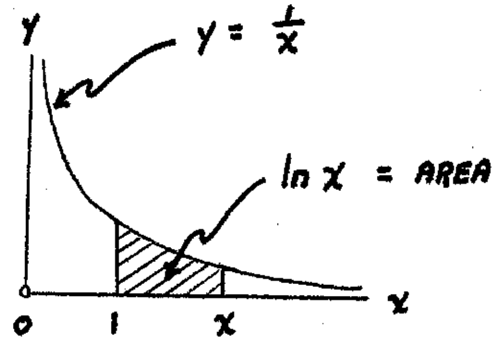
DERIVATIVES OF LOGARITHMIC, EXPONENTIAL AND INVERSE TRIG FUNCTIONS

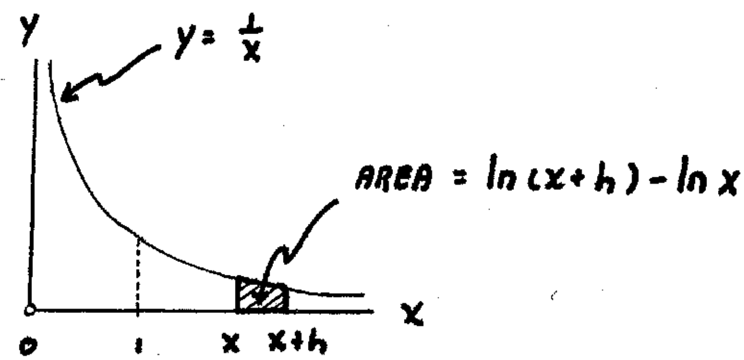
RECALL THE "CORRECT" DEFINITION OF $\ln x$ $(x > 0)$:



NOW WE WANT TO FIND THE DERIVATIVE OF THE NATURAL LOGARITHM FUNCTION.

$$(\ln x)' = \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h}$$

HERE'S A PICTURE OF $\ln(x+h) - \ln x$:



THUS,

$$h\left(\frac{1}{x+h}\right) \leq \ln(x+h) - \ln x \leq h\left(\frac{1}{x}\right)$$

DIVIDE EVERYTHING BY h TO GET

$$\frac{1}{x+h} \leq \frac{\ln(x+h) - \ln x}{h} \leq \frac{1}{x}$$

SINCE $\lim_{h\to 0} \frac{1}{x+h} = \frac{1}{x}$ AND $\lim_{h\to 0} \frac{1}{x} = \frac{1}{x}$, THE SQUEEZE THEOREM IMPLIES THAT

$$\lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} = \frac{1}{x}$$

AS WELL. THUS, FOR $x > 0$,

$$\boxed{(\ln x)' = \frac{1}{x}}$$

COMBINED WITH THE CHAIN RULE THIS GIVES THE USUAL THING:

$$\boxed{(\ln u(x))' = \frac{1}{u(x)}\, u'(x)}$$

FOR ANY POSITIVE FUNCTION $u(x)$.

E.G., $(\ln\sqrt{x})' = \frac{1}{\sqrt{x}}(\sqrt{x})' = \frac{1}{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2x}$

OR, EVEN BETTER,

$$(\ln\sqrt{x})' = (\ln x^{\frac{1}{2}})' = \left(\frac{1}{2}\ln x\right)' = \frac{1}{2}\,\frac{1}{x} = \frac{1}{2x}$$

$$(\ln x)' = \frac{1}{x} \qquad \text{FOR } x > 0$$

$$(\ln u(x))' = \frac{1}{u(x)} u'(x) \qquad \text{FOR } u(x) > 0$$

FOR $x < 0$, $-x > 0$ SO $\ln(-x)$ IS DEFINED AND

$$(\ln(-x))' = \frac{1}{-x}(-x)' = \frac{1}{-x}(-1) = \frac{1}{x}$$

THUS,

$$(\ln x)' = \frac{1}{x} \quad \text{FOR } x > 0$$

$$(\ln(-x))' = \frac{1}{x} \quad \text{FOR } x < 0$$

SO

$$\boxed{(\ln |x|)' = \frac{1}{x} \quad \text{FOR } x \neq 0}$$

WITH THE CHAIN RULE,

$$\boxed{(\ln |u(x)|)' = \frac{1}{u(x)} u'(x)}$$

E.G., $(\ln|\sin x|)' = \frac{1}{\sin x} \cos x = \cot x$

LOGARITHMS WITH OTHER BASES : $\log_b x = \frac{\ln x}{\ln b}$ SO

$$\boxed{(\log_b x)' = \frac{1}{\ln b} \frac{1}{x}}$$

E.G., $(\log_2 x)' = \frac{1}{\ln 2} \frac{1}{x}$

WITH THE CHAIN RULE,

$$\boxed{(\log_b u(x))' = \frac{1}{\ln b}\,\frac{1}{u(x)}\,u'(x)}$$

WITH THE ABSOLUTE VALUES,

$$\boxed{(\log_b |u(x)|)' = \frac{1}{\ln b}\,\frac{1}{u(x)}\,u'(x)}$$

E.G., $$(\log_3 |\tan x|)' = \frac{1}{\ln 3}\,\frac{1}{\tan x}\,(\sec^2 x)$$

EXPONENTIAL FUNCTIONS:

$$y = e^x$$
$$\ln y = x$$

DIFFERENTIATE IMPLICITLY WITH RESPECT TO x:

$$(\ln y)' = (x)'$$
$$\frac{1}{y}\,y' = 1$$
$$y' = y$$
$$y' = e^x$$
$$\boxed{(e^x)' = e^x}$$

WITH THE CHAIN RULE

$$\boxed{(e^{u(x)})' = e^{u(x)}\,u'(x)}$$

E.G., $(e^{\csc x})' = e^{\csc x}(-\csc x \cot x)$

EXPONENTIAL FUNCTIONS WITH OTHER BASES :

$$b^x = e^{x \ln b}$$

SO

$$(b^x)' = e^{x \ln b}(x \ln b)'$$
$$= b^x \ln b$$

$$\boxed{(b^x)' = b^x \ln b}$$

WITH THE CHAIN RULE,

$$\boxed{(b^{u(x)})' = b^{u(x)} \ln b \, u'(x)}$$

E.G.,

$$(5^{x^2})' = 5^{x^2} \ln 5 \, (2x)$$

<u>EXAMPLES</u> :

1. COMPUTE y' IF $y = \sin(e^x)$

$$y' = \cos(e^x)(e^x)' = \cos(e^x)e^x$$
$$= e^x \cos(e^x)$$

2. COMPUTE y' IF $y = \exp\sqrt{1+5x^3}$

$$y' = \left(e^{\sqrt{1+5x^3}}\right)' = e^{\sqrt{1+5x^3}}\left(\sqrt{1+5x^3}\right)'$$
$$= e^{\sqrt{1+5x^3}}\left(\tfrac{1}{2}(1+5x^3)^{-\frac{1}{2}}(1+5x^3)'\right)$$
$$= e^{\sqrt{1+5x^3}}\left(\frac{15x^2}{2\sqrt{1+5x^3}}\right)$$
$$= \frac{15x^2\exp\sqrt{1+5x^3}}{2\sqrt{1+5x^3}}$$

3. COMPUTE y' IF $y = \ln\left(\frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\right)$

SIMPLIFY FIRST !

$$y = \ln(x^2\sqrt[3]{7x-14}) - \ln(1+x^2)^4$$
$$= \ln(x^2) + \ln(7x-14)^{\frac{1}{3}} - 4\ln(1+x^2)$$
$$= 2\ln x + \tfrac{1}{3}\ln(7x-14) - 4\ln(1+x^2)$$

$$y' = 2\left(\frac{1}{x}\right) + \frac{1}{3}\frac{1}{7x-14}(7x-14)' - 4\frac{1}{1+x^2}(1+x^2)'$$
$$= \frac{2}{x} + \frac{7}{3(7x-14)} - \frac{8x}{1+x^2}$$

4. COMPUTE y' IF $y = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}$

LOGARITHMIC DIFFERENTIATION : TAKE $\ln$ OF BOTH SIDES, DIFFERENTIATE IMPLICITLY AND SOLVE FOR y'.

$$y = \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4} \Rightarrow \ln y = \ln\left(\frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}\right)$$

$$\ln y = 2\ln x + \tfrac{1}{3}\ln(7x-14) - 4\ln(1+x^2)$$

DIFFERENTIATE :

$$\frac{1}{y}y' = \frac{2}{x} + \frac{7}{3(7x-14)} - \frac{8x}{1+x^2}$$

$$y' = \left(\frac{2}{x} + \frac{7}{3(7x-14)} - \frac{8x}{1+x^2}\right)y$$

$$y' = \left(\frac{2}{x} + \frac{7}{3(7x-14)} - \frac{8x}{1+x^2}\right)\frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}$$

5. $y = x^{\sin x}$

$$\ln y = \ln(x^{\sin x})$$

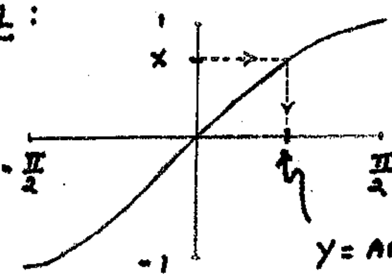
$$\ln y = (\sin x)(\ln x)$$

$$\frac{1}{y}y' = (\sin x)\left(\frac{1}{x}\right) + (\ln x)(\cos x)$$

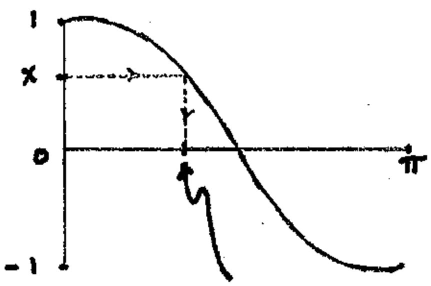
$$y' = \left(\frac{\sin x}{x} + (\ln x)(\cos x)\right)y$$

$$y' = \left(\frac{\sin x}{x} + (\ln x)(\cos x)\right)x^{\sin x}$$

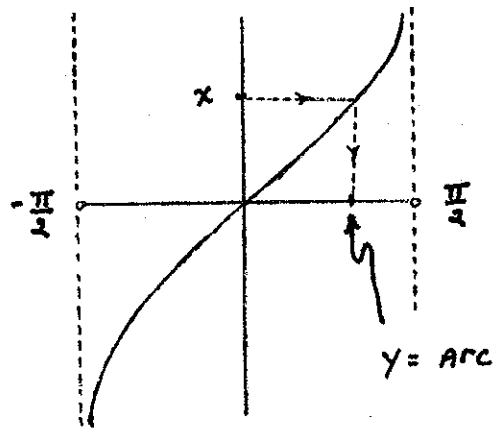
RECALL :



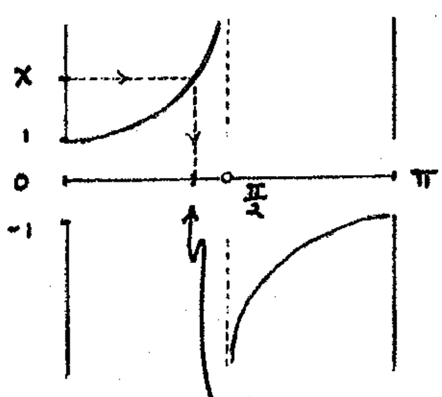
$y = \text{Arcsin}\, x$ = Angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ WHOSE SINE IS x



$y = \text{Arccos}\, x$ = ANGLE IN $[0, \pi]$ WHOSE COSINE IS x



$y = \text{Arctan}\, x$ = ANGLE IN $(-\frac{\pi}{2}, \frac{\pi}{2})$ WHOSE TANGENT IS x



$y = \text{Arcsec}\, x$ = ANGLE IN $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ WHOSE SECANT IS x

DERIVATIVES :

$$y = \arcsin x$$

$$\sin y = x$$

DIFFERENTIATE IMPLICITLY WITH RESPECT TO x :

$$(\sin y)' = (x)'$$

$$(\cos y)\, y' = 1$$

$$y' = \frac{1}{\cos y}$$

$$y' = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\boxed{(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}}$$

SIMILARLY,

$$\boxed{(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{(\arctan x)' = \frac{1}{1+x^2}}$$

$$\boxed{(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}}}$$

COMBINED WITH THE CHAIN RULE WE GET THE USUAL THING :

$(\text{Arcsin}\, u(x))' = \dfrac{1}{\sqrt{1-(u(x))^2}}\, u'(x)$
$(\text{Arccos}\, u(x))' = -\dfrac{1}{\sqrt{1-(u(x))^2}}\, u'(x)$
$(\text{Arctan}\, u(x))' = \dfrac{1}{1+(u(x))^2}\, u'(x)$
$(\text{Arcsec}\, u(x))' = \dfrac{1}{\lvert u(x)\rvert\sqrt{(u(x))^2-1}}\, u'(x)$

EXAMPLES :

1. $$(\text{Arcsin}\,(x^3))' = \frac{1}{\sqrt{1-(x^3)^2}}\,(x^3)' = \frac{3x^2}{\sqrt{1-x^6}}$$

2. $$(\text{Arcsec}\,(e^x))' = \frac{1}{\lvert e^x\rvert\sqrt{(e^x)^2-1}}\,(e^x)' = \frac{e^x}{e^x\sqrt{e^{2x}-1}} = \frac{1}{\sqrt{e^{2x}-1}}$$

3. $f(x) = \ln(\text{Arctan}\sqrt{x}) \Rightarrow$

$$f'(x) = \frac{1}{\text{Arctan}\sqrt{x}}\,(\text{Arctan}\sqrt{x})'$$

$$= \frac{1}{\text{Arctan}\sqrt{x}}\,\frac{1}{1+(\sqrt{x})^2}\,(\sqrt{x})' = \frac{1}{\text{Arctan}\sqrt{x}}\,\frac{1}{1+x}\,\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}\,(1+x)\,\text{Arctan}\sqrt{x}}$$

4. FIND ALL VALUES OF X AT WHICH $y = \text{Arctan}\,(x^2)$ HAS A HORIZONTAL TANGENT LINE.

$$y' = \frac{1}{1+(x^2)^2}\,(x^2)' = \frac{2x}{1+x^4}$$

$$y' = 0 \Rightarrow \frac{2x}{1+x^4} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$$

5. FIND y' BY IMPLICIT DIFFERENTIATION IF

$$\arcsin(xy) = \arccos(x-y)$$

$$(\arcsin(xy))' = (\arccos(x-y))'$$

$$\frac{1}{\sqrt{1-(xy)^2}}(xy)' = -\frac{1}{\sqrt{1-(x-y)^2}}(x-y)'$$

$$\frac{1}{\sqrt{1-x^2y^2}}(xy'+y) = -\frac{1}{\sqrt{1-(x-y)^2}}(1-y')$$

$$\frac{x}{\sqrt{1-x^2y^2}}y' + \frac{y}{\sqrt{1-x^2y^2}} = -\frac{1}{\sqrt{1-(x-y)^2}} + \frac{1}{\sqrt{1-(x-y)^2}}y'$$

MULTIPLY THROUGH BY $\sqrt{1-x^2y^2}\sqrt{1-(x-y)^2}$:

$$x\sqrt{1-(x-y)^2}\,y' + y\sqrt{1-(x-y)^2} = -\sqrt{1-x^2y^2} + \sqrt{1-x^2y^2}\,y'$$

$$\left(x\sqrt{1-(x-y)^2} - \sqrt{1-x^2y^2}\right)y' = -\sqrt{1-x^2y^2} - y\sqrt{1-(x-y)^2}$$

$$y' = -\frac{y\sqrt{1-(x-y)^2} + \sqrt{1-x^2y^2}}{x\sqrt{1-(x-y)^2} - \sqrt{1-x^2y^2}}$$

6. FIND $\frac{d^2y}{dx^2}$ IF $y = \arccos(3x)$

$$\frac{dy}{dx} = (\arccos(3x))' = -\frac{1}{\sqrt{1-(3x)^2}}(3x)' = -\frac{3}{\sqrt{1-9x^2}}$$

$$\frac{d^2y}{dx^2} = \left(-\frac{3}{\sqrt{1-9x^2}}\right)' = -3\left((1-9x^2)^{-\frac{1}{2}}\right)'$$

$$= -3\left(-\tfrac{1}{2}(1-9x^2)^{-\frac{3}{2}}(1-9x^2)'\right)$$

$$= \frac{3}{2}(1-9x^2)^{-\frac{3}{2}}(-18x) = -27x\,(1-9x^2)^{-3/2}$$