)$ AND $[a,b]$.



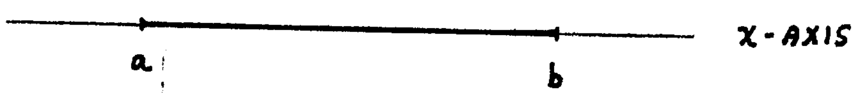
IF $f(x)$ TAKES BOTH POSITIVE AND NEGATIVE VALUES ON $[a,b]$, THEN THE PROCEDURE YIELDS THE <u>NET</u> <u>SIGNED</u> <u>AREA</u> BETWEEN THE GRAPH OF $f(x)$ AND THE INTERVAL $[a,b]$.



THE REAL SIGNIFICANCE OF THIS RATHER ELABORATE <u>RIEMANN SUM PROCEDURE</u>, HOWEVER, IS THAT IT SOLVES A HUGE NUMBER OF TOTALLY UNRELATED PROBLEMS. FOR THE MOMENT WE WILL LOOK AT JUST ONE EXAMPLE.

<u>THE MASS OF A WIRE</u> :

CONSIDER A THIN (1-DIMENSIONAL) METAL WIRE LYING ALONG THE X-AXIS FROM $x = a$ TO $x = b$.

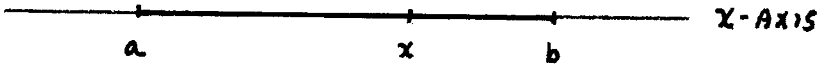


IF THE METAL IS HOMOGENEOUS (CONSTANT DENSITY, SAY, ρ_0 g/cm) THEN THE MASS M OF THE WIRE IS JUST

$$\begin{aligned} M &= (\#\ g/cm)(\#\ cm) \\ &= (\text{DENSITY})(\text{LENGTH}) \\ &= \rho_0 (b-a) \end{aligned}$$

SUPPOSE, HOWEVER, THAT THE METAL IS INHOMOGENEOUS (DENSITY VARIES FROM POINT-TO-POINT ALONG THE WIRE, SAY,

$$\rho = \rho(x) \;).$$



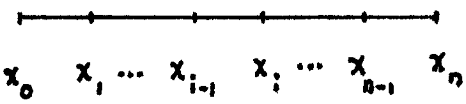
$\rho(x)$ = DENSITY AT LOCATION x
FOR EACH x IN $[a, b]$

HOW CAN ONE COMPUTE THE MASS M NOW?

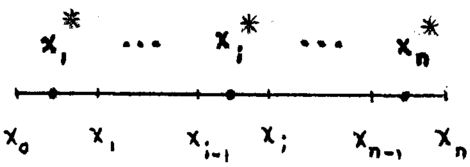
STANDARD OPERATING PROCEDURE : GENERATE A SEQUENCE OF BETTER AND BETTER APPROXIMATIONS TO M AND TAKE THEIR LIMIT.

IDEA : OVER A SMALL SEGMENT OF THE WIRE THE DENSITY IS NEARLY CONSTANT SO THE MASS OF THE SEGMENT IS APPROXIMATELY "DENSITY TIMES LENGTH". APPROXIMATION BECOMES BETTER AS LENGTH OF SEGMENT GOES TO ZERO.

1. CARVE THE WIRE UP INTO n SEGMENTS



2. SELECT A SAMPLE POINT IN EACH



EVALUATE THE DENSITY AT EACH SAMPLE POINT

$$\rho(x_1^*), \ldots, \rho(x_i^*), \ldots, \rho(x_n^*)$$

AND COMPUTE THE APPROXIMATE MASS OF EACH SEGMENT

$$\rho(x_1^*)\Delta x_1, \ldots, \rho(x_i^*)\Delta x_i, \ldots, \rho(x_n^*)\Delta x_n$$

3. COMPUTE THE APPROXIMATE MASS OF THE WIRE

$$\sum_{i=1}^{n} \rho(x_i^*)\Delta x_i$$

4. REPEAT # 1-3 OVER AND OVER WITH SMALLER AND SMALLER SEGMENTS AND TAKE THE LIMIT

$$M = \lim_{\Delta x_{max} \to 0} \sum_{i=1}^{n} \rho(x_i^*)\Delta x_i$$

PRECISELY THE SAME SEQUENCE OF STEPS (RIEMANN SUM PROCEDURE) HAS SOLVED TWO QUITE DIFFERENT PROBLEMS.

THIS IS JUST THE BEGINNING. WE WILL SEE THAT THE SAME IDEAS SOLVE A WIDE VARIETY OF PROBLEMS IN MATHEMATICS AND THE APPLICATIONS.

THE POINT IS THAT THE NUMBER THAT RESULTS FROM APPLYING THE RIEMANN SUM PROCEDURE TO SOME FUNCTION ON SOME INTERVAL, BECAUSE IT MEANS MANY DIFFERENT THINGS IN DIFFERENT

CONTEXTS, DESERVES A NAME AND A SYMBOL.

GIVEN $f(x)$ ON $[a,b]$, THE NUMBER

$$\lim_{\Delta x_{max} \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

THAT RESULTS FROM THE RIEMANN SUM PROCEDURE IS CALLED THE <u>DEFINITE</u> <u>INTEGRAL</u> <u>OF</u> $f(x)$ <u>OVER</u> $[a,b]$ AND IS DENOTED

$$\int_a^b f(x)dx .$$

a IS CALLED THE <u>LOWER</u>, AND b THE <u>UPPER</u> <u>LIMIT</u> <u>OF</u> <u>INTEGRATION</u>.

<u>NOTE</u> : DESPITE THE SIMILARITY OF THE SYMBOLS, $\int_a^b f(x)dx$ AND $\int f(x)dx$ ARE ENTIRELY TYPES OF THINGS. THE FIRST IS A <u>NUMBER</u>. THE SECOND IS A <u>COLLECTION</u> <u>OF</u> <u>FUNCTIONS</u>. THE RELATIONSHIP BETWEEN THE TWO WILL BE DESCRIBED IN A MOMENT. THIS RELATIONSHIP IS, IN A SENSE, THE REASON WE ARE ALL HERE NOW, I.E., THE REASON PEOPLE STUDY CALCULUS.

EXAMPLES :

1. IF $f(x)$ IS NON-NEGATIVE ON $[a,b]$, THEN

$$\int_a^b f(x)\,dx = \text{AREA UNDER THE GRAPH OF } f(x) \text{ AND ABOVE } [a,b]$$

E.G.,
$$\int_0^1 x^2\,dx = \frac{1}{3}$$

2. FOR ANY $f(x)$ ON $[a,b]$,

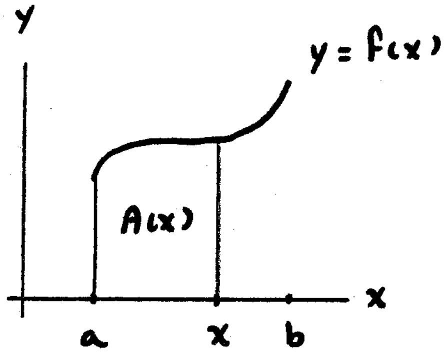
$$\int_a^b f(x)\,dx = \text{NET SIGNED AREA BETWEEN THE GRAPH OF } f(x) \text{ AND } [a,b]$$

3. IF $\rho(x)$ = DENSITY AT X OF A STRAIGHT WIRE LAYING ALONG THE X-AXIS FROM $x = a$ TO $x = b$, THEN

$$\int_a^b \rho(x)\,dx = \text{MASS OF THE WIRE}$$

THERE ARE MANY MORE SUCH EXAMPLES TO COME LATER.

NOW RECALL THAT



$$A'(x) = f(x)$$

SO $A(x)$ IS AN ANTIDERIVATIVE FOR $f(x)$ (I.E., IT IS ONE OF THE FUNCTIONS IN $\int f(x)dx$).

NOW LET $F(x)$ BE <u>ANY</u> ANTIDERIVATIVE FOR $f(x)$. THEN

$$F(x) = A(x) + C$$

FOR SOME CONSTANT C. NOW NOTICE THAT

$$\begin{aligned} F(b) - F(a) &= [A(b) + C] - [A(a) + C] \\ &= A(b) + C - 0 - C \\ &= A(b) \\ &= \text{AREA UNDER THE GRAPH OF } f(x) \\ &\quad \text{AND ABOVE } [a,b] \\ &= \int_a^b f(x)dx \end{aligned}$$

THE FUNDAMENTAL THEOREM OF CALCULUS : IF $F(x)$ IS ANY ANTIDERIVATIVE FOR $f(x)$, THEN

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$= F(x)\Big|_a^b$$

EXAMPLES :

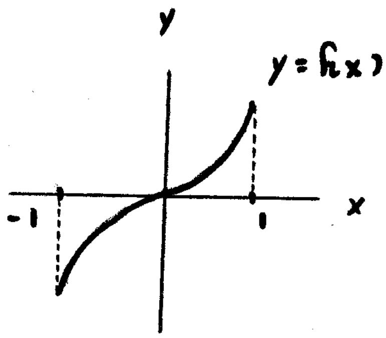
1. AREA UNDER THE GRAPH OF $f(x) = x^2$ AND ABOVE $[0,1]$ IS

$$\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}\cdot 1^3 - \frac{1}{3}\cdot 0^3 = \frac{1}{3}$$

2. NET SIGNED AREA BETWEEN THE GRAPH OF $f(x) = x^3$ AND THE INTERVAL $[-1,1]$ IS

$$\int_{-1}^1 x^3\,dx = \frac{1}{4}x^4\Big|_{-1}^1 = \frac{1}{4}\cdot 1^4 - \frac{1}{4}\cdot(-1)^4$$

$$= \frac{1}{4} - \frac{1}{4} = 0$$



3. $$\int_{-1}^{2} 4t(1-t^2)\,dt = \int_{-1}^{2} (4t-4t^3)\,dt = 4(\tfrac{1}{2}t^2)-4(\tfrac{1}{4}t^4)\Big|_{-1}^{2}$$

$$= 2t^2 - t^4 \Big|_{-1}^{2} = (2\cdot 2^2 - 2^4) - (2(-1)^2 - (-1)^4)$$

$$= -8 - 1$$

$$= -9$$

4. A WIRE LAYS ALONG THE X-AXIS FROM $X = 0$ TO $X = 1$ AND HAS DENSITY AT EACH POINT THAT IS PROPORTIONAL TO ITS DISTANCE FROM THE LEFT-HAND ENDPOINT. FIND THE MASS M OF THE WIRE.

$$\rho(x) = kx \qquad \text{(FOR SOME CONSTANT } k\text{)}$$

$$M = \int_0^1 \rho(x)\,dx = \int_0^1 kx\,dx = k(\tfrac{1}{2}x^2)\Big|_0^1$$

$$= \tfrac{1}{2}k\cdot 1^2 - \tfrac{1}{2}k\cdot 0^2 = \tfrac{1}{2}k$$

<u>ASSIGNMENT</u> # 4 : READ SECTIONS 6.1, 6.4, 6.5 (PAGES 386 - 388 ONLY) AND 6.6 (THROUGH EXAMPLE 2, PAGE 398) AND WORK THE FOLLOWING PROBLEMS.

EXERCISE SET 6.1 (PAGE 354) : # 1, 3, 5, 11, 13, 15, 17

EXERCISE SET 6.4 (PAGE 383) : # 1, 9, 31, 41

EXERCISE SET 6.6 (PAGE 406) : # 1, 3, 9, 10

CHAPTER REVIEW EXERCISES (PAGE 437) : # 4, 31, 32, 37, 47, 48, 75, 77