DERIVATIVES

IN CALCULUS THE DERIVATIVE OF $f : \mathbb{R} \to \mathbb{R}$ AT A POINT p IS GENERALLY DEFINED TO BE A NUMBER, BUT THE IMPORTANT OBJECT IS REALLY THE STRAIGHT LINE WITH THAT SLOPE THROUGH $(p, f(p))$ (THE "BEST LINEAR APPROXIMATION TO f NEAR p").

SUPPOSE X AND Y ARE SMOOTH MANIFOLDS OF DIMENSION n AND m, RESPECTIVELY, AND

$$F : X \to Y$$

IS A SMOOTH MAP.

FIX $p \in X$. THEN $F(p) \in Y$. WE WANT TO DEFINE A LINEAR TRANSFORMATION

$$F_{*p} : T_p(X) \longrightarrow T_{F(p)}(Y)$$

CALLED THE <u>DERIVATIVE</u> <u>OF</u> F <u>AT</u> p (THOUGHT OF INTUITIVELY AS THE "BEST LINEAR APPROXIMATION TO F NEAR p")

<u>NOTE</u>: SOME TEXTS CALL WHAT WE ARE GOING TO DEFINE THE "DIFFERENTIAL OF F AT p" AND DENOTE IT dF_p.

THERE ARE TWO USEFUL (AND EQUIVALENT) WAYS TO DEFINE F_{*p}.
I'LL GIVE BOTH AND SHOW THAT THEY ARE EQUIVALENT.

$\boxed{1}$ $$F_{*p} : T_p(X) \longrightarrow T_{F(p)}(Y)$$

FIX $v_p \in T_p(X)$. REGARD v_p AS THE VELOCITY VECTOR OF SOME SMOOTH CURVE α IN X:

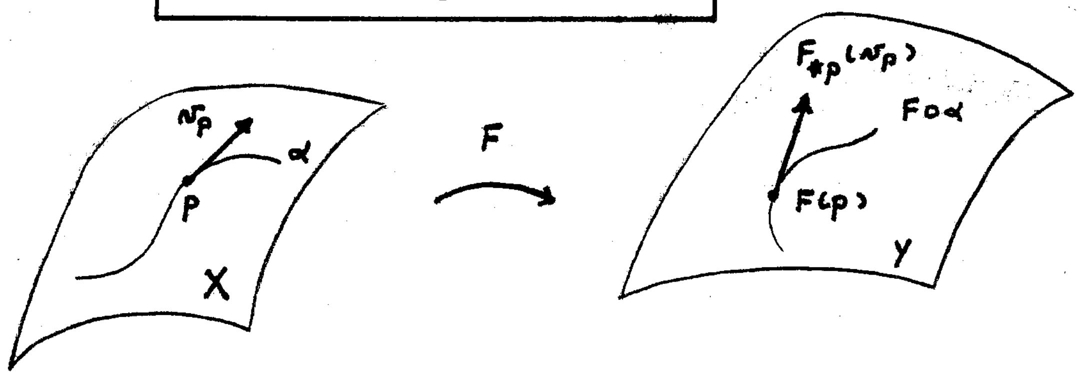
$$v_p = \alpha'(t_0)$$

THEN $F \circ \alpha$ IS A SMOOTH CURVE IN Y THROUGH $F(p)$

$$(F \circ \alpha)(t) = F(\alpha(t))$$
$$(F \circ \alpha)(t_0) = F(\alpha(t_0)) = F(p)$$

DEFINE

$$\boxed{\begin{aligned} F_{*p}(v_p) &= F_{*p}(\alpha'(t_0)) \\ &= (F \circ \alpha)'(t_0) \end{aligned}}$$



$\boxed{2}$ $$F_{*p} : T_p(X) \longrightarrow T_{F(p)}(Y)$$

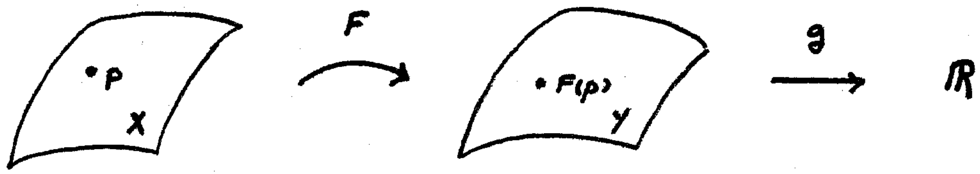
FIX $v_p \in T_p(X)$. REGARD

$$v_p : C^\infty(X) \longrightarrow \mathbb{R}$$

THEN $F_{*p}(v_p)$ IS TO BE REGARDED AS

$$F_{*p}(v_p) : C^\infty(Y) \longrightarrow \mathbb{R}$$

$$g \in C^\infty(Y) \implies g \circ F \in C^\infty(X)$$



DEFINE

$$\boxed{\left(F_{*p}(v_p) \right)(g) = v_p(g \circ F)}$$

TO SEE THAT GIVES THE SAME RESULT AS THE 1ST DEFINITION WHEN $v_p = \alpha'(t_0)$ NOTE THAT, FOR EVERY $g \in C^\infty(Y)$,

$$\begin{aligned}
\left(F_{*p}(\alpha'(t_0)) \right)(g) &= \alpha'(t_0)(g \circ F) \\
&= \frac{d}{dt}\left((g \circ F) \circ \alpha \right)\Big|_{t=t_0} \\
&= \frac{d}{dt}\left(g \circ (F \circ \alpha) \right)\Big|_{t=t_0} \\
&= \left((F \circ \alpha)'(t_0) \right)(g)
\end{aligned}$$

SO

$$F_{*p}(\alpha'(t_0)) = (F \circ \alpha)'(t_0)$$

WE WILL COMPUTE SOME EXAMPLES OF DERIVATIVES SHORTLY, BUT FIRST SOME ELEMENTARY PROPERTIES :

1. $F_{*p} : T_p(X) \longrightarrow T_{F(p)}(Y)$ IS A LINEAR TRANSFORMATION

<u>PROOF</u> :
$$\begin{aligned}
(F_{*p}(a v_p + b w_p))(g) &= (a v_p + b w_p)(g \circ F) \\
&= a\, v_p(g \circ F) + b\, w_p(g \circ F) \\
&= a\,(F_{*p}(v_p))(g) + b\,(F_{*p}(w_p))(g) \\
&= (a\, F_{*p}(v_p) + b\, F_{*p}(w_p))(g)
\end{aligned}$$

SO

$$F_{*p}(a v_p + b w_p) = a\, F_{*p}(v_p) + b\, F_{*p}(w_p) \qquad \square$$

2. IF $id : X \to X$ IS THE IDENTITY MAP, THEN, FOR ANY $p \in X$,

$$id_{*p} : T_p(X) \longrightarrow T_{id(p)}(X) = T_p(X)$$

IS $id_{T_p(X)}$.

<u>EXERCISE</u> 85 : PROVE THIS.

3. IF (U, φ) IS A CHART AT $p \in X$ WITH COORDINATE FUNCTIONS $x^1, \ldots, x^n$ AND (V, ψ) IS A CHART AT $F(p)$ WITH COORDINATE FUNCTIONS $y^1, \ldots, y^m$, THEN THE MATRIX OF F_{*p} WITH RESPECT TO THE BASES

$$\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{i=1,\ldots,n} \text{ FOR } T_p(X)$$

AND

$$\left\{ \frac{\partial}{\partial y^j}\Big|_{F(p)} \right\} \quad \text{FOR} \quad T_{F(p)}(Y)$$

IS THE JACOBIAN OF THE COORDINATE EXPRESSION

$$\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$$

AT $\varphi(p)$.

<u>PROOF</u>: BY THE BASIS THEOREM (FOR $T_{F(p)}(Y)$),

$$F_{*p}\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \left(F_{*p}\left(\frac{\partial}{\partial x^i}\Big|_p\right)\right)(y^j)\, \frac{\partial}{\partial y^j}\Big|_{F(p)}$$

$$= \left(\frac{\partial}{\partial x^i}\Big|_p (y^j \circ F)\right) \frac{\partial}{\partial y^j}\Big|_{F(p)}$$

$$= \left(\frac{\partial(y^j \circ F \circ \varphi^{-1})}{\partial x^i}(\varphi(p))\right) \frac{\partial}{\partial y^j}\Big|_{F(p)}$$

BUT

$$y^j \circ F \circ \varphi^{-1} = \pi^j \circ (\psi \circ F \circ \varphi^{-1})$$

$= j^{\text{TH}}$ COORDINATE FUNCTION OF $\psi \circ F \circ \varphi^{-1}$

SO THE PARTIAL DERIVATIVES OF THESE COORDINATE FUNCTIONS WITH RESPECT TO x^i FORM THE i^{TH} COLUMN OF THE MATRIX OF F_{*p} AND THIS IS JUST THE JACOBIAN OF $\psi \circ F \circ \varphi^{-1}$. $\square$

4. (<u>CHAIN RULE</u>) LET $F : X \to Y$ AND $G : Y \to Z$ BE SMOOTH MAPS OF DIFFERENTIABLE MANIFOLDS AND LET $p \in X$. THEN $G \circ F : X \to Z$ IS SMOOTH AND

$$(G \circ F)_{*p} = G_{*F(p)} \circ F_{*p} .$$

<u>PROOF</u>: SMOOTHNESS OF $G \circ F$ HAS ALREADY BEEN PROVED. LET $v_p \in T_p(X)$. THEN, FOR EVERY $g \in C^\infty(X)$,

$$\begin{aligned}
((G \circ F)_{*p}(v_p))(g) &= v_p(g \circ (G \circ F)) \\
&= v_p((g \circ G) \circ F) \\
&= (F_{*p}(v_p))(g \circ G) \\
&= (G_{*F(p)}(F_{*p}(v_p)))(g)
\end{aligned}$$

SO

$$(G \circ F)_{*p}(v_p) = (G_{*F(p)} \circ F_{*p})(v_p)$$

AND

$$(G \circ F)_{*p} = G_{*F(p)} \circ F_{*p} . \qquad \square$$

<u>EXERCISE</u> 86: WRITE THIS OUT IN TERMS OF THE MATRICES IN #3 AND EXPLAIN WHY IT IS CALLED THE "CHAIN RULE".

AS A COROLLARY WE HAVE THE MANIFOLD VERSION OF THE

<u>INVERSE</u> <u>FUNCTION</u> <u>THEOREM</u>: LET $F : X \to Y$ BE A SMOOTH MAP OF DIFFERENTIABLE MANIFOLDS AND LET $p \in X$. THEN

> $F_{*p} : T_p(X) \to T_{F(p)}(Y)$ IS AN <u>ISOMORPHISM</u> IF AND ONLY IF $F : X \to Y$ IS A <u>LOCAL</u> <u>DIFFEOMORPHISM</u> NEAR p (I.E., THERE EXIST OPEN NEIGHBORHOODS U OF p AND V OF $F(p)$ SUCH THAT $F|U$ IS A DIFFEOMORPHISM OF U ONTO V).

<u>PROOF</u>: SUPPOSE FIRST THAT F IS A LOCAL DIFFEOMORPHISM NEAR p. THEN, ON SOME NEIGHBORHOOD OF p, $F^{-1} \circ F = id$ SO THE CHAIN RULE GIVES

$$(F^{-1} \circ F)_{*p} = id_{*p}$$

$$F^{-1}_{*F(p)} \circ F_{*p} = id_{T_p(X)}$$

SIMILARLY, ON SOME NEIGHBORHOOD OF $F(p)$, $F \circ F^{-1} = id$ GIVES

$$(F \circ F^{-1})_{*F(p)} = id_{*F(p)}$$

$$F_{*F^{-1}(F(p))} \circ F^{-1}_{*F(p)} = id_{T_{F(p)}(Y)}$$

$$F_{*p} \circ F^{-1}_{*F(p)} = id_{T_{F(p)}(Y)}$$

SO

$$F^{-1}_{*F(p)} = (F_{*p})^{-1}$$

AND F_{*p} IS AN ISOMORPHISM.

NEXT SUPPOSE F_{*p} IS AN ISOMORPHISM. CHOOSING CHARTS (U, φ) AT p AND (V, ψ) AT $F(p)$, THE MATRIX OF F_{*p} IS THE JACOBIAN OF THE CORRESPONDING COORDINATE EXPRESSION FOR F. THIS JACOBIAN IS THEREFORE NONSINGULAR.

<u>ASIDE</u> : AT THIS POINT WE NEED TO APPEAL TO THE "ORDINARY" INVERSE FUNCTION THEOREM FROM CALCULUS SO I WILL PROVIDE A STATEMENT OF IT.

<u>INVERSE FUNCTION THEOREM</u>: LET A BE AN OPEN SET IN $\mathbb{R}^n$ AND $G : A \to \mathbb{R}^n$ A SMOOTH MAP. SUPPOSE $a \in A$ AND THE JACOBIAN OF G IS NONSINGULAR AT a. THEN $\exists$ OPEN SETS B_a AND $B_{G(a)}$ IN $\mathbb{R}^n$ SUCH THAT $a \in B_a$, $G(a) \in B_{G(a)}$,

$$G|B_a : B_a \longrightarrow B_{G(a)}$$

IS A SMOOTH BIJECTION AND HAS A SMOOTH INVERSE

$$(G|B_a)^{-1} : B_{G(a)} \longrightarrow B_a .$$

IN A NUTSHELL, $\det(J_G(a)) \neq 0 \Rightarrow G$ IS A LOCAL DIFFEOMORPHISM NEAR a.

NOW, RETURNING TO THE PROOF WE CONCLUDE THAT THE COORDINATE EXPRESSION

$$\psi \circ F \circ \varphi^{-1}$$

MUST BE A

DIFFEOMORPHISM NEAR $\varphi(p)$, I.E., ON SOME OPEN NEIGHBORHOOD U' OF $\varphi(p)$,

$$\psi \circ F \circ \varphi^{-1} : U' \longrightarrow V' = \psi(F(\varphi^{-1}(U')))$$

IS A DIFFEOMORPHISM. LET $U = \varphi^{-1}(U')$ AND $V = \psi^{-1}(V')$. THEN

$$F|U : U \to V$$

IS A DIFFEOMORPHISM. ▯

COROLLARY : IF TWO SMOOTH MANIFOLDS ARE DIFFEOMORPHIC, THEN THEY HAVE THE SAME DIMENSION.

NOTE : AS "OBVIOUS" AS THIS MAY SOUND TO YOU, REALIZE THAT THE PROOF DEPENDS CRUCIALLY ON THE DIFFERENTIABLE STRUCTURE. THE CORRESPONDING STATEMENT FOR TOPOLOGICAL MANIFOLDS (HOMEOMORPHIC $\Rightarrow$ SAME DIMENSION) IS TRUE, BUT MUCH MORE DIFFICULT TO PROVE.

NOTE : IN THE PROOF OF THE INVERSE FUNCTION THEOREM WE SHOWED THAT, FOR A (LOCAL) DIFFEOMORPHISM F

$$(F_{*p})^{-1} = (F^{-1})_{*F(p)}$$

AND THIS IN ITSELF IS WORTH REMEMBERING.

EXERCISE 87 : USE THE INVERSE FUNCTION THEOREM TO FILL IN THE LAST DETAIL IN OUR PROOF OF SMOOTHNESS FOR THE GAUSS MAP OF THE TORUS (SEE PAGES 12-13 OF THE LECTURE ON " SMOOTH MAPPINGS ON MANIFOLDS ").

WE WOULD NOW LIKE TO COMPUTE SOME EXPLICIT EXAMPLES OF TANGENT SPACES TO MANIFOLDS AND DERIVATIVES OF SMOOTH MAPS.

BEGIN BY TAKING ADVANTAGE OF THE FACT THAT THE TANGENT SPACE TO A SUBMANIFOLD OF EUCLIDEAN SPACE CAN, IN A CANONICAL WAY, BE IDENTIFIED WITH A LINEAR SUBSPACE OF THAT EUCLIDEAN SPACE. YOU WILL TAKE THE FIRST STEP.

<u>EXERCISE</u> 88: LET $\mathbb{R}^n$ HAVE ITS STANDARD DIFFERENTIABLE STRUCTURE. FIX A $p \in \mathbb{R}^n$. FOR EACH $v \in \mathbb{R}^n$ LET $v_p \in T_p(\mathbb{R}^n)$ BE DEFINED BY

$$v_p = \alpha'(0)$$

WHERE

$$\alpha : \mathbb{R} \longrightarrow \mathbb{R}^n$$

$$\alpha(t) = p + tv$$

SHOW THAT

$$v \longrightarrow v_p$$

IS AN ISOMORPHISM OF $\mathbb{R}^n$ ONTO $T_p(\mathbb{R}^n)$.

HENCEFORTH WE CALL THIS THE <u>CANONICAL</u> <u>ISOMORPHISM</u> AND USE IT TO IDENTIFY $T_p(\mathbb{R}^n)$ WITH $\mathbb{R}^n$.

ANY TANGENT SPACE TO A SUBMANIFOLD OF $\mathbb{R}^n$ CAN LIKEWISE BE IDENTIFIED WITH A LINEAR SUBSPACE OF $\mathbb{R}^n$. IN FACT, MORE IS TRUE :

<u>LEMMA</u> : LET X' BE A SUBMANIFOLD OF X, $p \in X'$ AND $\iota : X' \hookrightarrow X$ THE INCLUSION MAP. THEN

$$\iota_{*p} : T_p(X') \longrightarrow T_{\iota(p)}(X) = T_p(X)$$

IS AN ISOMORPHISM OF $T_p(X')$ ONTO A SUBSPACE OF $T_p(X)$.

<u>PROOOF</u> : $\iota_{*p} : T_p(X') \to T_p(X)$ IS LINEAR SO WE NEED ONLY SHOW THAT IT IS ONE-TO-ONE. IF $\dim X = n$ AND $\dim X' = k \leq n$, THEN THERE IS A CHART (U, φ) AT p IN X WITH COORDINATE FUNCTIONS $x^1, \ldots, x^k, \ldots, x^n$ SUCH THAT $(U \cap X', \varphi | U \cap X')$ IS A CHART AT p IN X' AND $\varphi(U \cap X') = \{x = (x^1, \ldots, x^k, \ldots, x^n) \in \varphi(U) : x^{k+1} = \cdots = x^n = 0\}$.

THUS, THE COORDINATE FUNCTIONS $y^1, \ldots, y^k$ FOR $(U \cap X', \varphi | U \cap X')$ ARE

$$y^1 = x^1$$
$$\vdots$$
$$y^k = x^k$$

RELATIVE TO THESE CHARTS THE COORDINATE EXPRESSION FOR

$\iota : X' \hookrightarrow X$ IS JUST

$$(y^1, \dots, y^k) \longrightarrow (y^1, \dots, y^k, 0, \dots, 0).$$

THE JACOBIAN CONTAINS THE $k \times k$ IDENTITY MATRIX SO IT HAS RANK k AND ι_{*p} MUST BE ONE-TO-ONE. $\square$

WE USE ι_{*p} TO IDENTIFY $T_p(X')$ WITH A SUBSPACE OF $T_p(X)$.

$$T_p(X') \subseteq T_p(X)$$

IN PARTICULAR, IF X' IS A SUBMANIFOLD OF $\mathbb{R}^n$,

$$T_p(X') \subseteq \mathbb{R}^n.$$

HERE'S THE POINT TO ALL OF THIS: IF X' IS A SUBMANIFOLD OF $\mathbb{R}^n$, EVERY ELEMENT OF $T_p(X')$ IS THE VELOCITY VECTOR OF A SMOOTH CURVE α IN X', WHICH CAN BE REGARDED AS A SMOOTH CURVE IN $\mathbb{R}^n$, WHOSE VELOCITY VECTOR CAN BE COMPUTED RELATIVE TO STANDARD COORDINATES AND IDENTIFIED WITH A POINT IN $\mathbb{R}^n$. BASICALLY, WE'RE BACK IN CALCULUS. HERE'S AN EXAMPLE:

EXAMPLE: S^n IS A SUBMANIFOLD OF $\mathbb{R}^{n+1}$. LET $p \in S^n$.

ANY ELEMENT OF $T_p(S^n)$ IS $\alpha'(0)$ FOR SOME SMOOTH CURVE α IN S^n.

REGARD α AS A SMOOTH CURVE IN $\mathbb{R}^{n+1}$. IN STANDARD COORDINATES,

$$\alpha(t) = (\alpha^1(t), \dots, \alpha^{n+1}(t)).$$

SINCE

$$\alpha'(0) = \alpha'(0)(x^1)\frac{\partial}{\partial x^1}\Big|_p + \dots + \alpha'(0)(x^{n+1})\frac{\partial}{\partial x^{n+1}}\Big|_p$$

AND

$$\alpha'(0)(x^i) = \frac{d}{dt}(x^i \circ \alpha)\Big|_{t=0} = \frac{d\alpha^i}{dt}(0),$$

AS A POINT IN $\mathbb{R}^{n+1}$ WE HAVE

$$\alpha'(0) = \left(\frac{d\alpha^1}{dt}(0), \dots, \frac{d\alpha^{n+1}}{dt}(0)\right)$$

BUT, SINCE $\alpha(t)$ LIES IN S^n,

$$(\alpha^1(t))^2 + \dots + (\alpha^{n+1}(t))^2 = 1$$

DIFFERENTIATING AT $t = 0$ GIVES

$$2\alpha^1(0)\frac{d\alpha^1}{dt}(0) + \dots + 2\alpha^{n+1}(0)\frac{d\alpha^{n+1}}{dt}(0) = 0$$

I.E.,

$$(\alpha^1(0), \dots, \alpha^{n+1}(0)) \cdot \left(\frac{d\alpha^1}{dt}(0), \dots, \frac{d\alpha^{n+1}}{dt}(0)\right) = 0$$

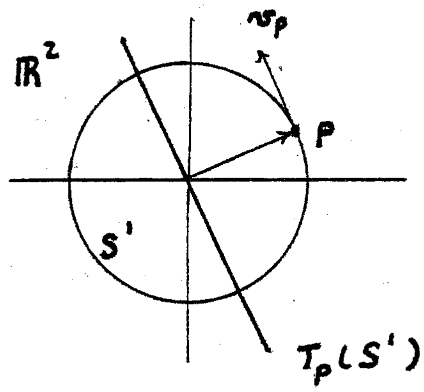
$$\alpha(0) \cdot \alpha'(0) = 0$$

$$p \cdot \alpha'(0) = 0$$

THUS, EVERY ELEMENT v_p OF $T_p(S^n)$ IS, AS A VECTOR IN $\mathbb{R}^{n+1}$, ORTHOGONAL TO (THE POSITION VECTOR OF) p :

$$p \cdot v_p = 0$$

SINCE $\dim(S^n) = n$, $\dim T_p(S^n) = n$ AND SINCE THE ORTHOGONAL COMPLEMENT OF p IN $\mathbb{R}^{n+1}$ HAS DIMENSION n, THIS MUST BE PRECISELY $T_p(S^n)$ (AS A SUBSPACE OF $\mathbb{R}^{n+1}$), E.G., FOR S^1,



NOW LET'S COMPUTE THE <u>DERIVATIVE</u> <u>OF</u> <u>THE</u> <u>HOPF</u> <u>MAP</u>

$$H : S^3 \to S^2$$

$$H(z^1, z^2) = \left(2\,\mathrm{Re}(\bar{z}^1 z^2),\ 2\,\mathrm{Im}(\bar{z}^1 z^2),\ |z^2|^2 - |z^1|^2 \right)$$

LET

$$z_1 = x^1 + i y^1 \qquad z_2 = x^2 + i y^2 .$$

THEN

$$H(x^1, y^1, x^2, y^2) = \left(2(x^1x^2 + y^1y^2),\ 2(x^1y^2 - x^2y^1),\ (x^2)^2 + (y^2)^2 - (x^1)^2 - (y^1)^2 \right)$$

THIS IS THE RESTRICTION TO S^3 OF THE SMOOTH MAP $\tilde{H} : \mathbb{R}^4 \to \mathbb{R}^3$ GIVEN BY THE SAME FORMULA, I.E.,

$$H = \tilde{H} \circ \iota$$

WHERE $\iota : S^3 \hookrightarrow \mathbb{R}^4$ IS THE INCLUSION MAP.

<u>EXERCISE</u> 89 : SHOW THAT, WHEN $T_p(S^3)$ IS IDENTIFIED WITH A SUBSPACE OF $T_p(\mathbb{R}^4)$ AS IN THE LEMMA ON PAGE 11, H_{*p} IS THE RESTRICTION TO $T_p(S^3)$ OF $\tilde{H}_{*p}$.

<u>EXERCISE</u> 90 : GENERALIZE EXERCISE 89.

THUS, WE NEED ONLY COMPUTE $\tilde{H}_{*p}$ AND RESTRICT IT TO $T_p(S^3)$. COMPUTING THE PARTIAL DERIVATIVES OF $2(x^1x^2+y^1y^2)$, $2(x^1y^2-x^2y^1)$, AND $(x^2)^2+(y^2)^2-(x^1)^2-(y^1)^2$ WE OBTAIN THE JACOBIAN OF $\tilde{H}$:

$$2\begin{pmatrix} x^2 & y^2 & x^1 & y^1 \\ y^2 & -x^2 & -y^1 & x^1 \\ -x^1 & -y^1 & x^2 & y^2 \end{pmatrix}$$

NOTICE THAT ANY PAIR OF ROWS (THOUGHT OF AS VECTORS IN $\mathbb{R}^4$) ARE ORTHOGONAL. IN PARTICULAR, THE ROWS ARE LINEARLY INDEPENDENT AT ANY $(x^1, y^1, x^2, y^2) \neq (0,0,0,0)$. THUS, THE JACOBIAN HAS RANK 3 AT ANY POINT OF S^3. WE CONCLUDE THAT $\forall p \in S^3$

$$\dim(\mathbb{R}^4) = \dim(\operatorname{Im} \tilde{H}_{p*}) + \dim(\ker \tilde{H}_{p*})$$

$$4 = 3 + \dim(\ker \tilde{H}_{p*})$$

SO

$$\dim(\ker \tilde{H}_{p*}) = 1$$

SINCE $H_{p*} = \tilde{H}_{p*} \mid T_p(S^3)$ IS CERTAINLY NOT AN ISOMORPHISM,

$$\dim(\ker H_{p*}) = 1$$

AS WELL. THUS,

$$\dim(T_p(S^3)) = \dim(\operatorname{Im} H_{p*}) + \dim(\ker H_{p*})$$

IMPLIES

$$\dim(\operatorname{Im} H_{p*}) = 2$$

I.E.,

$$H_{p*} : T_p(S^3) \to T_{H(p)}(S^2)$$

IS SURJECTIVE.

THUS, THE HOPF MAP IS AN EXAMPLE OF THE FOLLOWING:

LET X BE A SMOOTH n-MANIFOLD AND Y A SMOOTH m-MANIFOLD WITH $n > m$. A SMOOTH MAP

$$F : X \to Y$$

IS A <u>SUBMERSION AT</u> $p \in X$ IF $F_{*p} : T_p(X) \to T_{F(p)}(Y)$ IS SURJECTIVE. F IS A <u>SUBMERSION</u> IF THIS IS TRUE FOR EVERY $p \in X$.

AND, WHY DO WE CARE ?

<u>THEOREM</u> : X A SMOOTH n-MANIFOLD, Y A SMOOTH m-MANIFOLD, $n > m$, $F : X \to Y$ A SMOOTH MAP. IF $q \in F(X)$ AND F IS A SUBMERSION AT EACH $p \in F^{-1}(q)$, THEN $F^{-1}(q)$ IS A SUBMANIFOLD OF X OF DIMENSION $n-m$.

> <u>NOTE</u> : THIS IS A HUGE GENERALIZATION OF OUR EARLIER RESULT ON LEVEL HYPERSURFACES. I HAVE PUT THE PROOF IN AN APPENDIX BECAUSE I WOULD LIKE TO BE CERTAIN THAT WE HAVE TIME TO SEE SOME NONTRIVIAL APPLICATIONS.

FIRST, WHAT DOES IT SAY ABOUT THE HOPF MAP

$$H : S^3 \to S^2 \quad ?$$

<u>EXERCISE</u> 91 : SHOW THAT H MAPS S^3 <u>ONTO</u> S^2.

THUS, FOR ANY $p \in S^2$, $H^{-1}(p)$ IS A SUBMANIFOLD OF S^3 OF DIMENSION $3-2 = 1$. IT IS, OF COURSE, JUST THE COPY OF S^1 THAT, IN OUR ORIGINAL CONSTRUCTION (" QUOTIENT SPACES ", PAGES 19-26) WE IDENTIFIED TO A POINT TO GET p, BUT NOW WE KNOW THAT THESE CIRCLES ARE ACTUALLY SUBMANIFOLDS OF S^3.

NOW FOR A MUCH MORE INTERESTING EXAMPLE. WE WILL SHOW THAT THE ORTHOGONAL GROUP $O(3)$ AND THE SPECIAL ORTHOGONAL GROUP $SO(3)$ ARE SMOOTH MANIFOLDS (AND THEREFORE LIE GROUPS).

WE WILL IDENTIFY THE SET OF ALL 3×3 REAL MATRICES WITH $\mathbb{R}^9$ IN THE FOLLOWING WAY :

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longrightarrow$$

$$(a_{11}, a_{21}, a_{22}, a_{31}, a_{32}, a_{33}, a_{12}, a_{13}, a_{23})$$

(STRING OUT THE ENTRIES "LOWER TRIANGLE FIRST")

CONSIDER THE SUBSET OF ALL 3×3 REAL MATRICES S THAT ARE SYMMETRIC ($S^T = S$)

$$S = \begin{pmatrix} s_{11} & s_{21} & s_{31} \\ s_{21} & s_{22} & s_{32} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} \longrightarrow$$

$$(s_{11}, s_{21}, s_{22}, s_{31}, s_{32}, s_{33}, \underbrace{s_{21}, s_{31}, s_{32}}_{\text{REPITITIONS}})$$

THUS, PROJECTING THIS SUBSET OF $\mathbb{R}^9$ ONTO THE FIRST 6 COORDINATES GIVES A MAP FROM THE SYMMETRIC MATRICES TO $\mathbb{R}^6$ THAT IS ONE-TO-ONE, ONTO, CONTINUOUS AND HAS A CONTINUOUS INVERSE, I.E., A HOMEOMORPHISM.

NOW WE DEFINE A MAP

$$
\begin{array}{ccc}
F : \mathbb{R}^9 & \longrightarrow & \mathbb{R}^6 \\
\uparrow & & \uparrow \\
\text{ALL } 3\times 3 \text{ REAL MATRICES} & & \text{SYMMETRIC } 3\times 3 \text{ REAL MATRICES}
\end{array}
$$

BY

$$F(A) = AA^T$$

(<u>NOTE</u> : AA^T IS SYMMETRIC BECAUSE $(AA^T)^T = (A^T)^T A^T = AA^T$.)

WRITING OUT THE MATRIX PRODUCT AA^T AND STRINGING OUT ITS ENTRIES AS DESCRIBED ABOVE ONE FINDS THAT

$$A = (a_{11}, a_{21}, a_{22}, a_{31}, a_{32}, a_{33}, a_{12}, a_{13}, a_{23}) \xrightarrow{F}$$

$$
\begin{aligned}
F(A) = \big(& a_{11}^2 + a_{12}^2 + a_{13}^2 \;, \; a_{21}a_{11} + a_{22}a_{12} + a_{23}a_{13} \;, \\
& a_{21}^2 + a_{22}^2 + a_{23}^2 \;, \; a_{31}a_{11} + a_{32}a_{12} + a_{33}a_{13} \;, \\
& a_{31}a_{21} + a_{32}a_{22} + a_{33}a_{23} \;, \; a_{31}^2 + a_{32}^2 + a_{33}^2 \big)
\end{aligned}
$$

THE 6 COORDINATE FUNCTIONS $F^1, \ldots, F^6$ OF THE COORDINATES $(x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9) = (a_{11}, a_{21}, a_{22}, a_{31}, a_{32}, a_{33}, a_{12}, a_{13}, a_{23})$ ARE THEREFORE CLEARLY SMOOTH:

$$F^1(a_{11}, \ldots, a_{23}) = a_{11}^2 + a_{12}^2 + a_{13}^2$$

$$F^2(a_{11}, \ldots, a_{23}) = a_{21}a_{11} + a_{22}a_{12} + a_{23}a_{13}$$

$$\vdots$$

THE JACOBIAN OF F AT A IS

$$J_F(A) = \begin{pmatrix} \frac{\partial F^1}{\partial a_{11}} & \frac{\partial F^1}{\partial a_{21}} & \cdots & \frac{\partial F^1}{\partial a_{23}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F^6}{\partial a_{11}} & \frac{\partial F^6}{\partial a_{21}} & \cdots & \frac{\partial F^6}{\partial a_{23}} \end{pmatrix} =$$

$$\begin{pmatrix} 2a_{11} & 0 & 0 & 0 & 0 & 0 & 2a_{12} & 2a_{13} & 0 \\ a_{21} & a_{11} & a_{12} & 0 & 0 & 0 & a_{22} & a_{23} & a_{13} \\ 0 & 2a_{21} & 2a_{22} & 0 & 0 & 0 & 0 & 0 & 2a_{23} \\ a_{31} & 0 & 0 & a_{11} & a_{12} & a_{13} & a_{32} & a_{33} & 0 \\ 0 & a_{31} & a_{32} & a_{21} & a_{22} & a_{23} & 0 & 0 & a_{33} \\ 0 & 0 & 0 & 2a_{31} & 2a_{32} & 2a_{33} & 0 & 0 & 0 \end{pmatrix}$$

THIS IS JUST THE MATRIX OF F_{*A} IN STANDARD COORDINATES ON $\mathbb{R}^9$ AND $\mathbb{R}^6$.

NOW RECALL THAT THE ORTHOGONAL GROUP $O(3)$ CONSISTS OF THOSE 3×3 REAL MATRICES A SATISFYING $AA^T = I$ (THE 3×3 IDENTITY MATRIX), I.E.,

$$O(3) = F^{-1}(I)$$

THUS, TO SHOW THAT $O(3)$ IS A SUBMANIFOLD OF $\mathbb{R}^9$ WE NEED ONLY PROVE THAT

$$A \in O(3) \Rightarrow J_F(A) \text{ HAS RANK } 6$$

BUT THE RANK OF A MATRIX IS EQUAL TO ITS ROW RANK, I.E., THE DIMENSION OF THE SPACE SPANNED BY ITS ROWS, SO IT WILL SUFFICE TO SHOW THAT

$$A \in O(3) \Rightarrow \text{THE ROWS OF } J_F(A) \text{ ARE LINEARLY INDEPENDENT IN } \mathbb{R}^9.$$

BUT $A \in O(3) \Rightarrow$ THE ROWS OF A ARE ORTHOGONAL AND THIS IMPLIES THAT THE ROWS OF $J_F(A)$ ARE ORTHOGONAL, E.G., THE DOT PRODUCT OF THE 2^{ND} AND 3^{RD} ROWS IS

$$a_{21}\cdot 0 + 2a_{11}a_{21} + 2a_{12}a_{22} + 0\cdot 0 + 0\cdot 0 + 0\cdot 0 + a_{22}\cdot 0 + a_{23}\cdot 0 + 2a_{13}a_{23}$$

$$= 2(a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23}) = 0$$

SIMILARLY FOR THE REMAINING ROWS.

THUS, AT EVERY $A \in O(3) \subseteq \mathbb{R}^9$ THE JACOBIAN OF $F(A) = AA^T$ HAS RANK 6 AND SO F_{*A} IS SURJECTIVE. OUR THEOREM (PAGE 17) THEREFORE IMPLIES THAT $O(3) = F^{-1}(I)$ IS A SUBMANIFOLD OF $\mathbb{R}^9$ OF DIMENSION $9-6=3$.

$SO(3)$ IS AN OPEN SUBSET OF $O(3)$ SO IT IS ALSO A 3-DIMENSIONAL SUBMANIFOLD OF $\mathbb{R}^9$.

> NOTE: IT CAN BE SHOWN THAT $SO(3)$ IS ACTUALLY DIFFEOMORPHIC TO REAL PROJECTIVE 3-SPACE $\mathbb{RP}^3$.

SINCE MATRIX MULTIPLICATION AND INVERSION ARE SMOOTH ON $GL(3, \mathbb{R})$ IT FOLLOWS FROM EXERCISE 7 (3) & (4) THAT THE SAME IS TRUE OF $O(3)$ AND $SO(3)$ SO THESE ARE ALSO LIE GROUPS.

REMARK: ALL OF THIS GENERALIZES TO $O(n)$ AND $SO(n)$ FOR ANY $n > 1$.

A SUBMERSION SQUASHES A HIGH DIMENSIONAL MANIFOLD INTO A LOWER DIMENSIONAL MANIFOLD WITH (INTUITIVELY) "MINIMAL SQUASHING" (DERIVATIVE OF MAXIMAL RANK).

TURNING THE DIMENSIONS AROUND ONE ARRIVES AT THE FOLLOWING ANALOGOUS NOTIONS.

X AN n-MANIFOLD, Y AN m-MANIFOLD, $n < m$

A SMOOTH MAP

$$F : X \to Y$$

IS AN <u>IMMERSION</u> IF, FOR EACH $p \in X$,

$$F_{*p} : T_p(X) \longrightarrow T_{F(p)}(Y)$$

IS ONE-TO-ONE. IF, IN ADDITION, F IS A HOMEOMORPHISM ONTO ITS IMAGE $F(X)$, THEN F IS CALLED AN <u>EMBEDDING</u>.

USING TECHNIQUES VERY MUCH LIKE THOSE WE USED TO PROVE THE LAST THEOREM ONE CAN SHOW THAT

1. ANY IMMERSION IS LOCALLY AN EMBEDDING
2. IF $F : X \to Y$ IS AN EMBEDDING, THEN $F(X)$ IS A SMOOTH MANIFOLD AND $F : X \to F(X)$ IS A DIFFEOMORPHISM.

AN OBVIOUS EXAMPLE OF AN EMBEDDING IS

$$F : \mathbb{R}^n \to \mathbb{R}^m \qquad (n < m)$$

$$F(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0).$$

HERE ARE SOME EXAMPLES OF IMMERSIONS : FIX A $k = \pm 1, \pm 2, \ldots$

DEFINE

$$F_k : S^1 \to \mathbb{R}^2 \quad (\equiv \mathbb{C})$$

BY

$$F_k(z) = z^k \qquad \forall z \in S^1$$

$$F_k(e^{i\theta}) = (e^{i\theta})^k = e^{ik\theta}$$

$$F_k(\cos\theta, \sin\theta) = (\cos k\theta, \sin k\theta)$$

THE POLAR COORDINATE EXPRESSION FOR F_k IS

$$\theta \to (\cos k\theta, \sin k\theta)$$

SO THE JACOBIAN IS

$$\begin{pmatrix} -k \sin k\theta \\ k \cos k\theta \end{pmatrix}$$

WHICH HAS RANK 1 AT EVERY POINT SO $(F_k)_{*p}$ IS INJECTIVE AT EACH $p \in S^1$ AND F_k IS AN IMMERSION.

F_k IS AN EMBEDDING IF AND ONLY IF $k = \pm 1$.

ALTHOUGH WE WILL HAVE NO TIME FOR THE PROOF (AND NO REAL NEED FOR THE RESULT) I WOULD BE REMISS IN MY DUTY IF I DID NOT MAKE YOU AWARE OF THE FOLLOWING FAMOUS OLD RESULT.

<u>WHITNEY</u> <u>EMBEDDING</u> <u>THEOREM</u>: LET X BE A SMOOTH MANIFOLD OF DIMENSION n. THEN THERE EXISTS A SMOOTH EMBEDDING

$$F : X \longrightarrow \mathbb{R}^{2n+1}$$

(SO $\mathbb{R}^{2n+1}$ CONTAINS A DIFFEOMORPHIC COPY OF EVERY SMOOTH n-MANIFOLD).

IN THE NEXT SECTION ON RIEMANNIAN MANIFOLDS WE WILL HAVE OCCASION TO MENTION AN EVEN MORE FAMOUS EMBEDDING THEOREM DUE TO JOHN NASH (YES, <u>THAT</u> JOHN NASH).

APPENDIX :

THEOREM : LET X BE A SMOOTH MANIFOLD OF DIMENSION n, Y A SMOOTH MANIFOLD OF DIMENSION m WITH $n > m$ AND $F : X \to Y$ A SMOOTH MAP. IF $q \in F(X)$ AND F IS A SUBMERSION AT EACH $p \in F^{-1}(q)$, THEN $F^{-1}(q)$ IS A SUBMANIFOLD OF X OF DIMENSION $n-m$.

PROOF :

WE FIRST PROVE THE RESULT IN THE SPECIAL CASE IN WHICH $X = U$ IS AN OPEN SET IN $\mathbb{R}^n$ AND $Y = \mathbb{R}^m$ $(n > m)$.

$$F : U \longrightarrow \mathbb{R}^m$$

$$F(x^1, \ldots, x^n) = (F^1(x^1, \ldots, x^n), \ldots, F^m(x^1, \ldots, x^n))$$

FIX A $q \in F(U)$ WITH THE PROPERTY THAT FOR EACH $p \in F^{-1}(q)$ THE JACOBIAN

$$J_F(p) = \begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}$$

HAS RANK m (I.E., F_{*p} IS SURJECTIVE).

FIX A $p \in F^{-1}(q)$. WE MUST FIND A CHART (V, ψ) FOR $\mathbb{R}^n$ WITH $p \in V$ SUCH THAT

$$\psi(V \cap F^{-1}(q)) = \{ y = (y^1, \ldots, y^n) \in \psi(V) : y^{n-m+1} = \cdots = y^n = 0 \}$$

$p \in F^{-1}(q) \Rightarrow$ $J_F(p)$ HAS RANK m SO SOME $m \times m$ SUBMATRIX IS NONSINGULAR. BY RENUMBERING THE COORDINATES IF NECESSARY WE MAY ASSUME IT IS THE ONE IN THE DOTTED BOX :

$$J_F(p) = \left(\begin{array}{ccc|ccc} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^{n-m}}(p) & \frac{\partial F^1}{\partial x^{n-m+1}}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^{n-m}}(p) & \frac{\partial F^m}{\partial x^{n-m+1}}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{array} \right)$$

WE WISH TO USE THE INVERSE FUNCTION THEOREM SO WE DEFINE A SMOOTH MAP

$$\tilde{F} : U \longrightarrow \mathbb{R}^n = \mathbb{R}^{n-m} \times \mathbb{R}^m$$

BY

$$\begin{aligned} \tilde{F}(x) &= \tilde{F}(x^1, \ldots, x^{n-m}, x^{n-m+1}, \ldots, x^n) \\ &= (x^1, \ldots, x^{n-m}, F^1(x), \ldots, F^m(x)) \end{aligned}$$

THE JACOBIAN $J_{\tilde{F}}(p)$ IS GIVEN BY

$$\begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^{n-m}}(p) & \frac{\partial F^1}{\partial x^{n-m+1}}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & & \vdots & & & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^{n-m}}(p) & \frac{\partial F^m}{\partial x^{n-m+1}}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}$$

WHICH CLEARLY HAS n INDEPENDENT COLUMNS, I.E., RANK n, SO IT IS NONSINGULAR.

THE INVERSE FUNCTION THEOREM THEREFORE IMPLIES THAT THERE EXIST OPEN NEIGHBORHOODS V OF

$$p = (p^1, \ldots, p^{n-m}, p^{n-m+1}, \ldots, p^n)$$

AND W OF

$$\tilde{F}(p) = (p^1, \ldots, p^{n-m}, F^1(p), \ldots, F^m(p))$$

$$= (p^1, \ldots, p^{n-m}, q^1, \ldots, q^m)$$

SUCH THAT

$$\tilde{F}|V : V \to W$$

IS A DIFFEOMORPHISM.

NOTICE THAT $(V, \tilde{F}|V)$ IS A CHART ON $\mathbb{R}^n$ WITH $p \in V$ AND

IT CARRIES ANY

$$(x^1, \dots, x^{n-m}, x^{n-m+1}, \dots, x^n) \in V \cap F^{-1}(q)$$

TO

$$(x^1, \dots, x^{n-m}, q^1, \dots, q^m).$$

THUS, IF WE DEFINE

$$\psi : V \to \mathbb{R}^n$$

BY

$$\psi(x^1, \dots, x^{n-m}, x^{n-m+1}, \dots, x^n) = (y^1, \dots, y^{n-m}, y^{n-m+1}, \dots, y^n) =$$

$$(x^1, \dots, x^{n-m}, x^{n-m+1} - q^1, \dots, x^n - q^m)$$

WE HAVE A CHART (V, ψ) WITH THE REQUIRED PROPERTY.

NOW WE'LL USE THE SPECIAL CASE WE HAVE JUST ESTABLISHED TO PROVE THE THEOREM IN GENERAL.

X AN n-MANIFOLD $\qquad$ Y AN m-MANIFOLD

$$n > m$$

$$F : X \to Y$$

$q \in F(X)$ SUCH THAT F IS A SUBMERSION AT EACH $p \in F^{-1}(q)$

LET $p \in F^{-1}(q)$ BE ARBITRARY. CHOOSE CHARTS (U, φ) AT p AND (V, ψ) AT q AND CONSIDER THE COORDINATE EXPRESSION

$$\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \longrightarrow \psi(V)$$

NOTE THAT

$$(\psi \circ F \circ \varphi^{-1})(\varphi(p)) = \psi(q)$$

AND

$$(\psi \circ F \circ \varphi^{-1})_{*\varphi(p)} = \psi_{*q} \circ F_{*p} \circ \varphi^{-1}_{*\varphi(p)}.$$

ALL THREE DERIVATIVES ON THE RIGHT-HAND SIDE ARE SURJECTIVE SO

$$(\psi \circ F \circ \varphi^{-1})_{*\varphi(p)} : T_{\varphi(p)}(\varphi(U \cap F^{-1}(V))) \rightarrow T_{\psi(q)}(\psi(V))$$

IS SURJECTIVE.

THIS IS TRUE FOR EVERY $p \in F^{-1}(q) \cap (U \cap F^{-1}(V))$. NOTE THAT

$$p \in F^{-1}(q) \cap (U \cap F^{-1}(V)) \quad \Leftrightarrow \quad \varphi(p) \in (\psi \circ F \circ \varphi^{-1})^{-1}(\psi(q))$$

(CONVINCE YOURSELF OF THIS)

THUS,

$\psi \circ F \circ \varphi^{-1}$ IS A SUBMERSION AT EACH POINT OF $(\psi \circ F \circ \varphi^{-1})^{-1}(\psi(q))$

SO WE CAN APPLY THE RESULT FOR EUCLIDEAN SPACES TO CONCLUDE THAT

$$(\psi \circ F \circ \varphi^{-1})^{-1}(\psi(q)) = \varphi(F^{-1}(q) \cap U)$$

IS A SUBMANIFOLD OF $\varphi(U \cap F^{-1}(V)) \overset{\text{OPEN}}{\subseteq} \mathbb{R}^n$ OF DIMENSION $n-m$.

THUS, FOR EACH $\varphi(p) \in (\psi \circ F \circ \varphi^{-1})^{-1}(\psi(q))$ THERE IS A CHART (W, ξ) FOR $\mathbb{R}^n$ WITH COORDINATE FUNCTIONS $x^1, \dots, x^n$ SUCH THAT

$$\varphi(p) \in W \subseteq \varphi(U \cap F^{-1}(V))$$

AND

$$\xi(W \cap (\psi \circ F \circ \varphi^{-1})^{-1}(\psi(q))) =$$

$$\{(x^1, \dots, x^n) \in \xi(W) : x^{n-m+1} = \dots = x^n = 0\}$$

NOW IT IS EASY TO CHECK THAT

$$(\varphi^{-1}(W), \xi \circ \varphi | \varphi^{-1}(W))$$

IS A CHART OF THE REQUIRED TYPE AT $p \in F^{-1}(q)$. $\square$