DIFFERENTIABLE STRUCTURES

RECALL :

1. $S^2 = \{ x = (x^1, x^2, x^3) \in \mathbb{R}^3 : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 \}$ CAN BE COVERED BY TWO CHARTS (U_S, φ_S) AND (U_N, φ_N) :

$$U_S = S^2 - \{(0,0,1)\} \qquad\qquad U_N = S^2 - \{(0,0,-1)\}$$

$$\varphi_S(x^1, x^2, x^3) = \left(\frac{x^1}{1-x^3}, \frac{x^2}{1-x^3} \right) \qquad\qquad \varphi_N(x^1, x^2, x^3) = \left(\frac{x^1}{1+x^3}, \frac{x^2}{1+x^3} \right)$$

$$\varphi_S \circ \varphi_N^{-1} : \mathbb{R}^2 - \{(0,0)\} \longrightarrow \mathbb{R}^2 - \{(0,0)\}$$

$$\varphi_S \circ \varphi_N^{-1}(y^1, y^2) = \left(\frac{y^1}{(y^1)^2 + (y^2)^2}, \frac{y^2}{(y^1)^2 + (y^2)^2} \right)$$

$$\varphi_N \circ \varphi_S^{-1} : \mathbb{R}^2 - \{(0,0)\} \longrightarrow \mathbb{R}^2 - \{(0,0)\}$$

$$\varphi_N \circ \varphi_S^{-1}(y^1, y^2) = \left(\frac{y^1}{(y^1)^2 + (y^2)^2}, \frac{y^2}{(y^1)^2 + (y^2)^2} \right)$$

2. $\mathbb{RP}^2 = S^2/$ANTIPODAL POINTS CAN BE COVERED BY THREE CHARTS

$$U_k = \{ [p^1, p^2, p^3] \in \mathbb{RP}^2 : p^k \neq 0 \}$$

$$k = 1, 2, 3$$

$$\varphi_k : U_k \to \mathbb{R}^2 \quad , \quad k = 1, 2, 3$$

$$\varphi_1([p^1, p^2, p^3]) = \left(\frac{p^2}{p^1}, \frac{p^3}{p^1} \right)$$

$$\varphi_2([p^1, p^2, p^3]) = \left(\frac{p^1}{p^2}, \frac{p^3}{p^2} \right)$$

$$\varphi_3([p^1, p^2, p^3]) = \left(\frac{p^1}{p^3}, \frac{p^2}{p^3} \right)$$

$$\varphi_2 \circ \varphi_1^{-1} : \{ (x^1, x^2) \in \mathbb{R}^2 : x^1 \neq 0 \} \rightarrow \{ (x^1, x^2) \in \mathbb{R}^2 : x^1 \neq 0 \}$$

$$(\varphi_2 \circ \varphi_1^{-1})(x^1, x^2) = \left(\frac{1}{x^1} , \frac{x^2}{x^1} \right)$$

AND SIMILARLY FOR THE REST

IT IS THE OVERLAP MAPS (COORDINATE TRANSFORMATIONS) THAT ARE OF INTEREST TO US AT THE MOMENT.

THESE ARE ALL HOMEOMORPHISMS BETWEEN OPEN SETS IN EUCLIDEAN SPACE (BY DEFINITION OF A "CHART").

THEY ARE, HOWEVER, MUCH MORE.

RECALL (FROM CALCULUS):

IF U IS OPEN IN $\mathbb{R}^n$ AND $f : U \rightarrow \mathbb{R}$ IS A REAL-VALUED FUNCTION ON U, THEN THE i^{TH} PARTIAL DERIVATIVE OF f AT $p \in U$ IS DEFINED BY

$$D_i f(p) = \lim_{h \to 0} \frac{f(p^1, \ldots, p^i + h, \ldots, p^n) - f(p^1, \ldots, p^i, \ldots, p^n)}{h}$$

AND SIMILARLY FOR HIGHER ORDER PARTIAL DERIVATIVES. f IS SAID TO BE SMOOTH, OR C^∞, IF IT AND ALL OF ITS PARTIAL DERIVATIVES, OF ALL ORDERS AND TYPES, EXIST AND ARE CONTINUOUS.

IF $F : U \to \mathbb{R}^m$ WITH COORDINATE FUNCTIONS $F^1, \ldots, F^m$, I.E.,

$$F(p) = (F^1(p), \ldots, F^m(p))$$

$\forall\, p \in U$, THEN F IS SAID TO BE <u>SMOOTH</u>, OR $\underline{C}^\infty$, IF EACH F^i, $i = 1, \ldots, m$, IS SMOOTH.

<u>EXAMPLES</u> : ALL OF THE OVERLAP MAPS IN THE TWO EXAMPLES ABOVE, E.G., FOR

$$\varphi_S \circ \varphi_N^{-1} : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^2 - \{(0,0)\}$$

THE COORDINATE FUNCTIONS ARE

$$(y^1, y^2) \longrightarrow \frac{y^1}{(y^1)^2 + (y^2)^2}$$

AND

$$(y^1, y^2) \longrightarrow \frac{y^2}{(y^1)^2 + (y^2)^2}$$

AND THESE CLEARLY HAVE CONTINUOUS PARTIAL DERIVATIVES OF ALL ORDERS AND TYPES ON $\mathbb{R}^2 - \{(0,0)\}$.

NOW, SUPPOSE X IS A TOPOLOGICAL MANIFOLD (HAUSDORFF, SECOND COUNTABLE AND LOCALLY EUCLIDEAN). TWO CHARTS (U_1, φ_1) AND (U_2, φ_2) ON X ARE SAID TO BE $\underline{C^\infty}$-<u>RELATED</u> IF EITHER $U_1 \cap U_2 = \emptyset$ OR $U_1 \cap U_2 \neq \emptyset$ AND THE OVERLAP FUNCTIONS

$$\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$$

AND

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

ARE SMOOTH.

<u>EXAMPLES</u>: (U_S, φ_S) AND (U_N, φ_N) ARE C^∞-RELATED CHARTS ON S^2. ANY TWO OF (U_k, φ_k), $k = 1,2,3$, ARE C^∞-RELATED CHARTS ON $\mathbb{RP}^2$.

AN <u>ATLAS</u> $\mathcal{A}$ OF DIMENSION n ON X IS A COLLECTION OF n-DIMENSIONAL CHARTS ON X WHOSE DOMAINS COVER ALL OF X AND ANY TWO OF WHICH ARE C^∞-RELATED.

<u>EXAMPLES</u>: $(\mathbb{R}^n, id)$ FOR $\mathbb{R}^n$

(U_S, φ_S) AND (U_N, φ_N) FOR S^2

(U_k, φ_k), $k = 1,2,3$, FOR $\mathbb{RP}^2$

IF $\mathcal{A}$ IS AN ATLAS FOR X, THEN A CHART (U, φ) IS <u>ADMISSIBLE TO</u> $\mathcal{A}$ IF IT IS C^∞- RELATED TO EVERY CHART IN $\mathcal{A}$.

<u>EXAMPLES</u> :

1. LET $\mathcal{A} = \{ (\mathbb{R}^n, id) \}$ BE THE ATLAS FOR $\mathbb{R}^n$ WITH JUST ONE CHART.

   ANOTHER CHART (U, φ) ON $\mathbb{R}^n$ IS C^∞- RELATED TO THE SINGLE CHART $(\mathbb{R}^n, id)$, I.E., ADMISSIBLE TO $\mathcal{A}$, IF

   $$\varphi \circ id^{-1} = \varphi : U \to \varphi(U)$$

   AND

   $$id \circ \varphi^{-1} = \varphi^{-1} : \varphi(U) \to U$$

   ARE BOTH SMOOTH.

2. LET $\mathcal{A} = \{ (U_S, \varphi_S), (U_N, \varphi_N) \}$ BE THE STEREOGRAPHIC PROJECTION ATLAS FOR S^2.

   DEFINE ANOTHER CHART (U, φ) ON S^2 BY " PROJECTING THE OPEN UPPER HEMISPHERE VERTICALLY DOWN INTO THE XY - PLANE ".

   MORE PRECISELY :

LET

$$U = \{ (x^1, x^2, x^3) \in S^2 \ : \ x^3 > 0 \}$$

AND

$$\varphi : U \to \mathbb{R}^2$$

$$\varphi(x^1, x^2, x^3) = (x^1, x^2)$$

NOTE THAT

$$\varphi(U) = \{ (x^1, x^2) \in \mathbb{R}^2 \ : \ (x^1)^2 + (x^2)^2 < 1 \}$$

AND

$$\varphi^{-1} : \varphi(U) \to U$$

$$\varphi^{-1}(x^1, x^2) = (x^1, x^2, \sqrt{1 - (x^1)^2 - (x^2)^2})$$

SO (U, φ) REALLY IS A CHART ON S^2.

WE SHOW THAT (U, φ) IS ADMISSIBLE TO $\mathcal{A}$, I.E., C^∞-RELATED TO BOTH (U_S, φ_S) AND (U_N, φ_N).

I'LL PROVE THIS FOR (U_S, φ_S). <u>EXERCISE</u> 61: YOU DO IT FOR (U_N, φ_N).

WE MUST SHOW THAT

$$\varphi_S \circ \varphi^{-1} : \varphi(U \cap U_S) \to \varphi_S(U \cap U_S)$$

AND

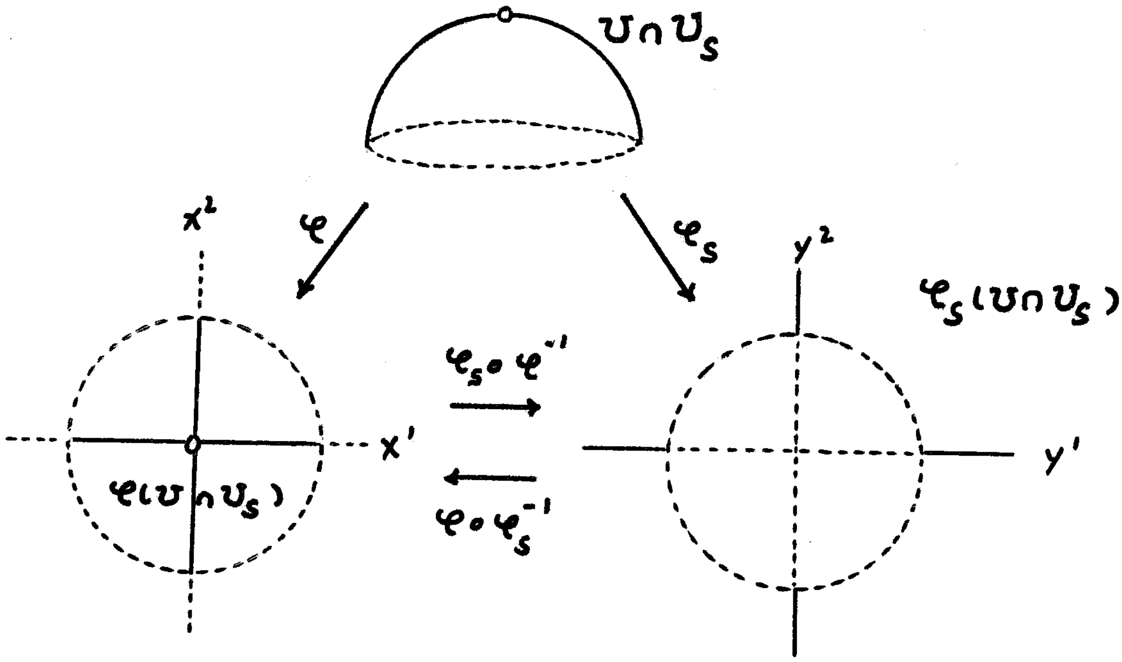
$$\varphi \circ \varphi_S^{-1} : \varphi_S(U \cap U_S) \to \varphi(U \cap U_S)$$

ARE BOTH SMOOTH.

$$U \cap U_S = \{ (x^1, x^2, x^3) \in S^2 : x^3 > 0,\ x^3 \neq 1 \}$$

$$\varphi(U \cap U_S) = \{ (x^1, x^2) \in \mathbb{R}^2 : 0 < (x^1)^2 + (x^2)^2 < 1 \}$$

$$\varphi_S(U \cap U_S) = \{ (y^1, y^2) \in \mathbb{R}^2 : (y^1)^2 + (y^2)^2 > 1 \}$$



NOW JUST COMPUTE :

$$(\varphi_S \circ \varphi^{-1})(x^1, x^2) = \varphi_S\left(x^1, x^2, \sqrt{1-(x^1)^2-(x^2)^2}\right)$$

$$= \left(\frac{x^1}{1-\sqrt{1-(x^1)^2-(x^2)^2}} \ , \ \frac{x^2}{1-\sqrt{1-(x^1)^2-(x^2)^2}} \right)$$

WHICH IS SMOOTH ON THE PUNCTURED DISC $\varphi(U \cap U_S)$

$$(\varphi \circ \varphi_S^{-1})(y^1, y^2) = \varphi\left(\frac{2y^1}{(y^1)^2+(y^2)^2+1}, \frac{2y^2}{(y^1)^2+(y^2)^2+1}, \frac{(y^1)^2+(y^2)^2-1}{(y^1)^2+(y^2)^2+1} \right)$$

$$= \left(\frac{2y^1}{(y^1)^2+(y^2)^2+1}, \frac{2y^2}{(y^1)^2+(y^2)^2+1} \right)$$

WHICH IS SMOOTH EVERYWHERE AND SO, IN PARTICULAR, ON $\varphi_S(U \cap U_S)$.

AN ATLAS $\mathcal{A}$ FOR X IS SAID TO BE <u>MAXIMAL</u> IF IT CONTAINS EVERY CHART THAT IS ADMISSIBLE TO IT

<u>NOTE</u>: EVERY ATLAS IS CONTAINED IN A UNIQUE MAXIMAL ATLAS (THROW IN EVERYTHING ADMISSIBLE TO IT).

A MAXIMAL ATLAS FOR X IS CALLED A <u>DIFFERENTIABLE STRUCTURE FOR</u> X

A TOPOLOGICAL MANIFOLD X TOGETHER WITH A CHOICE OF DIFFERENTIABLE STRUCTURE FOR X IS CALLED A <u>SMOOTH</u>, OR <u>C^∞</u>, OR <u>DIFFERENTIABLE MANIFOLD</u>.

<u>NOTE</u>: TO DETERMINE A SMOOTH MANIFOLD IT IS ENOUGH TO GIVE AN <u>ATLAS</u> SINCE THIS IS CONTAINED IN A UNIQUE DIFFERENTIABLE STRUCTURE.

EXAMPLES OF DIFFERENTIABLE MANIFOLDS :

1. $\mathbb{R}^n$ WITH ITS STANDARD DIFFERENTIABLE STRUCTURE :

$$\text{ATLAS : } \mathcal{A} = \{ (\mathbb{R}^n, id) \}$$

2. S^2 WITH ITS STANDARD DIFFERENTIABLE STRUCTURE :

$$\text{ATLAS : } \mathcal{A} = \{ (U_S, \varphi_S), (U_N, \varphi_N) \}$$

SAME FOR ANY SPHERE S^n, $n \geq 1$.

3. $\mathbb{RP}^2$ WITH ITS STANDARD DIFFERENTIABLE STRUCTURE :

$$\text{ATLAS : } \mathcal{A} = \{ (U_k, \varphi_k) : k = 1,2,3 \}$$

SIMILARLY FOR ANY PROJECTIVE SPACE $\mathbb{RP}^n$.

4. (OPEN SUBMANIFOLDS) IF X IS ANY DIFFERENTIABLE MANIFOLD WITH ATLAS $\mathcal{A}$ AND W IS ANY OPEN SUBSET OF X (WITH THE SUBSPACE TOPOLOGY), THEN W IS CLEARLY HAUSDORFF AND SECOND COUNTABLE AND THE COLLECTION

$$\mathcal{A}_W = \{ (U \cap W, \varphi | U \cap W) : (U, \varphi) \in \mathcal{A} \}$$

IS AN ATLAS FOR W. WITH THE CORRESPONDING DIFFERENTIABLE STRUCTURE, W IS CALLED AN OPEN SUBMANIFOLD OF X

FOR EXAMPLE, THE GENERAL LINEAR GROUP

$$GL(n, \mathbb{R})$$

IS AN OPEN SUBMANIFOLD OF $\mathbb{R}^{n^2}$. WE WILL SEE LATER THAT THIS MAKES $GL(n, \mathbb{R})$ AN EXAMPLE OF A "LIE GROUP".

<u>NOTE</u>: THE NOTION OF A GENERAL "SUBMANIFOLD" (WITHOUT "OPEN") IS ONE WE WILL DISCUSS SHORTLY.

5. (A <u>NONSTANDARD</u> <u>DIFFERENTIABLE</u> <u>STRUCTURE</u> <u>ON</u> $\mathbb{R}$)

CONSIDER THE TOPOLOGICAL MANIFOLD $\mathbb{R}$. THE CHART THAT DETERMINES THE STANDARD DIFFERENTIABLE STRUCTURE ON $\mathbb{R}$ IS $(\mathbb{R}, id)$.

HERE'S ANOTHER CHART ON $\mathbb{R}$: $(U, \varphi) = (\mathbb{R}, \varphi)$, WHERE

$$\varphi(x) = x^3$$

THIS REALLY IS A CHART SINCE φ AND φ^{-1} $(\varphi^{-1}(y) = \sqrt[3]{y})$ ARE BOTH CONTINUOUS.

NOTICE, HOWEVER, THAT THIS CHART IS <u>NOT</u> ADMISSIBLE TO THE STANDARD ATLAS $\mathcal{A} = \{(\mathbb{R}, id)\}$ FOR $\mathbb{R}$ BECAUSE $\varphi^{-1}: \mathbb{R} \to \mathbb{R}$ IS <u>NOT</u> SMOOTH ON ALL OF $\mathbb{R}$ (THE DERIVATIVE IS $\frac{1}{3} y^{-\frac{2}{3}}$ WHICH IS UNDEFINED AT $y = 0$).

THUS, WE CAN DEFINE A DIFFERENT ("NONSTANDARD") DIFFERENTIABLE STRUCTURE ON $\mathbb{R}$ FROM THE ATLAS

$$\mathcal{A}' = \{(U, \varphi)\}$$

$$U = \mathbb{R}$$
$$\varphi(x) = x^3$$

<u>NOTE</u>: ALTHOUGH THIS REALLY IS A DIFFERENT DIFFERENTIABLE STRUCTURE ON $\mathbb{R}$, WE WILL SEE SOON THAT IT IS NOT "TOO DIFFERENT".

6. (<u>LEVEL</u> <u>HYPERSURFACES</u> IN $\mathbb{R}^n$)

HERE WE WILL DESCRIBE A GENERAL PROCEDURE FOR PRODUCING EXAMPLES THAT INCLUDES MOST OF THE OBJECTS STUDIED IN CLASSICAL DIFFERENTIAL GEOMETRY.

LET $f : \mathbb{R}^{n+1} \to \mathbb{R}$ BE A SMOOTH FUNCTION. A POINT $p = (p^1, \ldots, p^{n+1})$ IN $\mathbb{R}^{n+1}$ IS CALLED A <u>REGULAR POINT OF</u> f IF AT LEAST ONE OF THE PARTIAL DERIVATIVES $D_1 f(p), \ldots, D_{n+1} f(p)$ IS NONZERO. A POINT WHERE ALL OF THESE PARTIAL DERIVATIVES VANISH IS CALLED A <u>CRITICAL POINT OF</u> f. A <u>REGULAR VALUE OF</u> f IS A $c \in \mathbb{R}$ FOR WHICH $f^{-1}(c)$ CONSISTS ENTIRELY OF REGULAR POINTS OF f (THIS INCLUDES THE CASE $f^{-1}(c) = \emptyset$); ANY OTHER $c \in \mathbb{R}$ IS A <u>CRITICAL VALUE OF</u> f.

EXAMPLE : $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x,y,z) = x^2 + y^2 - z^2$$

$$D_1 f = 2x \ , \ D_2 f = 2y \ , \ D_3 f = -2z$$

THE ONLY CRITICAL POINT IS $(x,y,z) = (0,0,0)$

THE ONLY CRITICAL VALUE IS $c = 0$

NOTE : FOR ANY $c \in \mathbb{R}$, $f^{-1}(c)$ IS

$$x^2 + y^2 - z^2 = c$$

WHICH IS A

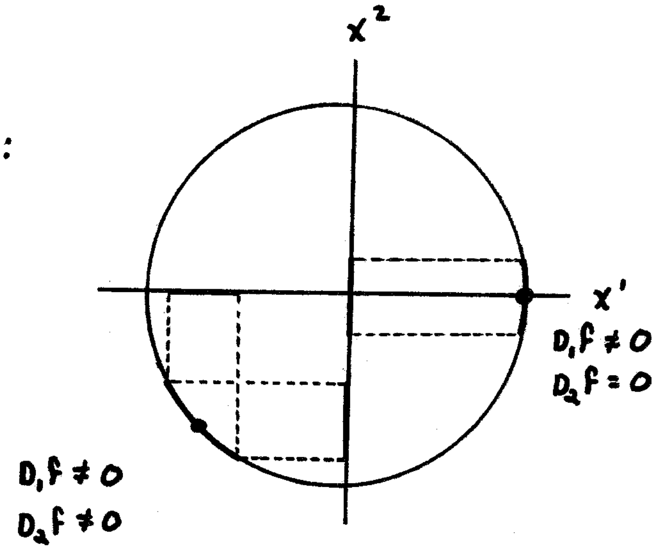
$$\begin{cases} \text{HYPERBOLOID OF 1-SHEET IF } c > 0 \\ \text{CONE IF } c = 0 \\ \text{HYPERBOLOID OF 2-SHEETS IF } c < 0 \end{cases}$$

NOTE : SARD'S THEOREM : THE SET OF CRITICAL VALUES OF A SMOOTH FUNCTION $f : \mathbb{R}^{n+1} \to \mathbb{R}$ HAS "MEASURE ZERO" IN $\mathbb{R}$.

WE WILL SHOW THAT, IF c IS A REGULAR VALUE OF $f : \mathbb{R}^{n+1} \to \mathbb{R}$, THEN $f^{-1}(c)$ HAS A NATURAL MANIFOLD STRUCTURE (PROVIDED IT'S NONEMPTY).

MORE PRECISELY, AT ANY $p \in f^{-1}(c)$ ONE OF THE PARTIAL DERIVATIVES IS NONZERO, SAY, $D_i f(p) \neq 0$. WE WILL PROVE THAT IN SOME NEIGHBORHOOD OF p IN THE RELATIVE TOPOLOGY ON $f^{-1}(p)$ IN $\mathbb{R}^{n+1}$, THE PROJECTION INTO THE $\mathbb{R}^n$ OPPOSITE x^i (SPACE OF $(x^1, \dots, x^{i-1}, 0, x^{i+1}, \dots, x^{n+1})$) IS A CHART AND THAT THESE CHARTS OVERLAP SMOOTHLY TO GIVE AN ATLAS FOR $f^{-1}(c)$.

ILLUSTRATION :



$f : \mathbb{R}^2 \to \mathbb{R}$

$f(x^1, x^2) = (x^1)^2 + (x^2)^2$

$D_1 f = 2x^1$

$D_2 f = 2x^2$

$f^{-1}(1) = S^1$

THUS, LET $f : \mathbb{R}^{n+1} \to \mathbb{R}$ BE SMOOTH, c A REGULAR VALUE OF f AND ASSUME

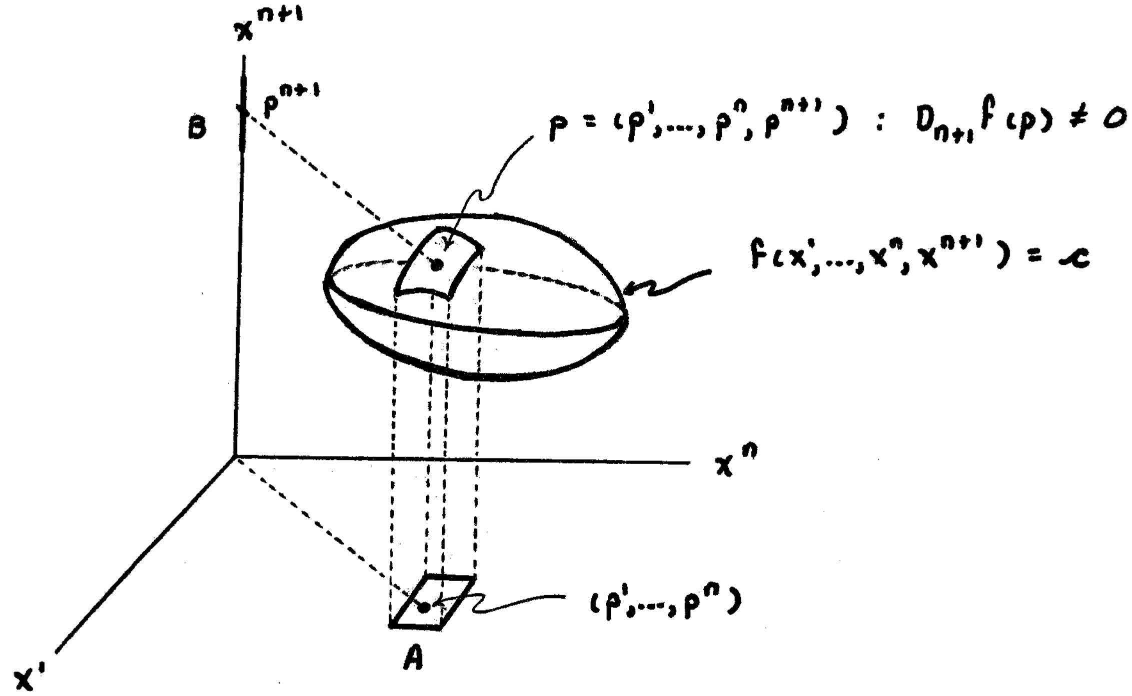
$$f^{-1}(c) \neq \phi.$$

FIX SOME $p = (p^1, \dots, p^n, p^{n+1}) \in f^{-1}(c)$. BY ASSUMPTION, SOME PARTIAL DERIVATIVE OF f IS NONZERO AT p.

BY RENUMBERING, IF NECESSARY, WE MAY ASSUME

$$D_{n+1} f(p) \neq 0.$$

NOW WE NEED TO APPEAL TO A RESULT FROM REAL ANALYSIS CALLED THE <u>IMPLICIT</u> <u>FUNCTION</u> <u>THEOREM</u> WHICH ASSERTS THE FOLLOWING (WE'LL WRITE $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, WHERE THE "$\mathbb{R}$" IS THE x^{n+1}- COORDINATE SPACE) :



$\exists$ OPEN NEIGHBORHOOD A OF $(p^1, \ldots, p^n)$ IN $\mathbb{R}^n$ AND AN OPEN INTERVAL B ABOUT p^{n+1} IN $\mathbb{R}$ SUCH THAT

$\forall \, (x^1, \ldots, x^n) \in A \;\; \exists! \;\; g(x^1, \ldots, x^n) \in B$ SUCH THAT

$$f(x^1, \ldots, x^n, g(x^1, \ldots, x^n)) = c$$

AND, MOREOVER, $g : A \to B$ IS SMOOTH.

THUS,

$$f(x^1, \ldots, x^n, x^{n+1}) = c \iff x^{n+1} = g(x^1, \ldots, x^n)$$

(NEAR p, $f^{-1}(c)$ IS JUST THE GRAPH OF THE SMOOTH FUNCTION g).

NOW, $A \times B$ IS OPEN IN $\mathbb{R}^{n+1}$ SO

$$U = f^{-1}(c) \cap (A \times B)$$

IS OPEN IN THE SUBSPACE TOPOLOGY ON $f^{-1}(c)$. MOREOVER, THE

MAP $\varphi : U \to \mathbb{R}^n$ DEFINED BY

$$\varphi(x^1,\dots,x^n, g(x^1,\dots,x^n)) = (x^1,\dots,x^n)$$

IS CONTINUOUS (RESTRICTION OF A PROJECTION), ONE-TO-ONE, ONTO $A \overset{\text{open}}{\subseteq} \mathbb{R}^n$ AND HAS A CONTINUOUS INVERSE

$$\varphi^{-1} : A \to U$$

$$\varphi^{-1}(x^1,\dots,x^n) = (x^1,\dots,x^n, g(x^1,\dots,x^n))$$

I.E., (U, φ) IS A CHART AT p IN $f^{-1}(c)$.

SINCE $p \in f^{-1}(c)$ WAS ARBITRARY WE HAVE PRODUCED A CHART OF DIMENSION n AT EACH POINT OF $f^{-1}(c)$.

<u>EXERCISE 62</u>: USE THE SMOOTHNESS OF THE MAPS g TO SHOW THAT ANY TWO SUCH CHARTS ARE C^∞-RELATED.

THUS, WE HAVE AN ATLAS FOR $f^{-1}(c)$ AND THIS DETERMINES A DIFFERENTIABLE STRUCTURE FOR $f^{-1}(c)$.

ANY SUCH MANIFOLD $f^{-1}(c)$ IS CALLED A <u>LEVEL</u> <u>HYPERSURFACE</u> IN $\mathbb{R}^{n+1}$.

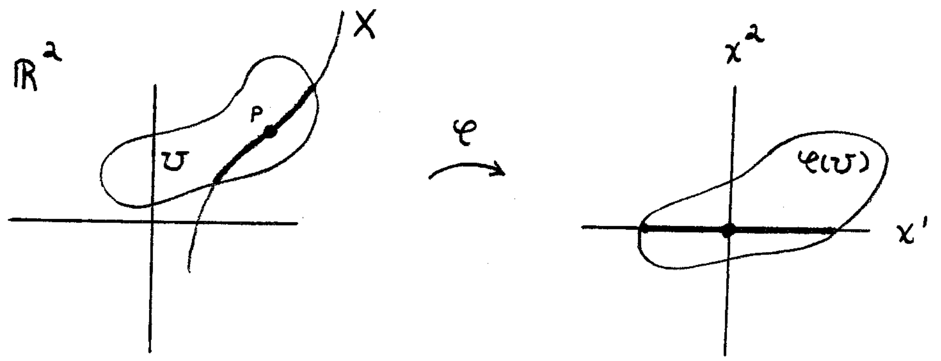
IN PARTICULAR, WHEN $n+1 = 3$ THESE ARE THE <u>SMOOTH</u> <u>LEVEL</u> <u>SURFACES</u> IN SPACE.

THESE SMOOTH LEVEL HYPERSURFACES ARE NOT JUST SUBSPACES OF $\mathbb{R}^n$ THAT ADMIT MANIFOLD STRUCTURES. THEY "INHERIT" THEIR MANIFOLD STRUCTURE IN A NATURAL WAY FROM $\mathbb{R}^n$. WE NOW MAKE THIS PRECISE.

LET X BE A TOPOLOGICAL SUBSPACE OF $\mathbb{R}^n$. WE SAY THAT X IS A <u>k-DIMENSIONAL SMOOTH SUBMANIFOLD OF</u> $\mathbb{R}^n$ (FOR SOME INTEGER $0 \le k \le n$) IF

> FOR EACH $p \in X$ THERE IS A CHART (U, φ) IN THE STANDARD DIFFERENTIABLE STRUCTURE FOR $\mathbb{R}^n$ SUCH THAT
>
> $$\varphi(p) = (0, \dots, 0) \in \mathbb{R}^n$$
>
> AND
>
> $$\varphi(U \cap X) = \{ x \in \varphi(U) : x^{k+1} = \dots = x^n = 0 \}$$

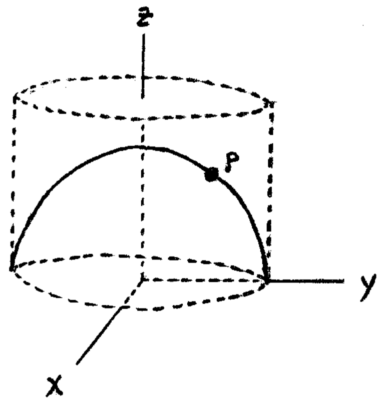
(φ "FLATTENS" $U \cap X$ ONTO AN OPEN SET IN THE COORDINATE HYPERPLANE $(x^1, \dots, x^k, 0, \dots, 0)$ IN $\mathbb{R}^n$)



<u>NOTE</u> : BY COMPOSING WITH A TRANSLATION IN $\mathbb{R}^n$ ONE CAN ALLOW $\varphi(p)$ TO BE ANY POINT IN $\mathbb{R}^n$. BY COMPOSING WITH A ROTATION OF $\mathbb{R}^n$ ONE CAN ALLOW $\varphi(U \cap X)$ TO LIE IN ANY k-DIMENSIONAL HYPERPLANE.

<u>EXAMPLE</u> : S^2 IS A SUBMANIFOLD OF $\mathbb{R}^3$

SUPPOSE, FOR EXAMPLE, THAT $p \in S^2$ IS IN THE UPPER HEMISPHERE $p^3 > 0$. DEFINE A CHART (U, φ) ON $\mathbb{R}^3$ AS FOLLOWS:



$$U = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2 < 1,\ z > 0 \}$$

$$\varphi : U \to \mathbb{R}^3$$

$$\varphi(x,y,z) = (x, y, z - \sqrt{1-x^2-y^2})$$

<u>EXERCISE 63</u> : SHOW THAT (U, φ) REALLY IS A CHART FOR THE STANDARD DIFFERENTIABLE STRUCTURE FOR $\mathbb{R}^3$.

$$\varphi(U \cap S^2) = \{ (x,y,0) : x^2+y^2 < 1 \}$$

SIMILAR ARGUMENTS WORK FOR ANY POINT IN S^2.

SIMILARLY, S^n IS A SUBMANIFOLD OF $\mathbb{R}^{n+1}$.

<u>EXERCISE 64</u> : LET A BE AN OPEN SET IN $\mathbb{R}^n$ AND $g : A \to \mathbb{R}$ A SMOOTH, REAL-VALUED FUNCTION ON A. SHOW THAT THE GRAPH OF g

$$X = \{ (x^1, \ldots, x^n, g(x^1, \ldots, x^n)) : (x^1, \ldots, x^n) \in A \}$$

IS A SMOOTH, n-DIMENSIONAL SUBMANIFOLD OF $\mathbb{R}^{n+1}$.

<u>EXERCISE 65</u> : SHOW THAT ANY SMOOTH LEVEL HYPERSURFACE IN $\mathbb{R}^{n+1}$ IS A SMOOTH SUBMANIFOLD OF $\mathbb{R}^{n+1}$.

<u>EXERCISE 66</u> : IF X IS A SUBMANIFOLD OF $\mathbb{R}^n$, THEN FOR EACH $p \in X$ $\exists$ CHART (U_p, φ_p) FOR $\mathbb{R}^n$ WITH $p \in U_p$ SUCH THAT $(U_p \cap X, \varphi_p | U_p \cap X)$ IS A CHART ON X AT p. SHOW THAT THE COLLECTION OF ALL SUCH CHARTS ON X FORMS AN ATLAS FOR X. THE CORRESPONDING DIFFERENTIABLE STRUCTURE FOR X IS THE <u>SUBMANIFOLD</u> <u>DIFFERENTIABLE</u> <u>STRUCTURE</u>.

<u>NOTE</u> : WE HAVE NOT YET DEFINED A SUBMANIFOLD OF AN ARBITRARY MANIFOLD, BUT WE WILL SOON.

PRODUCTS: LET X AND Y BE DIFFERENTIABLE MANIFOLDS OF DIMENSION n AND m, RESPECTIVELY.

LET $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ BE A CHART IN THE DIFFERENTIABLE STRUCTURE FOR X AND $\psi : V \to \psi(V) \subseteq \mathbb{R}^m$ A CHART IN THE DIFFERENTIABLE STRUCTURE FOR Y.

THEN $U \times V$ IS OPEN IN THE PRODUCT TOPOLOGY OF $X \times Y$ AND $\varphi(U) \times \psi(V)$ IS OPEN IN $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$.

DEFINE

$$\varphi \times \psi : U \times V \to \varphi(U) \times \psi(V)$$

$$(\varphi \times \psi)(u, v) = (\varphi(u), \psi(v))$$

EXERCISE 67: SHOW THAT $(U \times V, \varphi \times \psi)$ IS A CHART ON $X \times Y$ AND THAT ANY TWO SUCH CHARTS ARE C^∞-RELATED.

THUS, $X \times Y$ IS A DIFFERENTIABLE MANIFOLD OF DIMENSION $n+m$.

THE SAME PROCEDURE OBVIOUSLY EXTENDS TO LARGER PRODUCTS $X_1 \times \cdots \times X_k$ OF DIFFENTIABLE MANIFOLDS. FOR EXAMPLE, THE TORI $S^1 \times \cdots \times S^1$ ARE ALL DIFFERENTIABLE MANIFOLDS.

THE DEFINITION OF A GENERAL "SUBMANIFOLD" IS VIRTUALLY IDENTICAL TO THAT OF A "SUBMANIFOLD OF $\mathbb{R}^n$", BUT MEANINGFUL EXAMPLES WILL HAVE TO WAIT UNTIL WE HAVE ASSEMBLED A BIT MORE MACHINERY.

LET X BE AN n-DIMENSIONAL DIFFERENTIABLE MANIFOLD. A TOPOLOGICAL SUBSPACE X' OF X IS CALLED A <u>k-DIMENSIONAL SMOOTH SUBMANIFOLD OF</u> X (FOR SOME INTEGER $0 \leq k \leq n$) IF

> FOR EACH $p \in X'$ THERE IS A CHART (U, φ) IN THE DIFFERENTIABLE STRUCTURE FOR X SUCH THAT
> $$\varphi(p) = (0, \ldots, 0) \in \mathbb{R}^n$$
> AND
> $$\varphi(U \cap X') = \{ x \in \varphi(U) : x^{k+1} = \cdots = x^n = 0 \}$$

THE COLLECTION OF ALL CHARTS $(U \cap X', \varphi | U \cap X')$ IS AN ATLAS FOR X' DETERMINING THE <u>SUBMANIFOLD DIFFERENTIABLE STRUCTURE FOR</u> X'.

<u>EXERCISE</u> 68: SHOW THAT IF $k = 0$, THEN X' IS A DISCRETE SPACE AND IF $k = n$, THEN X' IS AN OPEN SUBMANIFOLD OF X.