DERIVATIVES

In Calculus the derivative of \( f : \mathbb{R} \rightarrow \mathbb{R} \) at a point \( p \) is generally defined to be a number, but the important object is really the straight line with that slope through \( (p, f(p)) \) (the "best linear approximation to \( f \) near \( p \)).

Suppose \( X \) and \( Y \) are smooth manifolds of dimension \( n \) and \( m \), respectively, and

\[
F : X \rightarrow Y
\]

is a smooth map.

Fix \( p \in X \). Then \( F(p) \in Y \). We want to define a linear transformation

\[
F_p^* : T_p(X) \rightarrow T_{F(p)}(Y)
\]

called the derivative of \( F \) at \( p \) (thought of intuitively as the "best linear approximation to \( F \) near \( p \)).

Note: Some texts call what we are going to define the "differential of \( F \) at \( p \)" and denote it \( dF_p \).
There are two useful (and equivalent) ways to define $F_{*p}$.
I'll give both and show that they are equivalent.

1

$F_{*p} : T_p(X) \rightarrow T_{F(p)}(Y)$

Fix $n_p \in T_p(X)$. Regard $n_p$ as the velocity vector of some smooth curve $\alpha$ in $X$:

$n_p = \alpha'(t_0)$

Then $F \circ \alpha$ is a smooth curve in $Y$ through $F(p)$

$(F \circ \alpha)(t) = F(\alpha(t))$

$(F \circ \alpha)(t_0) = F(\alpha(t_0)) = F(p)$

Define

$F_{*p}(n_p) = F_{*p}(\alpha'(t_0))$

$= (F \circ \alpha)'(t_0)$

2

$F_{*p} : T_p(X) \rightarrow T_{F(p)}(Y)$
Fix $\nu_p \in T_p(X)$. Regard

$$\nu_p : C^\infty(X) \rightarrow \mathbb{R}$$

Then $F_*\nu_p(\nu_p)$ is to be regarded as

$$F_*\nu_p(\nu_p) : C^\infty(Y) \rightarrow \mathbb{R}$$

$$g \in C^\infty(Y) \Rightarrow g \circ F \in C^\infty(X)$$

Define

$$\left( F_*\nu_p(\nu_p) \right)(g) = \nu_p(\circ F)$$

To see that gives the same result as the 1st definition when $\nu_p = \omega(\mathbf{u}_o)$ note that, for every $g \in C^\infty(Y)$,

$$\left( F_*\nu_p(\omega(\mathbf{u}_o)) \right)(g) = \omega(\mathbf{u}_o)(\circ F)$$

$$= \frac{d}{dt} \left( g \circ (F \circ \mathbf{u}_o) \right) \bigg|_{t=t_0}$$

$$= \frac{d}{dt} \left( \circ (F \circ \mathbf{u}_o) \right) \bigg|_{t=t_0}$$

$$= \left( (F \circ \mathbf{u}_o)'(t_0) \right)(g)$$

So

$$F_*\nu_p(\omega(\mathbf{u}_o)) = (F \circ \mathbf{u}_o)'(t_0)$$
We will compute some examples of derivatives shortly, but first some elementary properties:

1. \( F_p : T_p(X) \rightarrow T_{F(p)}(Y) \) is a linear transformation

   \[
   \text{Proof: } (F_p(a \cdot v_p + b \cdot w_p))(g) = (a \cdot v_p + b \cdot w_p)(g \circ F) \\
   = a \cdot v_p(g \circ F) + b \cdot w_p(g \circ F) \\
   = a \cdot (F_p(v_p))(g) + b \cdot (F_p(w_p))(g) \\
   = (a \cdot F_p(v_p) + b \cdot F_p(w_p))(g)
   \]

   So

   \[
   F_p(a \cdot v_p + b \cdot w_p) = a \cdot F_p(v_p) + b \cdot F_p(w_p)
   \]

2. If \( \text{Id} : X \rightarrow X \) is the identity map, then, for any \( p \in X \),

   \[
   \text{Id}_p : T_p(X) \rightarrow T_{\text{Id}(p)}(X) = T_p(X)
   \]

   is \( \text{Id}_{T_p(X)} \).

   Exercise 85: Prove this.

3. If \( (U, \psi) \) is a chart at \( p \in X \) with coordinate functions \( x'_1, \ldots, x'_n \) and \( (V, \psi) \) is a chart at \( F(p) \) with coordinate functions \( y'_1, \ldots, y'_m \), then the matrix of \( F_p \) with respect to the bases

   \[
   \left\{ \frac{\partial}{\partial x'_i} \bigg|_p \right\}_{i=1,\ldots,n}
   \]

   and

   \[
   \left\{ \frac{\partial}{\partial y'_i} \bigg|_{F(p)} \right\}_{i=1,\ldots,m}
   \]

   for \( T_p(X) \)
\[ \left\{ \frac{\partial}{\partial y^j} \right\}_{F(p)} \text{ for } T_{F(p)}(\mathcal{L}) \]

is the Jacobian of the coordinate expression

\[ \psi \circ F \circ \mathcal{L}^{-1} : \mathcal{L}(\mathcal{U} \cap F^{-1}(V)) \to \psi(V) \]

at \( \mathcal{L}(p) \).

**Proof:** By the Basis Theorem (for \( T_{F(p)}(\mathcal{L}) \)),

\[ F_{\#p} \left( \frac{\partial}{\partial x^i} \big|_p \right) = \left( F_{\#p} \left( \frac{\partial}{\partial x^i} \big|_p \right) \left( y^j \big|_{\mathcal{L}(p)} \right) \frac{\partial}{\partial y^j} \big|_{F(p)} \right) \]

\[ = \left( \frac{\partial}{\partial x^i} \big|_p (y^j \circ F) \right) \frac{\partial}{\partial y^j} \big|_{F(p)} \]

\[ = \left( \frac{\partial (y^j \circ F \circ \mathcal{L}^{-1})}{\partial x^i} \big|_{\mathcal{L}(p)} \right) \frac{\partial}{\partial y^j} \big|_{F(p)} \]

But

\[ y^j \circ F \circ \mathcal{L}^{-1} = \pi^j \circ (\psi \circ F \circ \mathcal{L}^{-1}) \]

\[ = j^{th} \text{ coordinate function of } \psi \circ F \circ \mathcal{L}^{-1} \]

So the partial derivatives of these coordinate functions with respect to \( x^i \) form the \( i^{th} \) column of the matrix of \( F_{\#p} \) and this is just the Jacobian of \( \psi \circ F \circ \mathcal{L}^{-1} \). \( \square \)
4. **(Chain Rule)** Let $F : X \to Y$ and $G : Y \to Z$ be smooth maps of differentiable manifolds and let $p \in X$. Then $G \circ F : X \to Z$ is smooth and

$$(G \circ F)_p = G_{F(p)} \circ F_p.$$ \hfill \square

**Proof:** Smoothness of $G \circ F$ has already been proved. Let $N_p \in T_p(X)$. Then, for every $g \in C^\infty(X)$,

$$(G \circ F)_p (N_p)(g) = N_p (g \circ (G \circ F))$$

$$= N_p ((g \circ G) \circ F)$$

$$= (F_p(N_p))(g \circ G)$$

$$= (G_{F(p)}(F_p(N_p)))(g)$$

So

$$(G \circ F)_p (N_p) = (G_{F(p)} \circ F_p)(N_p)$$

and

$$(G \circ F)_p = G_{F(p)} \circ F_p.$$ 

**Exercise 86:** Write this out in terms of the matrices in #3 and explain why it is called the "Chain Rule."
As a corollary we have the manifold version of the

**Inverse Function Theorem:** Let $F : X \rightarrow Y$ be a smooth map of differentiable manifolds and let $p \in X$. Then

$$F_\ast p : T_p(X) \rightarrow T_{F(p)}(Y)$$

is an isomorphism if and only if $F : X \rightarrow Y$ is a local diffeomorphism near $p$ (i.e., there exist open neighborhoods $U$ of $p$ and $V$ of $F(p)$ such that $F|U$ is a diffeomorphism of $U$ onto $V$).

**Proof:** Suppose first that $F$ is a local diffeomorphism near $p$. Then, on some neighborhood of $p$, $F^{-1} \circ F = 1_p$ so the chain rule gives

$$(F^{-1} \circ F)_\ast p = 1_{T_p(X)}$$

Similarly, on some neighborhood of $F(p)$, $F \circ F^{-1} = 1_{T_{F(p)}(Y)}$ gives

$$(F \circ F^{-1})_{\ast F(p)} = 1_{T_{F(p)}(Y)}$$

$${F_\ast}_{F(p)} \circ {F^{-1}_\ast}_{F(p)} = 1_{T_{F(p)}(Y)}$$

$$F_\ast p \circ F^{-1}_\ast p = 1_{T_{F(p)}(Y)}$$

So

$$F^{-1}_\ast p = (F_\ast p)^{-1}$$

and $F_\ast p$ is an isomorphism.

Next suppose $F_\ast p$ is an isomorphism. Choosing charts $(U, \phi)$ at $p$ and $(V, \psi)$ at $F(p)$, the matrix of $F_\ast p$ is the Jacobian of the corresponding coordinate expression for $F$. This Jacobian is therefore nonsingular.
ASIDE: AT THIS POINT WE NEED TO APPEAL TO THE
"ORDINARY" INVERSE FUNCTION THEOREM FROM
CALCULUS SO I WILL PROVIDE A STATEMENT
OF IT.

**INVERSE FUNCTION THEOREM:** LET $A$ BE AN
OPEN SET IN $\mathbb{R}^n$ AND $G: A \rightarrow \mathbb{R}^n$ A SMOOTH MAP.
SUPPOSE $a \in A$ AND THE JACOBIAN OF $G$ IS
NONSINGULAR AT $a$. THEN THERE EXIST OPEN SETS
$B_a$ AND $B_{G(a)}$ IN $\mathbb{R}^n$ SUCH THAT $a \in B_a,
G(a) \in B_{G(a)}$;

$$G|_{B_a} : B_a \rightarrow B_{G(a)}$$

IS A SMOOTH BIJECTION AND HAS A SMOOTH
INVERSE

$$(G|_{B_a})^{-1} : B_{G(a)} \rightarrow B_a.$$

IN A NUTSHELL, $\det (J_G(a)) \neq 0 \Rightarrow G$
IS A LOCAL Diffeomorphism Near $a$.

NOW, RETURNING TO THE PROOF WE CONCLUDE THAT THE COORDINATE
EXPRESSION

$$\psi \circ F \circ \xi^{-1}$$

MUST BE A
DIFFEOMORPHISM NEAR \( e(p) \), I.E., ON SOME OPEN NEIGHBORHOOD \( U' \) OF \( e(p) \),
\[
\psi \circ \Phi^{-1} : U' \rightarrow V' = \psi(\Phi^{-1}(U'))
\]
is a DIFFEOMORPHISM. LET \( U = \psi^{-1}(U') \) AND \( V = \psi^{-1}(V') \). THEN
\[
F|_U : U \rightarrow V
\]
is a DIFFEOMORPHISM.

COROLLARY: IF TWO SMOOTH MANIFOLDS ARE DIFFEOMORPHIC, THEN THEY
HAVE THE SAME DIMENSION.

NOTE: AS "OBVIOUS" AS THIS MAY SOUND TO YOU, REALIZE THAT THE
PROOF DEPENDS CRUCIALLY ON THE DIFFERENTIABLE STRUCTURE.
The corresponding statement for topological manifolds
(Homeomorphic \( \Rightarrow \) Same Dimension) IS TRUE, BUT MUCH
MORE DIFFICULT TO PROVE.

NOTE: IN THE PROOF OF THE INVERSE FUNCTION THEOREM WE SHOWED
THAT, FOR A (LOCAL) DIFFEOMORPHISM \( F \)
\[
(F*_{p})^{-1} = (F^{-1})*_{F(p)}
\]
AND THIS IN ITSELF IS WORTH REMEMBERING.

EXERCISE 87: USE THE INVERSE FUNCTION THEOREM TO FILL IN THE LAST DETAIL
IN OUR PROOF OF SMOOTHNESS FOR THE GAUSS MAP OF THE TORUS (SEE PAGES 12-13 OF
THE LECTURE ON "SMOOTH MAPPINGS ON MANIFOLDS").
WE WOULD NOW LIKE TO COMPUTE SOME EXPPLICIT EXAMPLES OF TANGENT
SPACES TO MANIFOLDS AND DERIVATIVES OF SMOOTH MAPS.

BEGIN BY TAKING ADVANTAGE OF THE FACT THAT THE TANGENT SPACE
TO A SUBMANIFOLD OF EUCLIDEAN SPACE CAN, IN A CANONICAL
WAY, BE IDENTIFIED WITH A LINEAR SUBSPACE OF THAT EUCLIDEAN
SPACE. YOU WILL TAKE THE FIRST STEP.

EXERCISE 88: LET IR^n HAVE ITS STANDARD DIFFERENTIABLE
STRUCTURE. FIX A p ∈ IR^n. FOR EACH n ∈ IR^n
LET n_p ∈ T_p(IR^n) BE DEFINED BY

n_p = α'(10)

WHERE

α : IR → IR^n

α(k) = p + kn

SHOW THAT

n → n_p

IS AN ISOMORPHISM OF IR^n ONTO T_p(IR^n).

HENCEFORTH WE CALL THIS THE CANONICAL ISOMORPHISM AND USE IT
TO IDENTIFY T_p(IR^n) WITH IR^n.
ANY TANGENT SPACE TO A SUBMANIFOLD OF $\mathbb{R}^n$ CAN LIKewise BE IDENTIFIED WITH A LINEAR SUBSPACE OF $\mathbb{R}^n$. IN FACT, MORE IS TRUE:

**Lemma:** Let $X'$ be a submanifold of $X$, $p \in X'$ and $\mathcal{L} : X' \to X$ the inclusion map. Then

$$\mathcal{L}_p : T_p(X') \to T_{\mathcal{L}(p)}(X) = T_p(X)$$

is an isomorphism of $T_p(X')$ onto a subspace of $T_p(X)$.

**Proof:** $\mathcal{L}_p : T_p(X') \to T_p(X)$ is linear so we need only show that it is one-to-one. If $\dim X = n$ and $\dim X' = k \leq n$, then there is a chart $(U, \varphi)$ at $p$ in $X$ with coordinate functions $x_1', \ldots, x_k', \ldots, x_n'$ such that $(U \cap X', \varphi|_{U \cap X'})$ is a chart at $p$ in $X'$ and

$$\varphi(U \cap X') = \{ x = (x_1', \ldots, x_k', \ldots, x_n') \in \varphi(U) : x_1' = \ldots = x_k' = 0 \}.$$

Thus, the coordinate functions $y_1', \ldots, y_k'$ for $(U \cap X', \varphi|_{U \cap X'})$ are

$$y_1' = x_1'$$

$$\vdots$$

$$y_k' = x_k'$$

Relative to these charts the coordinate expression for
\( g : X' \to X \) is just

\[
(y', \ldots, y^k) \to (y', \ldots, y^k, 0, \ldots, 0).
\]

The Jacobian contains the \( \mathbb{R} \times \mathbb{R} \) identity matrix so it has rank \( \mathbb{R} \) and \( g \) is one-to-one.

We use \( g \) to identify \( T_p(X') \) with a subspace of \( T_p(X) \).

\[
T_p(X') \subset T_p(X)
\]

In particular, if \( X' \) is a submanifold of \( \mathbb{R}^n \),

\[
T_p(X') \subset \mathbb{R}^n.
\]

Here's the point to all of this: If \( X' \) is a submanifold of \( \mathbb{R}^n \), every element of \( T_p(X') \) is the velocity vector of a smooth curve \( \alpha \) in \( X' \), which can be regarded as a smooth curve in \( \mathbb{R}^n \), whose velocity vector can be computed relative to standard coordinates and identified with a point in \( \mathbb{R}^n \). Basically, we're back in calculus. Here's an example:
EXAMPLE: \( S^n \) is a submanifold of \( \mathbb{R}^{n+1} \). Let \( p \in S^n \).

Any element of \( T_p(S^n) \) is \( \alpha'(0) \) for some smooth curve \( \alpha \) in \( S^n \).

Regard \( \alpha \) as a smooth curve in \( \mathbb{R}^{n+1} \). In standard coordinates,

\[
\alpha(t) = (\alpha^1(t), \ldots, \alpha^{n+1}(t)).
\]

Since

\[
\alpha'(0) = \alpha'(0)(x^i) \frac{\partial}{\partial x^i} \bigg|_p + \cdots + \alpha'(0)(x^{n+1}) \frac{\partial}{\partial x^{n+1}} \bigg|_p,
\]

and

\[
\alpha'(0)(x^i) = \frac{d}{dt} (x^i \circ \alpha) \bigg|_{t=0} = \frac{d\alpha^i}{dt}(0),
\]

as a point in \( \mathbb{R}^{n+1} \) we have

\[
\alpha'(0) = (\frac{d\alpha^1}{dt}(0), \ldots, \frac{d\alpha^{n+1}}{dt}(0)).
\]

But, since \( \alpha(t) \) lies in \( S^n \),

\[
(\alpha^1(t))^2 + \cdots + (\alpha^{n+1}(t))^2 = 1
\]

Differentiating at \( t = 0 \) gives

\[
2 \cdot \alpha'(0) \frac{d\alpha^i}{dt}(0) + \cdots + 2 \cdot \alpha^{n+1}(0) \frac{d\alpha^{n+1}}{dt}(0) = 0
\]

I.e.,

\[
(\alpha^1(0), \ldots, \alpha^{n+1}(0)) \cdot (\frac{d\alpha^1}{dt}(0), \ldots, \frac{d\alpha^{n+1}}{dt}(0)) = 0
\]

\[
\alpha(0) \cdot \alpha'(0) = 0
\]

\[
p \cdot \alpha'(0) = 0
\]
Thus, every element $\mathbf{n}_p$ of $T_p(S^n)$ is, as a vector in $\mathbb{R}^{n+1}$, orthogonal to (the position vector of) $p$:

$$p \cdot \mathbf{n}_p = 0$$

Since $\dim(S^n) = n$, $\dim T_p(S^n) = n$ and since the orthogonal complement of $p$ in $\mathbb{R}^{n+1}$ has dimension $n$, this must be precisely $T_p(S^n)$ (as a subspace of $\mathbb{R}^{n+1}$), e.g., for $S'$,

![Diagram showing $\mathbb{R}^2$, $\mathbf{n}_p$, $p$, $S'$, and $T_p(S')$.]

Now let's compute the derivative of the Hopf map

$$H : S^3 \rightarrow S^2$$

$$H(z',z^2) = (2 \Re(z'z^2), 2 \Im(z'z^2), |z|^2 - 1z'^2 - 1z^2)$$

Let

$$z_1 = x' + iy' \quad z_2 = x^2 + iy^2.$$ 

Then

$$H(x',y',x^2,y^2) = (2(x'x^2 + y'y^2), 2(y'y^2 - x'^2), (x^2)^2 + (y^2)^2 - (x'^2 - y'^2)^2)$$
This is the restriction to \( S^3 \) of the smooth map \( \tilde{H} : \mathbb{R}^4 \to \mathbb{R}^2 \) given by the same formula, i.e.,
\[
\tilde{H} = \tilde{h} \circ \tilde{e}
\]
where \( \tilde{e} : S^3 \to \mathbb{R}^4 \) is the inclusion map.

**Exercise 89:** Show that, when \( T_p(S^3) \) is identified with a subspace of \( T_p(\mathbb{R}^4) \) as in the lemma on page 11, \( H_{\ast p} \) is the restriction to \( T_p(S^3) \) of \( \tilde{h}_{\ast p} \).

**Exercise 90:** Generalize Exercise 89.

Thus, we need only compute \( \tilde{h}_{\ast p} \) and restrict it to \( T_p(S^3) \).

Computing the partial derivatives of \( 2(x'x^2 + y'y^2), 2(x'y^2 - x'y'), \)
and \( (x^2)^2 + (y^2)^2 - (x')^2 - (y')^2 \), we obtain the Jacobian of \( \tilde{H} : \)
\[
2 \begin{pmatrix}
x^2 & y^2 & x' & y' \\
y^2 & -x^2 & -y' & x' \\
-x' & -y' & x^2 & y^2
\end{pmatrix}
\]

Notice that any pair of rows (thought of as vectors in \( \mathbb{R}^4 \)) are orthogonal. In particular, the rows are linearly independent at any \( (x', y', x^2, y^2) \neq (0, 0, 0, 0) \). Thus, the Jacobian has rank 3 at any point of \( S^3 \). We conclude that \( \forall p \in S^3 \).
\[ \dim(\mathbb{R}^n) = \dim(\text{im} \, H_{p^+}) + \dim(\ker \, H_{p^+}) \]

\[ 4 = 3 + \dim(\ker \, H_{p^+}) \]

So
\[ \dim(\ker \, H_{p^+}) = 1 \]

Since \( H_{p^+} = \tilde{H}_{p^+} | T_p(S^3) \) is certainly not an isomorphism,
\[ \dim(\ker \, H_{p^+}) = 1 \]

as well. Thus,
\[ \dim(T_p(S^3)) = \dim(\text{im} \, H_{p^+}) + \dim(\ker \, H_{p^+}) \]

implies
\[ \dim(\text{im} \, H_{p^+}) = 2 \]

i.e.,
\[ H_{p^+} : T_p(S^3) \to T_{H(p)}(S^3) \]

is surjective.

Thus, the Hopf map is an example of the following:

Let \( X \) be a smooth \( n \)-manifold and \( Y \) a smooth \( m \)-manifold with \( n > m \). A smooth map
\[ F : X \to Y \]

is a submersion at \( p \in X \) if \( F_{*p} : T_p(X) \to T_{F(p)}(Y) \) is surjective.

\( F \) is a submersion if this is true for every \( p \in X \).
AND, WHY DO WE CARE?

**Theorem**: \( X \) a smooth \( n \)-manifold, \( Y \) a smooth \( m \)-manifold, \( n > m \), \( F : X \to Y \) a smooth map. If \( \beta \in F'(x) \) and \( F \) is a submersion at each \( p \in F^{-1}(\beta) \), then \( F^{-1}(\beta) \) is a submanifold of \( X \) of dimension \( n - m \).

**Note**: This is a huge generalization of our earlier result on level hypersurfaces. I have put the proof in an appendix because I would like to be certain that we have time to see some nontrivial applications.

**First, what does it say about the Hopf map**

\[
H : S^3 \to S^2
\]

**Exercise 91**: Show that \( H \) maps \( S^3 \) onto \( S^2 \).

Thus, for any \( p \in S^2 \), \( H^{-1}(p) \) is a submanifold of \( S^3 \) of dimension \( 3 - 2 = 1 \). It is, of course, just the copy of \( S^1 \) that, in our original construction ("quotient spaces", pages 19-26) we identified to a point to get \( p \), but now we know that these circles are actually submanifolds of \( S^3 \).
Now for a much more interesting example, we will show that the orthogonal group $O(3)$ and the special orthogonal group $SO(3)$ are smooth manifolds (and therefore Lie groups).

We will identify the set of all $3 \times 3$ real matrices with $\mathbb{R}^9$ in the following way:

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix} \rightarrow
(a_{11}, a_{21}, a_{22}, a_{31}, a_{32}, a_{33}, a_{12}, a_{13}, a_{23})
\]

(STRING OUT THE ENTRIES "LOWER TRIANGLE FIRST")

Consider the subset of all $3 \times 3$ real matrices $S$ that are symmetric ($S^T = S$)

\[
S = \begin{pmatrix}
a_{11} & a_{21} & a_{31} \\
a_{21} & a_{22} & a_{32} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix} \rightarrow
(a_{11}, a_{21}, a_{22}, a_{31}, a_{32}, a_{33}, a_{21}, a_{31}, a_{32})
\]

REPETITIONS
Thus, projecting this subset of $\mathbb{R}^9$ onto the first $6$ coordinates gives a map from the symmetric matrices to $\mathbb{R}^6$ that is one-to-one, onto, continuous and has a continuous inverse, i.e., a homeomorphism.

Now we define a map

$$F : \mathbb{R}^9 \longrightarrow \mathbb{R}^6$$

by

$$F(A) = A A^T$$

(Note: $A A^T$ is symmetric because $(A A^T)^T = (A^T)^T A^T = A A^T$.)

Writing out the matrix product $A A^T$ and stringing out its entries as described above one finds that

$$A = \begin{pmatrix} a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32}, a_{13}, a_{23}, a_{33} \end{pmatrix} \quad \longrightarrow$$

$$F(A) = \begin{pmatrix} a_{11}^2 + a_{12}^2 + a_{13}^2, a_{21} a_{11} + a_{22} a_{12} + a_{23} a_{13}, a_{31}^2 + a_{32}^2 + a_{33}^2, a_{31} a_{11} + a_{32} a_{12} + a_{33} a_{13}, a_{31} a_{21} + a_{32} a_{22} + a_{33} a_{23}, a_{31}^2 + a_{32}^2 + a_{33}^2 \end{pmatrix}$$
THE \( g \) COORDINATE FUNCTIONS \( F'_1, \ldots, F'_6 \) OF THE COORDINATES

\((x', x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9) = (a_{11}, a_{21}, a_{22}, a_{31}, a_{32}, a_{33}, a_{41}, a_{42}, a_{43})\)

ARE THEREFORE CLEARLY SMOOTH:

\[ F'_1(a_{11}, \ldots, a_{43}) = a_{11}^2 + a_{12}^2 + a_{13}^2 \]

\[ F'_2(a_{11}, \ldots, a_{43}) = a_{21}a_{11} + a_{22}a_{12} + a_{23}a_{13} \]

\[ \vdots \]

THE JACOBIAN OF \( F \) AT \( A \) IS

\[ J_F(A) = \begin{pmatrix}
\frac{\partial F'_1}{\partial a_{11}} & \frac{\partial F'_1}{\partial a_{21}} & \cdots & \frac{\partial F'_1}{\partial a_{43}} \\
\frac{\partial F'_2}{\partial a_{11}} & \frac{\partial F'_2}{\partial a_{21}} & \cdots & \frac{\partial F'_2}{\partial a_{43}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F'_6}{\partial a_{11}} & \frac{\partial F'_6}{\partial a_{21}} & \cdots & \frac{\partial F'_6}{\partial a_{43}}
\end{pmatrix} = \]

\[ \begin{pmatrix}
2a_{11} & 0 & 0 & 0 & 0 & 0 & 2a_{12} & 2a_{13} & 0 \\
a_{21} & a_{11} & a_{12} & 0 & 0 & 0 & a_{22} & a_{23} & a_{13} \\
0 & 2a_{21} & 2a_{22} & 0 & 0 & 0 & 0 & 2a_{23} \\
a_{31} & 0 & 0 & a_{11} & a_{12} & a_{13} & a_{32} & a_{33} & 0 \\
0 & a_{31} & a_{32} & a_{21} & a_{22} & a_{23} & 0 & 0 & a_{33} \\
0 & 0 & 0 & 2a_{31} & 2a_{32} & 2a_{33} & 0 & 0 & 0
\end{pmatrix} \]

THIS IS JUST THE MATRIX OF \( F_A \) IN STANDARD COORDINATES ON \( \mathbb{R}^9 \) AND \( \mathbb{R}^6 \).
Now recall that the orthogonal group $O(3)$ consists of those $3 \times 3$ real matrices $A$ satisfying $A A^T = I$ (the $3 \times 3$ identity matrix), i.e.,

$$O(3) = F^{-1}(I)$$

Thus, to show that $O(3)$ is a submanifold of $\mathbb{R}^9$, we need only prove that

$$A \in O(3) \Rightarrow J_f(A) \text{ has rank 6}$$

But the rank of a matrix is equal to its row rank, i.e., the dimension of the space spanned by its rows, so it will suffice to show that

$$A \in O(3) \Rightarrow \text{the rows of } J_f(A) \text{ are linearly independent in } \mathbb{R}^9.$$

But $A \in O(3) \Rightarrow$ the rows of $A$ are orthogonal and this implies that the rows of $J_f(A)$ are orthogonal, e.g., the dot product of the $2^{\text{nd}}$ and $3^{\text{rd}}$ rows is

$$a_{21} \cdot 0 + 2a_{41} a_{21} + 2a_{12} a_{22} + 0 \cdot 0 + 0 \cdot 0 + 0 \cdot 0 + a_{22} \cdot 0 + a_{23} \cdot 0 + 2a_{13} a_{23} = 2 (a_{11} a_{21} + a_{12} a_{22} + a_{13} a_{23}) = 0$$

Similarly for the remaining rows.
Thus, at every $A \in O(3) \subseteq \mathbb{R}^9$ the Jacobian of $f(A) = AA^T$ has rank 6 and so $f_A$ is surjective. Our theorem (page 17) therefore implies that $O(3) = f^{-1}(I)$ is a submanifold of $\mathbb{R}^9$ of dimension $9 - 6 = 3$.

$SO(3)$ is an open subset of $O(3)$ so it is also a 3-dimensional submanifold of $\mathbb{R}^9$.

**Note:** It can be shown that $SO(3)$ is actually diffeomorphic to real projective 3-space $\mathbb{RP}^3$.

Since matrix multiplication and inversion are smooth on $GL(3, \mathbb{R})$ it follows from Exercise 7 (3) & (4) that the same is true of $O(3)$ and $SO(3)$ so these are also Lie groups.

**Remark:** All of this generalizes to $O(n)$ and $SO(n)$ for any $n > 1$. 
A SUBMERSION SQUASHES A HIGH DIMENSIONAL MANIFOLD INTO A LOWER
DIMENSIONAL MANIFOLD WITH (INTUITIVELY) "MINIMAL SQUASHING"
(DERIVATIVE OF MAXIMAL RANK).

TURNING THE DIMENSIONS AROUND ONE ARRIVES AT THE FOLLOWING
ANALOGOUS NOTIONS.

\[ X \text{ AN } n\text{-MANIFOLD, } Y \text{ AN } m\text{-MANIFOLD, } n < m \]

A SMOOTH MAP
\[ F : X \to Y \]

IS AN INNERSION IF, FOR EACH \( p \in X \),
\[ F_p : T_p(X) \to T_{F(p)}(Y) \]

IS ONE-TO-ONE. IF, IN ADDITION, \( F \) IS A
HOMEOMORPHISM ONTO ITS IMAGE \( F(X) \), THEN
\( F \) IS CALLED AN EMBEDDING.

USING TECHNIQUES VERY MUCH LIKE THOSE WE USED TO PROVE THE LAST THEOREM
ONE CAN SHOW THAT

1. ANY INNERSION IS LOCALLY AN EMBEDDING

2. IF \( F : X \to Y \) IS AN EMBEDDING, THEN \( F(X) \) IS A SMOOTH
MANIFOLD AND \( F : X \to F(X) \) IS A DIFFEOMORPHISM.
An obvious example of an embedding is

\[ F : \mathbb{R}^n \to \mathbb{R}^m \quad (n < m) \]

\[ F(x', \ldots, x^n) = (x', \ldots, x^n, 0, \ldots, 0). \]

Here are some examples of immersions: Fix a \( k = \pm 1, \pm 2, \ldots \)

Define

\[ F_k : S' \to \mathbb{R}^2 \quad (\equiv \mathbb{C}) \]

by

\[ F_k(z) = z^k \quad \forall z \in S' \]

\[ F_k(e^{i\theta}) = (e^{i\theta})^k = e^{ik\theta} \]

\[ F_k(\cos \theta, \sin \theta) = (\cos k\theta, \sin k\theta) \]

The polar coordinate expression for \( F_k \) is

\[ \theta \to (\cos k\theta, \sin k\theta) \]

so the Jacobian is

\[
\begin{pmatrix}
-ik \sin k\theta \\
\phantom{-}ik \cos k\theta
\end{pmatrix}
\]

which has rank 1 at every point so \((F_k)_{*p}\) is injective at each \( p \in S' \) and \( F_k \) is an immersion.

\( F_k \) is an embedding if and only if \( k = \pm 1 \).
Although we will have no time for the proof (and no real need for the result) I would be remiss in my duty if I did not make you aware of the following famous old result.

**Whitney Embedding Theorem**: Let $X$ be a smooth manifold of dimension $n$. Then there exists a smooth embedding

$$F : X \rightarrow \mathbb{R}^{2n+1}$$

(since $\mathbb{R}^{2n+1}$ contains a diffeomorphic copy of every smooth $n$-manifold).

In the next section on Riemannian manifolds we will have occasion to mention an even more famous embedding theorem due to John Nash (yes, that John Nash).
Theorem: Let $X$ be a smooth manifold of dimension $n$, $Y$ a smooth manifold of dimension $m$ with $n \geq m$ and $F: X \to Y$ a smooth map. If $q \in F(X)$ and $F$ is a submersion at each $p \in F^{-1}(q)$, then $F^{-1}(q)$ is a submanifold of $X$ of dimension $n - m$.

Proof:

We first prove the result in the special case in which $X = U$ is an open set in $\mathbb{R}^n$ and $Y = \mathbb{R}^m$ ($n \geq m$).

$F: U \to \mathbb{R}^m$

$F(x', \ldots, x^n) = (F'(x', \ldots, x^n), \ldots, F^m(x', \ldots, x^n))$

Fix a $q \in F(U)$ with the property that for each $p \in F^{-1}(q)$ the Jacobian

$$J_F(p) = \begin{pmatrix} \frac{\partial F'}{\partial x'}(p) & \cdots & \frac{\partial F'}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x'}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}$$

has rank $m$ (i.e., $F_{*p}$ is surjective).
Fix a $p \in F^{-1}(q)$. We must find a chart $(V, \psi)$ for $\mathbb{R}^n$ with $p \in V$ such that

$$
\psi(V \cap F^{-1}(q)) = \{ y = (y'_1, ..., y'_n) \in \psi(V) : y'_{n-m+1} = ... = y'_n = 0 \}
$$

$p \in F^{-1}(q) \Rightarrow J_F(p)$ has rank $m$ so some $m \times m$ submatrix is nonsingular. By renumbering the coordinates if necessary we may assume it is the one in the dotted box:

$$
J_F(p) = \begin{pmatrix}
\frac{\partial F'}{\partial x'_1}(p) & ... & \frac{\partial F'}{\partial x'_{n-m}}(p) & \frac{\partial F'}{\partial x'_{n-m+1}}(p) & ... & \frac{\partial F'}{\partial x'_n}(p) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_m}{\partial x'_1}(p) & ... & \frac{\partial F_m}{\partial x'_{n-m}}(p) & \frac{\partial F_m}{\partial x'_{n-m+1}}(p) & ... & \frac{\partial F_m}{\partial x'_n}(p)
\end{pmatrix}
$$

We wish to use the inverse function theorem so we define a smooth map

$$
\tilde{F} : U \to \mathbb{R}^n = \mathbb{R}^{n-m} \times \mathbb{R}^m
$$

by

$$
\tilde{F}(x) = (x'_1, ..., x'_{n-m}, x_{n-m+1}, ..., x'_n) = (x'_1, ..., x'_{n-m}, F_1(x), ..., F_m(x))
$$

The Jacobian $J_{\tilde{F}}(p)$ is given by
\[
\begin{pmatrix}
1 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 1 & 0 & \cdots & 0 \\
\frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^m}(p) & \frac{\partial F^1}{\partial x^{m+1}}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^m}(p) & \frac{\partial F^m}{\partial x^{m+1}}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p)
\end{pmatrix}
\]

which clearly has \( n \) independent columns, i.e., rank \( n \), so it is nonsingular.

The inverse function theorem therefore implies that there exist open neighborhoods \( V \) of

\[ p = (p', \ldots, p^{n-m}, p^{n-m+1}, \ldots, p^n) \]

and \( W \) of

\[ \tilde{F}(p) = (p', \ldots, p^{n-m}, F(p), \ldots, F^m(p)) \]

\[ = (p', \ldots, p^{n-m}, q', \ldots, q^m) \]

such that

\[ \tilde{F} | V : V \to W \]

is a diffeomorphism.

Notice that \((V, \tilde{F}|V)\) is a chart on \( \mathbb{R}^n \) with \( p \in V \) and
IT CARRIES ANY
\[(x',...,x^{n-m},x^{n-m+1},...,x^n) \in \bigcap F(q')\]
to
\[(x',...,x^{n-m},q',...,q^m).\]

THUS, IF WE DEFINE
\[\psi : \mathcal{V} \to \mathbb{R}^n\]

BY
\[\psi(x',...,x^{n-m},x^{n-m+1},...,x^n) = (y',...,y^{n-m},y^{n-m+1},...,y^n) =
(x',...,x^{n-m},x^{n-m+1},q',...,x^n-q^m)\]

WE HAVE A CHART \((\mathcal{V},\psi)\) WITH THE REQUIRED PROPERTY.

NOW WE'LL USE THE SPECIAL CASE WE HAVE JUST ESTABLISHED TO PROVE
THE THEOREM IN GENERAL.

\[X\text{ AN } n\text{-MANIFOLD} \quad Y\text{ AN } m\text{-MANIFOLD}\]

\[n > m\]

\[F : X \to Y\]

\[q \in F(x)\] SUCH THAT \(F\) IS A SUBMERSION AT EACH \(p \in F(q)\)
LET \( p \in F^{-1}(q) \) BE ARBITRARY. CHOOSE CHARTS \((U, \varphi)\) AT \( p \) AND \((V, \psi)\) AT \( q \) AND CONSIDER THE COORDINATE EXPRESSION

\[
\psi \circ F \circ \varphi^{-1} : \psi(U \cap F^{-1}(V)) \to \psi(V)
\]

NOTE THAT

\[
(\psi \circ F \circ \varphi^{-1})(\varphi(p)) = \psi(q)
\]

AND

\[
(\psi \circ F \circ \varphi^{-1}) \circ \varphi(p) = \psi \circ F \circ p \circ \varphi^{-1} \circ \varphi(p)
\]

ALL THREE DERIVATIVES ON THE RIGHT-HAND SIDE ARE SURJECTIVE SO

\[
(\psi \circ F \circ \varphi^{-1}) \circ \varphi(p) : T_{\varphi(p)}(U \cap F^{-1}(V)) \to T_{\psi(q)}(\psi(V))
\]

IS SURJECTIVE.

THIS IS TRUE FOR EVERY \( p \in F^{-1}(q) \cap (\cup F^{-1}(V)) \). NOTE THAT

\[
p \in F^{-1}(q) \cap (\cup F^{-1}(V)) \Leftrightarrow \varphi(p) \in (\psi \circ F \circ \varphi^{-1})^{-1}(\psi(q))
\]

(CONVince yourself of this)

THUS,

\( \psi \circ F \circ \varphi^{-1} \) IS A SUBMERSION AT EACH POINT OF \((\psi \circ F \circ \varphi^{-1})^{-1}(\psi(q))\)

SO WE CAN APPLY THE RESULT FOR EUCLIDEAN SPACES TO CONCLUDE THAT
\[(\psi \circ F \circ \psi^{-1})(\psi(q)) = \psi(F(q) \cap U)\]

is a submanifold of \(\mathcal{C}(\mathbb{R}^n) \subseteq \mathbb{R}^n\) of dimension \(n-m\).

Thus, for each \(\psi(p) \in (\psi \circ F \circ \psi^{-1})(\psi(q))\) there is a chart \((W, \xi)\) for \(\mathbb{R}^n\) with coordinate functions \(x', \ldots, x^n\) such that \(\psi(p) \in W \subseteq \mathcal{C}(\mathbb{R}^n)\)

and

\[\xi(W \cap (\psi \circ F \circ \psi^{-1})(\psi(q))) = \{ (x', \ldots, x^n) \in \xi(W) : x_{m+1} = \ldots = x^n = 0 \}\]

Now it is easy to check that

\[(\xi^{-1}(W), \xi \circ \psi \circ \xi^{-1}(W))\]

is a chart of the required type at \(p \in F(q)\). \(\square\)