DIFFERENTIATION AND INTEGRATION OF SERIES

TWO THINGS WE KNOW :

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

FOR ALL x.

NOW NOTICE THAT

$$(x)' = 1$$

$$\left(\frac{1}{3!}x^3\right)' = \frac{1}{3!}(3x^2) = \frac{1}{2!}x^2$$

$$\left(\frac{1}{5!}x^5\right)' = \frac{1}{5!}(5x^4) = \frac{1}{4!}x^4$$

$$\left(\frac{1}{7!}x^7\right)' = \frac{1}{7!}(7x^6) = \frac{1}{6!}x^6$$

$$\vdots$$

SO

$$(x)' - \left(\frac{1}{3!}x^3\right)' + \left(\frac{1}{5!}x^5\right)' - \left(\frac{1}{7!}x^7\right)' + \cdots = \cos x$$

$$= (\sin x)'$$

THE DERIVATIVE OF $\sin x$ CAN BE COMPUTED BY DIFFENTIATING ITS MACLAURIN SERIES <u>TERM-BY-TERM</u> JUST AS IF IT WERE A POLYNOMIAL.

THEOREM : IF

$$f(x) = \sum_{k=0}^{\infty} c_k (x-x_0)^k = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots + c_k(x-x_0)^k + \cdots$$

ON THE OPEN INTERVAL $(x_0 - R, x_0 + R)$, THEN $f(x)$ IS DIFFERENTIABLE ON THIS INTERVAL AND

$$f'(x) = (c_0)' + (c_1(x-x_0))' + (c_2(x-x_0)^2)' + \cdots + (c_k(x-x_0)^k)' + \cdots$$

$$= c_1 + 2c_2(x-x_0) + \cdots kc_k(x-x_0)^{k-1} + \cdots$$

$$= \sum_{k=1}^{\infty} k c_k (x-x_0)^{k-1}$$

HIGHER ORDER DERIVATIVES ARE COMPUTED IN THE SAME WAY.

EXAMPLE : $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$ FOR ALL X

$$(e^x)' = \sum_{k=1}^{\infty} k \frac{1}{k!} x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1}$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

$$= e^x$$

NOTE ON PREVIOUS EXAMPLE : DESPITE APPEARENCES THE TWO SERIES $\sum_{k=0}^{\infty} \frac{1}{k!} x^k$ AND $\sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1}$ ARE EXACTLY THE SAME, AS BECOMES APPARENT WHEN YOU START WRITING THE TERMS OUT. EACH CAN BE OBTAINED FROM THE OTHER BY RE-INDEXING, E.G.,

$$\sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1}$$: LET $n = k-1$. THEN

$$k=1 \Rightarrow n = 1-1 = 0$$

SO
$$\sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

WHICH IS THE SAME AS $\sum_{k=0}^{\infty} \frac{1}{k!} x^k$

ANOTHER EXAMPLE :

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \quad \Rightarrow \quad f'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

LET $n = k-1$. THEN $k = n+1$

AND $k=1 \Rightarrow n = 1-1 = 0$ SO

$$f'(x) = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

(THIS WILL BE USEFUL WHEN WE RETURN TO DIFFERENTIAL EQUATIONS.)

POWER SERIES CAN ALSO BE INTEGRATED TERM-BY-TERM :

THEOREM : IF

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$$

ON $(x_0 - R, x_0 + R)$, THEN

$$\int f(x)\,dx = \sum_{k=0}^{\infty} \frac{c_k}{k+1} (x - x_0)^{k+1} + C$$

ON $(x_0 - R, x_0 + R)$.

EXAMPLES :

1. $$\int \cos x \, dx = \int \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \, dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!\,(2k+1)} x^{2k+1} + C$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} + C$$

$$= \sin x + C$$

2. WE OBTAIN A SERIES EXPANSION FOR ARCTAN x ON $(-1, 1)$ BY INTEGRATING AN EXPANSION FOR $\frac{1}{1+x^2}$ AS FOLLOWS :

FOR $-1 < x < 1$,

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{(GEOMETRIC SERIES)}$$

SO

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-x)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k x^k$$

AND

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k (x^2)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

NOW INTEGRATE

$$\int \frac{1}{1+x^2}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + C$$

FOR SOME CHOICE OF C THIS GIVES THE ANTIDERIVATIVE ARCTAN X FOR $\frac{1}{1+x^2}$. TO SOLVE

$$\text{ARCTAN } x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + C$$

FOR C, PLUG IN $x = 0$.

$$\text{ARCTAN } 0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cdot 0^{2k+1} + C$$

$$0 = 0 + C$$

SO

$$\text{ARCTAN } x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \quad \text{ON } (-1, 1)$$

<u>NOTE</u> : THIS SERIES ACTUALLY CONVERGES AT $X = \pm 1$ AS WELL (ALTERNATING SERIES TEST). IT'S NOT SO OBVIOUS (BUT IT'S TRUE) THAT THE SERIES ALSO CONVERGES TO ARCTAN X FOR THESE TWO VALUES OF X, E.G.,

$$\arctan(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cdot 1^{2k+1}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

3. THE FUNCTION e^{-x^2} HAS NO ELEMENTARY ANTIDERIVATIVE SO WE HAVE NO MEANS OF CALCULATING $\int_0^1 e^{-x^2}dx$ EXACTLY. WE FIND A SERIES EXPANSION FOR THE VALUE AND APPROXIMATE IT BY A PARTIAL SUM AS FOLLOWS.

FOR ALL X,

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

SO

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{1}{k!} (-x^2)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k}$$

NOW INTEGRATE

$$\int e^{-x^2}dx = \int \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k} dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!} x^{2k+1} + C$$

THUS,

$$\int_0^1 e^{-x^2}dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!}$$

$$= 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7.3!} + \cdots$$

SINCE THE SERIES IS ALTERNATING IT IS EASY TO APPROXIMATE THE ERROR MADE IN APPROXIMATING $\int_0^1 e^{-x^2}dx$ BY THE n^{TH} PARTIAL SUM

$$S_n = 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \cdots + \frac{(-1)^n}{(2n+1)n!}$$

$$\left| \int_0^1 e^{-x^2}dx - S_n \right| < \frac{1}{(2(n+1)+1)(n+1)!} = \frac{1}{(2n+3)(n+1)!}$$

FOR EXAMPLE, IF WE APPROXIMATE

$$\int_0^1 e^{-x^2}dx \approx 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} \approx 0.743$$

THEN THE ERROR IS LESS THAN

$$\frac{1}{9\cdot 4!} \approx 0.0046$$

THERE ARE A FEW MORE ALGEBRAIC TECHNIQUES FOR GETTING SERIES EXPANSIONS (OR AT LEAST PARTIAL SUMS OF THEM) QUICKLY. I'LL ILLUSTRATE WITH A FEW EXAMPLES.

EXAMPLES :

1. $f(x) = x^2 \sin x$: $\qquad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$

$$f(x) = x^2 \sin x = x^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+3}$$

2. $f(x) = \cos^2 x$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x = \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(2x)^{2k}$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k}}{(2k)!} x^{2k}$$

$$= \frac{1}{2} + \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k-1}}{(2k)!} x^{2k}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1}}{(2k)!} x^{2k}$$

3. $f(x) = e^{-x^2} \arctan x$

$$= \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k}\right)\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}\right)$$

$$= \left(1 - x^2 + \tfrac{1}{2}x^4 - \cdots\right)\left(x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \cdots\right)$$

AT THIS POINT WE JUST START MULTIPLYING OUT, KEEPING TRACK OF WHICH PRODUCTS GIVE RISE TO WHICH POWERS OF x :

CONSTANT TERM : NONE

x : $(1)(x) = x$

x^2 : NONE

x^3 : $(1)(-\frac{1}{3}x^3) + (-x^2)(x) = -\frac{4}{3}x^3$

x^4 : NONE

x^5 : $(1)(\frac{1}{5}x^5) + (-x^2)(-\frac{1}{3}x^3) + (\frac{1}{2}x^4)(x) =$

$$\frac{1}{5}x^5 + \frac{1}{3}x^5 + \frac{1}{2}x^5 = \frac{31}{30}x^5$$

$\vdots$

$$e^{-x^2} \text{ ARCTAN } x = x - \frac{4}{3}x^3 + \frac{31}{30}x^5 + \cdots$$

4. $f(x) = \tan x = \dfrac{\sin x}{\cos x}$

$$= \frac{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}}{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}} = \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots}$$

HERE WE JUST DO A LONG DIVISION (VERY LONG DIVISION):

$$
\begin{array}{r l}
 & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\
\hline
1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \;\Big) & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots \\
 & x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots \\
\hline
 & \frac{1}{3}x^3 - \frac{1}{30}x^5 - \cdots \\
 & \frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots \\
\hline
 & \frac{2}{15}x^5 + \cdots \\
 & \frac{2}{15}x^5 + \cdots \\
\hline
\end{array}
$$

ARITHMETIC : $-\frac{1}{6}x^3 + \frac{1}{2}x^3 = -\frac{1}{6}x^3 + \frac{3}{6}x^3 = \frac{2}{6}x^3 = \frac{1}{3}x^3$

$\frac{1}{120}x^5 - \frac{1}{24}x^5 = \frac{1}{120}x^5 - \frac{5}{120}x^5 = -\frac{4}{120}x^5 = -\frac{1}{30}x^5$

$-\frac{1}{30}x^5 + \frac{1}{6}x^5 = -\frac{1}{30}x^5 + \frac{5}{30}x^5 = \frac{4}{30}x^5 = \frac{2}{15}x^5$

SO

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

<u>NOTE</u> : JUST TO SEE THE ADVANTAGES OF THE LONG DIVISION YOU SHOULD START COMPUTING DERIVATIVES OF $f(x) = \tan x$ AND TRY TO USE THE MACLAURIN FORMULAS $\frac{f^{(k)}(0)}{k!}$ TO GET THE COEFFICIENTS.