DIFFERENTIATION TECHNIQUES (FINDING DERIVATIVES):

1. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

NEW ONES:

2. THE DERIVATIVE OF ANY CONSTANT FUNCTION IS ZERO:

$$\boxed{(c)' = 0}$$

E.G., IF $f(x) = 5$ FOR EVERY x, THEN $f'(x) = 0$ FOR EVERY x.

THIS IS OBVIOUS, RIGHT? HORIZONTAL LINE HAS A HORIZONTAL TANGENT AT EACH POINT.

3. FOR ANY REAL NUMBER n,

$$\boxed{(x^n)' = nx^{n-1}}$$

E.G., IF $f(x) = \sqrt{x} = x^{\frac{1}{2}}$, THEN $f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

THIS IS ANYTHING BUT OBVIOUS. I'LL SHOW YOU WHY IT'S TRUE, BUT ONLY FOR POSITIVE INTEGERS n:

RECALL :

$$A^2 - B^2 = (A-B)(A+B)$$
$$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$$
$$A^4 - B^4 = (A-B)(A^3 + A^2B + AB^2 + B^3)$$

LOOK AT THE PATTERN

$$A^n - B^n = (A-B)(A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1})$$

(TO PROVE THIS, JUST MULTIPLY OUT)

WE'LL USE THIS WITH $A = x + h$ AND $B = x$:

$$\begin{aligned}
f(x) = x^n \Rightarrow f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \to 0} \frac{((x+h)-x)((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + x^{n-1})}{h} \\
&= \lim_{h \to 0} \left((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + x^{n-1} \right) \\
&= x^{n-1} + x^{n-1} + \cdots + x^{n-1} \\
&= n x^{n-1}
\end{aligned}$$

AS PROMISED.

4. FOR ANY CONSTANT c,

$$\boxed{(c f(x))' = c f'(x)}$$

E.G., $(5x^3)' = 5(x^3)' = 5(3x^2) = 15x^2$.

THIS ONE IS EASY ENOUGH FOR YOU TO PROVE FOR YOURSELF. PLEASE DO SO.

5. THE DERIVATIVE OF A SUM (DIFFERENCE) IS THE SUM (DIFFERENCE) OF THE DERIVATIVES :

$$\boxed{(f(x) \pm g(x))' = f'(x) \pm g'(x)}$$

(ALSO TRUE FOR LARGER SUMS AND DIFFERENCES).

$$\begin{aligned} \text{E.G.,} \quad (3x^5 - 2x^2 + 1)' &= (3x^5)' - (2x^2)' + 1' \\ &= 3(x^5)' - 2(x^2)' + 0 \\ &= 3(5x^4) - 2(2x) \\ &= 15x^4 - 4x \end{aligned}$$

THE PROOF IS EASY : FOR SUMS,

$$\begin{aligned} (f(x) + g(x))' &= \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\ &= \lim_{h \to 0} \frac{(f(x+h) - f(x)) + (g(x+h) - g(x))}{h} \\ &= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right) \\ &= f'(x) + g'(x) \end{aligned}$$

AND NOW, TWO MORE THAT ARE NOT QUITE SO EASY TO PROVE.

6. (<u>PRODUCT RULE</u>)

$$\boxed{(f(x)g(x))' = f(x)g'(x) + g(x)f'(x)}$$

7. (<u>QUOTIENT RULE</u>)

$$\boxed{\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}}$$

E.G.,

$$((3x^2+2x-7)(x^3+8x))' =$$

$$(3x^2+2x-7)(x^3+8x)' + (x^3+8x)(3x^2+2x-7)' =$$

$$(3x^2+2x-7)(3x^2+8) + (x^3+8x)(6x+2) =$$

$$9x^4+6x^3-21x^2+24x^2+16x-56+6x^4+48x^2+2x^3+16x =$$

$$15x^4+8x^3+51x^2+32x-56$$

AND

$$\left(\frac{x^2-4}{x^2+4}\right)' = \frac{(x^2+4)(x^2-4)' - (x^2-4)(x^2+4)'}{(x^2+4)^2}$$

$$= \frac{(x^2+4)(2x) - (x^2-4)(2x)}{(x^2+4)^2}$$

$$= \frac{2x(x^2+4-x^2+4)}{(x^2+4)^2}$$

$$= \frac{16x}{(x^2+4)^2}$$

FOR THE RECORD, HERE'S A PROOF OF THE PRODUCT RULE (I'LL LEAVE THE QUOTIENT RULE FOR YOU TO THINK ABOUT):

$$(f(x)g(x))' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left[f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h} \right]$$

$$= f(x)\,g'(x) + g(x)\,f'(x) .$$

<u>HIGHER</u> <u>ORDER</u> <u>DERIVATIVES</u> : THE DERIVATIVE OF THE DERIVATIVE OF THE ...

E.G., LET $y = f(x) = \frac{1}{2}x^4 - 3x^2 + 1$

DERIVATIVE : $y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}x^4 - 3x^2 + 1\right) = 2x^3 - 6x$

SECOND DERIVATIVE :

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}(2x^3 - 6x) = 6x^2 - 6$$

THIRD DERIVATIVE :

$$y''' = f'''(x) = \frac{d^3y}{dx^3} = \frac{d}{dx}(6x^2 - 6) = 12x$$

FOURTH DERIVATIVE :

$$y^{(4)} = f^{(4)}(x) = \frac{d^4y}{dx^4} = \frac{d}{dx}(12x) = 12$$

FROM THIS POINT ON ALL OF THE DERIVATIVES OF $\frac{1}{2}x^4 - 3x^2 + 1$ ARE ZERO.

EXAMPLES :

1. FIND THE POINTS ON THE GRAPH OF $f(x) = \frac{x^2-4}{x^2+4}$ WHERE THE TANGENT LINE IS HORIZONTAL.

HORIZONTAL TANGENT MEANS $f'(x) = 0$.

$$f'(x) = \left(\frac{x^2-4}{x^2+4} \right)' = \frac{16x}{(x^2+4)^2} \quad \text{(PREVIOUS EXAMPLE)}$$

$$f'(x) = 0$$

$$\frac{16x}{(x^2+4)^2} = 0$$

$$16x = 0$$

$$x = 0 \qquad f(0) = \frac{0^2-4}{0^2+4} = -1$$

POINT : $(0, -1)$

2. FIND $\frac{d^2R}{dt^2}$ IF $R(t) = \frac{t+1}{t}$

$$\frac{dR}{dt} = \frac{d}{dt}\left(\frac{t+1}{t} \right) = \frac{d}{dt}\left(\frac{t}{t} + \frac{1}{t} \right) = \frac{d}{dt}(1 + t^{-1})$$

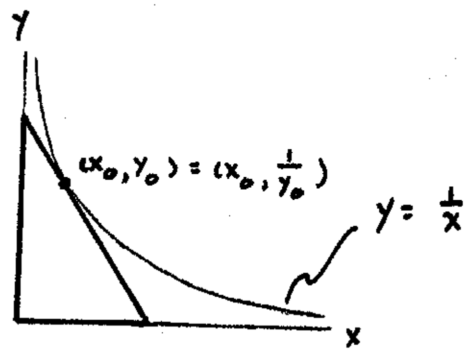
$$= 0 + (-1)t^{-1-1}$$

$$= -t^{-2}$$

$$\frac{d^2R}{dt^2} = \frac{d}{dt}\left(\frac{dR}{dt} \right) = \frac{d}{dt}(-t^{-2}) = -(-2)t^{-2-1}$$

$$= 2t^{-3} = \frac{2}{t^3}$$

3. SHOW THAT THE TRIANGLE FORMED BY ANY TANGENT LINE TO THE GRAPH OF $y = \frac{1}{x}$, $x > 0$, AND THE COORDINATE AXES HAS AN AREA OF 2 (SQUARE UNITS).



TANGENT LINE AT $(x_0, y_0) = (x_0, \frac{1}{x_0})$: SLOPE $= y'(x_0) = -\frac{1}{x_0^2}$

$$y - y_0 = m(x - x_0)$$

$$y - \frac{1}{x_0} = \left(-\frac{1}{x_0^2}\right)(x - x_0)$$

$$y - \frac{1}{x_0} = -\frac{1}{x_0^2}x + \frac{1}{x_0}$$

$$y = -\frac{1}{x_0^2}x + \frac{2}{x_0}$$

y - INTERCEPT : $x = 0$, $y = \frac{2}{x_0}$

x - INTERCEPT : $y = 0$, $0 = -\frac{1}{x_0^2}x + \frac{2}{x_0} \Rightarrow x = 2x_0$

$$\text{AREA} = \frac{1}{2}(\text{BASE})(\text{HEIGHT})$$

$$= \frac{1}{2}(2x_0)\left(\frac{2}{x_0}\right)$$

$$= 2$$