DISCRETE S^1-LOCALIZATION AND THE EQUIVARIANT EULER CLASS I

OBJECTIVE : TO REPHRASE THE DISCRETE S^1- EQUIVARIANT LOCALIZATION FORMULA

(1)
$$\int_M \alpha(\xi) = \sum_{p \in Z(\xi^\#)} (-2\pi)^k \frac{\alpha(\xi)_{[0]}(p)}{Pf(L_p(\xi))}$$

IN TERMS OF THE " S^1- EQUIVARIANT EULER CLASS OF THE NORMAL BUNDLE OF $Z(\xi^\#)$ "

(2)
$$\int_M \alpha(\xi) = \sum_{p \in Z(\xi^\#)} (-2\pi)^k \frac{\alpha(\xi)_{[0]}(p)}{e_{S^1}(N_p(Z(\xi^\#)))(\xi)}$$

IN ORDER TO MOTIVATE THE NONDISCRETE S^1- EQUIVARIANT LOCALIZATION FORMULA

$$\int_M \alpha(\xi) = \sum_{X \in \pi_0(Z(\xi^\#))} \int_X (-2\pi)^{\mathrm{RANK}(N(X))/2} \frac{\iota_X^*(\alpha(\xi)_{[0]})}{e_{S^1}(N(X))(\xi)}$$

CONTEXT FOR DISCRETE S^1-LOCALIZATION :

M COMPACT, ORIENTED, DIMENSION $2k$

ORIENTATION PRESERVING S^1-ACTION $(g,p) \in S^1 \times M \to g\cdot p = L_g(p) \in M$

$\alpha \in \Omega^*_{S^1}(M)$ S^1-EQUIVARIANTLY CLOSED

$\xi \in \mathrm{Lie}(S^1) = i\mathbb{R}$ SUCH THAT $Z(\xi^\#) = M^{S^1}$ IS FINITE

FIX AN S^1- INVARIANT RIEMANNIAN METRIC $\langle\,,\,\rangle_{S^1}$ ON M. THEN FOR $p \in Z(\xi^*)$

$$L_p(\xi) : T_p(M) \longrightarrow T_p(M)$$

$$L_p(\xi)(v_p) = (\mathcal{L}_{\xi^*} V)_p = [\xi^*, V]_p = -\frac{d}{dt}\left((L_{\exp(-t\xi)})_{*p}(v_p)\right)\Big|_{t=0}$$

IS SKEW - SYMMETRIC AND INVERTIBLE.

$$\boxed{\mathrm{EXP}\,(L_p(\xi)) = (L_{\exp(\xi)})_{*p}}$$

<u>REPRESENTATION</u> <u>OF</u> S^1 <u>ON</u> $T_p(M)$:

$$\rho = \rho_p : S^1 \longrightarrow GL(T_p(M))$$

$$\boxed{\rho(g) = (L_g)_{*p}}$$

$$\rho(g)(v_p) = g \cdot v_p = (L_g)_{*p}(v_p)$$

<u>NOTE</u> : $\quad g \cdot v_p = v_p \;\; \forall g \in S^1 \quad \Leftrightarrow \quad v_p = 0$

$$\rho_{*1} : \mathrm{Lie}(S^1) \longrightarrow \mathfrak{gl}(T_p(M))$$

$$\boxed{\rho_{*1}(\xi) = L_p(\xi)}$$

<u>LEMMA</u> : LET V BE A REAL VECTOR SPACE AND $\rho : S^1 \to GL(V)$ A REPRESENTATION OF S^1 ON V WHICH LEAVES NO NONZERO VECTOR FIXED ($\rho(g)(v) = v \;\; \forall g \in S^1 \iff v = 0$). LET $A \in End(V)$ BE GIVEN BY

$$A = \rho_{*1}(i) = \frac{d}{dt}(\rho(\exp(it)))\Big|_{t=0}$$

THEN THERE IS A UNIQUE DECOMPOSITION

$$V = V_1 \oplus \cdots \oplus V_\ell$$

OF V INTO S^1-INVARIANT SUBSPACES AND POSITIVE INTEGERS

$$0 < m_1 < \cdots < m_\ell$$

SUCH THAT

$$A_j := A|V_j = m_j J_j \qquad , \quad j = 1,\dots,\ell$$

WHERE

$$J_j^2 = -1 \qquad , \quad j = 1,\dots,\ell$$

AND SUCH THAT, ON V_j, THE S^1-ACTION IS GIVEN BY

$$\rho(e^{it}) = \exp(tA_j) = \exp(m_j t J_j) .$$

IN PARTICULAR, V HAS A CANONICAL COMPLEX STRUCTURE

$$J = J_1 \oplus \cdots \oplus J_\ell$$

AND SO IS EVEN DIMENSIONAL WITH A CANONICAL ORIENTATION.

<u>SKETCH OF THE PROOF</u> : $\langle\ ,\ \rangle$ = S^1-INVARIANT INNER PRODUCT ON V.

PRODUCT RULE $\Rightarrow$ A IS <u>SKEW-SYMMETRIC</u> WITH RESPECT TO $\langle\ ,\ \rangle$.

NO NONZERO VECTOR FIXED $\Rightarrow$ A <u>INVERTIBLE</u>

IF A DENOTES ALSO THE MATRIX WITH RESPECT TO A $\langle\,,\rangle$-ORTHONORMAL BASIS, THEN $A^T = -A$ AND $\det A \neq 0$ SO

$$A^T A = -A^2$$

IS SYMMETRIC AND POSITIVE DEFINITE. LET

$$0 < \lambda_1 < \cdots < \lambda_\ell$$

BE THE DISTINCT EIGENVALUES OF $-A^2$ AND

$$V = V_1 \oplus \cdots \oplus V_\ell$$

THE EIGENSPACE DECOMPOSITION OF V. THEN

$$A_j^2 = -\lambda_j 1 \qquad , \quad j = 1, \ldots, \ell$$

LET $m_j = \sqrt{\lambda_j}$ AND $J_j = \frac{1}{m_j} A_j$ TO GET $A_j = m_j J_j$ AND $J_j^2 = -1$. $\square$

WE WILL APPLY THIS TO $\rho : S^1 \to GL(T_p(M))$, BUT FIRST WE TAKE A SOMEWHAT PECULIAR VIEW OF $T_p(M)$ FOR $p \in Z(\xi^*)$.

<u>NOTE</u> : IT WILL NOT SEEM SO "PECULIAR" WHEN WE TURN TO THE NONDISCRETE CASE.

$Z(\xi^*) = M^{S^1}$ IS A 0-DIMENSIONAL SUBMANIFOLD OF M SO $T_p(Z(\xi^*)) = \{0\}$ FOR EACH $p \in Z(\xi^*)$. THE NORMAL BUNDLE OF $Z(\xi^*)$ IS THEREFORE

$$N(Z(\xi^*)) = \bigcup_{p \in Z(\xi^*)} N_p(Z(\xi^*)) = \bigcup_{p \in Z(\xi^*)} T_p(M)$$

THUS, ON EACH FIBER, WE HAVE A REPRESENTATION

$$\rho = \rho_p : S^1 \longrightarrow GL(N_p(Z(\xi^*)))$$

$$\rho(g) = (L_g)_{*p}$$

WITH

$$\rho_{*1}(\xi) = L_p(\xi)$$

AND SATISFYING THE CONDITIONS OF THE LEMMA WITH

$$A = \rho_{*1}(i) = L_p(i).$$

<u>CONCLUSION</u> : EACH FIBER SPLITS

$$N_p(Z(\xi^*)) = \bigoplus_{j=1}^{\ell_p} N_{p,j}(Z(\xi^*))$$

INTO S^1-INVARIANT SUBSPACES AND THERE EXIST POSITIVE INTEGERS $0 < m_{p,1} < \cdots < m_{p,\ell_p}$ AND OPERATORS $J_{p,j} : N_{p,j}(Z(\xi^*)) \to N_{p,j}(Z(\xi^*))$ SUCH THAT

$$J_{p,j}^{2} = -1$$

$$A \mid N_{p,j}(Z(\xi^{\#})) = m_{p,j} J_{p,j}$$

AND THE S^1-ACTION ON $N_{p,j}(Z(\xi^{\#}))$ IS GIVEN BY

$$\rho(e^{it}) = \exp(m_{p,j} t J_{p,j}).$$

$J_{p,j}$ IS A COMPLEX STRUCTURE FOR $N_{p,j}(Z(\xi^{\#}))$ WHICH IS THEREFORE EVEN DIMENSIONAL, CANONICALLY ORIENTED, AND SPLITS INTO A SUM

$$N_{p,j}(Z(\xi^{\#})) \cong \mathbb{C} \oplus \overset{r_{p,j}}{\cdots} \oplus \mathbb{C}$$

OF COPIES OF $\mathbb{C}$, ON EACH OF WHICH THE S^1-ACTION BY $e^{it} \in S^1$ IS MULTIPLICATION BY

$$e^{m_{p,j} t i}.$$

DOING THIS FOR EACH $j = 1, \ldots, \ell_p$ GIVES THE DECOMPOSITION

$$N_p(Z(\xi^{\#})) \cong \mathbb{C} \oplus \overset{k}{\cdots} \oplus \mathbb{C}$$

AND A SEQUENCE

$$0 < n_{p,1} \leq \cdots \leq n_{p,k}$$

OF (NOT NECESSARILY DISTINCT) POSITIVE INTEGERS SUCH THAT THE S^1-ACTION BY $e^{it} \in S^1$ ON THE i^{TH} SUMMAND IS MULTIPLICATION BY

$$e^{n_{p,i} t i} .$$

SINCE EACH CONNECTED COMPONENT OF $Z(\xi^{\#})$ IS A POINT $\{p\}$ WE CONCLUDE THAT

> THE NORMAL BUNDLE TO ANY CONNECTED COMPONENT OF $Z(\xi^{\#})$ IS OF THE FORM
>
> $$\begin{array}{c} \mathbb{C} \oplus \overset{k}{\cdots} \oplus \mathbb{C} \\ \downarrow \\ \{p\} \end{array}$$
>
> WITH AN S^1-ACTION ON $\mathbb{C} \oplus \overset{k}{\cdots} \oplus \mathbb{C}$ AS DESCRIBED ABOVE.

DEFINING THE OBVIOUS (TRIVIAL) S^1-ACTION ON THE BASE $\{p\}$ THIS IS AN EXAMPLE OF AN "S^1-EQUIVARIANT VECTOR BUNDLE", AS DEFINED BELOW.

<u>DEFINITION</u>: LET $\pi_E : E \to X$ BE A SMOOTH VECTOR BUNDLE AND G A COMPACT LIE GROUP. SUPPOSE G ACTS ON BOTH E AND X ON THE LEFT IN SUCH A WAY THAT

$$\pi_E(g \cdot v) = g \cdot \pi_E(v)$$

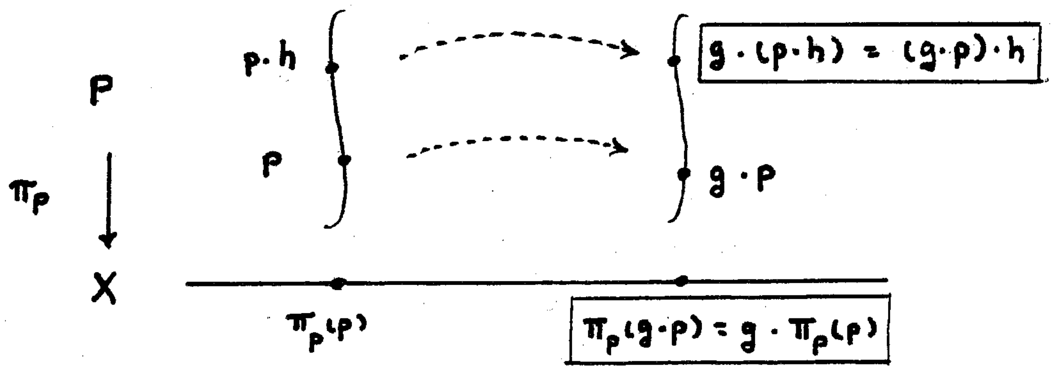
AND, FOR EACH $g \in G$, THE MAP

$$v \xrightarrow{\tau_g} g \cdot v \;:\; \pi_E^{-1}(x) \longrightarrow \pi_E^{-1}(g \cdot x)$$

IS LINEAR $\forall\, x \in X$. THEN $\pi_E : E \to X$ IS A <u>G-EQUIVARIANT VECTOR BUNDLE</u> OVER X.

> <u>NOTE</u>: FOLLOWS THAT τ_g IS AN ISOMORPHISM.
> IF $\pi_E : E \to X$ IS ORIENTED WE REQUIRE ALSO THAT THESE ISOMORPHISMS BE ORIENTATION PRESERVING.

<u>DEFINITION</u>: LET $H \hookrightarrow P \xrightarrow{\pi_P} X$ BE A SMOOTH PRINCIPAL H-BUNDLE AND G A LIE GROUP. THEN $H \hookrightarrow P \xrightarrow{\pi_P} X$ IS A <u>G-EQUIVARIANT PRINCIPAL H-BUNDLE</u> IF G ACTS ON BOTH P AND X ON THE LEFT AND



<u>EXAMPLE</u>: LET $\pi_E : E \to X$ BE ANY G-EQUIVARIANT, ORIENTED REAL VECTOR BUNDLE OF RANK (FIBER DIMENSION) $2k$. CHOOSE A G-INVARIANT FIBER METRIC ON $\pi_E : E \to X$ AND CONSIDER THE CORRESPONDING ORIENTED, ORTHONORMAL FRAME BUNDLE

$$SO(2k) \hookrightarrow F_{SO}(E) \xrightarrow{\pi_{SO}} X.$$

ACTION OF G ON $\pi_E : E \to X$ INDUCES AN ACTION OF G ON FRAMES. THIS, TOGETHER WITH THE ACTION OF G ON X FROM THE VECTOR BUNDLE MAKE THIS A G-EQUIVARIANT PRINCIPAL $SO(2k)$-BUNDLE.

THE ENTIRE APPARATUS OF THE CHERN-WEIL THEORY OF CHARACTERISTIC CLASSES GENERALIZES TO THIS CONTEXT TO PRODUCE "G-EQUIVARIANT CHARACTERISTIC CLASSES ON G-EQUIVARIANT PRINCIPAL H-BUNDLES". HERE WE OFFER ONLY A SYNOPSIS.

<u>RECALL</u> (ORDINARY CHERN-WEIL): $H \hookrightarrow P \xrightarrow{\pi_P} X$ A SMOOTH PRINCIPAL H-BUNDLE. CHOOSE A CONNECTION ω ON P WITH CURVATURE Ω. ANY INVARIANT POLYNOMIAL $\rho \in \mathbb{C}^k[\mathfrak{h}]^H \subseteq \mathbb{C}[\mathfrak{h}]^H$ ON THE LIE ALGEBRA $\mathfrak{h}$ CAN BE "EVALUATED AT Ω" AS FOLLOWS: CHOOSE A BASIS $\{\xi_1, \dots, \xi_n\}$ FOR $\mathfrak{h}$ AND WRITE $\omega = \omega^a \xi_a$ AND $\Omega = \Omega^a \xi_a$. ρ GIVES RISE (VIA POLARIZATION) TO

A UNIQUE SYMMETRIC, k-MULTILINEAR, H-INVARIANT FUNCTION $\mathfrak{h} \times \overset{k}{\cdots} \times \mathfrak{h} \to \mathbb{C}$ (ALSO DENOTED ρ). DEFINE

$$\rho(\Omega) = \rho(\Omega^{a_1}\xi_{a_1}, \dots, \Omega^{a_k}\xi_{a_k}) = \rho(\xi_{a_1}, \dots, \xi_{a_k})\Omega^{a_1}\wedge\cdots\wedge\Omega^{a_k}.$$

THEN $\rho(\Omega)$ IS A BASIC FORM ON P (H-INVARIANT AND HORIZONTAL) AND SO DESCENDS TO A $2k$-FORM $\bar{\rho}(\Omega)$ ON X:

$$\rho(\Omega) = \pi_p^*(\bar{\rho}(\Omega))$$

$\bar{\rho}(\Omega)$ IS CLOSED AND ITS COHOMOLOGY CLASS $[\bar{\rho}(\Omega)]$ DOES NOT DEPEND ON THE CHOICE OF ω OR $\{\xi_1, \dots, \xi_n\}$. IT IS A <u>CHARACTERISTIC CLASS</u> OF $H \hookrightarrow P \xrightarrow{\pi_p} X$.

<u>EQUIVARIANT CHERN-WEIL</u> (SKETCH):

$$H \hookrightarrow P \xrightarrow{\pi_p} X$$

A G-EQUIVARIANT PRINCIPAL H-BUNDLE.

COMBINE H- AND G-ACTIONS ON P TO OBTAIN A $G \times H$-ACTION:

$$(g,h)\cdot p = g\cdot(p\cdot h^{-1}) = (g\cdot p)\cdot h^{-1}$$

$$(g,1)\cdot p = g\cdot p$$

$$(1,h^{-1})\cdot p = p\cdot h$$

INDUCED ACTION ON $\Omega^*(P)$:

$$(g,h)\cdot \varphi = L_{(g,h)^{-1}}{}^{*}\varphi = R_h^* \circ L_{g^{-1}}^* \varphi$$

INDUCED ACTION ON $\mathbb{C}[\mathfrak{g} \oplus \mathfrak{h}] \otimes \Omega^*(P)$:

$$\alpha = \rho \otimes \varphi$$

$$((g,h)\cdot\alpha)(\xi,\zeta) = \rho(g^{-1}\xi g, h^{-1}\zeta h)\, R_h^* \circ L_{g^{-1}}^* \varphi$$

$\Omega_{G\times H}^*(P)$ = $G\times H$ - INVARIANT ELEMENTS :

$$\rho(g\xi g^{-1}, h\zeta h^{-1})\varphi = \rho(\xi,\zeta)\, R_h^* \circ L_{g^{-1}}^* \varphi$$

$(\Omega_{G\times H}^*(P), d_{G\times H})$ GIVES THE $G\times H$ - EQUIVARIANT COHOMOLOGY OF P.

<u>NOTE</u> : AN ELEMENT $\alpha = \rho \otimes \varphi$ OF $\Omega_{G\times H}^*(P)$ FOR WHICH ρ DEPENDS ONLY ON $\xi \in \mathfrak{g}$ CAN BE IDENTIFIED WITH AN ELEMENT OF $\Omega_G^*(P)$ (TAKE $h = 1$ ABOVE) THAT IS H - INVARIANT (TAKE $g = 1$ ABOVE). THE SET OF ALL SUCH <u>H - INVARIANT</u>, <u>G - EQUIVARIANT</u> <u>FORMS</u> <u>ON</u> P IS DENOTED

$$\Omega_G^*(P)^H \, .$$

THE ELEMENTS OF $\Omega^*_G(P)$ THAT ARE ALSO H-HORIZONTAL ($\Omega^*(P)$-PARTS VANISH ON H-VERTICAL VECTORS) DESCEND TO ELEMENTS OF $\Omega^*_G(X)$.

$$\Omega^*_G(P)_{H\text{-HOR}} = \Omega^*_G(P)_{H\text{-BASIC}} \cong \Omega^*_G(X)$$

EQUIVARIANT CHERN-WEIL PRODUCES AN ELEMENT OF THIS ALGEBRA BY EVALUATING A $p \in \mathbb{C}[\mathfrak{h}]^H$ ON THE "G-EQUIVARIANT CURVATURE OF A G-INVARIANT CONNECTION ON $H \hookrightarrow P \xrightarrow{\pi_P} X$".
HERE'S THE DEFINITION:

CHOOSE A CONNECTION ω ON $H \hookrightarrow P \xrightarrow{\pi_P} X$ THAT IS G-INVARIANT ($L_g^*\omega = \omega \ \ \forall g \in G$). THIS CAN BE OBTAINED FROM ANY CONNECTION BY AVERAGING OVER G.

THE CURVATURE Ω OF ω IS ALSO G-INVARIANT SO WE CAN IDENTIFY

$$\omega = 1 \otimes \omega \in \Omega^1_G(P) \otimes \mathfrak{h}$$

$$\Omega = 1 \otimes \Omega \in \Omega^2_G(P) \otimes \mathfrak{h}$$

EXTEND d_G AND ι_{ξ^*} FROM $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(P)$ TO

$$\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(P) \otimes \mathfrak{h}$$

COMPONENTWISE RELATIVE TO ANY BASIS FOR $\mathfrak{h}$.

THE <u>G-EQUIVARIANT</u> <u>CURVATURE</u> Ω_G OF ω IS

$$\Omega_G = d_G\omega + \tfrac{1}{2}[\omega,\omega] \in \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(P) \otimes \mathfrak{h}$$

THUS, $\forall\, \xi \in \mathfrak{g}$,

$$\Omega_G(\xi) = d\omega - \iota_{\xi^*}\omega + \tfrac{1}{2}[\omega,\omega]$$

$$= \Omega - \underbrace{\iota_{\xi^*}\omega}$$

DEFINE THE <u>EQUIVARIANT</u> <u>MOMENT</u> OF ω BY

$$\mathcal{J}_\omega \in \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(P) \otimes \mathfrak{h}$$

$$\mathcal{J}_\omega(\xi) = \iota_{\xi^*}\omega$$

$$\boxed{\Omega_G = \Omega - \mathcal{J}_\omega}$$

$\mathcal{J}_\omega(g\xi g^{-1}) = L_{g^{-1}}^{\,*}(\mathcal{J}_\omega(\xi))$ AND $L_g^*\Omega = \Omega$ IMPLY

$$\Omega_G \in \Omega_G^*(P) \otimes \mathfrak{h}$$

(SORRY FOR THE DOUBLE USE OF "Ω_G")

JUST AS Ω IS GENERALLY NOT CLOSED ($d\Omega \neq 0$), BUT IS COVARIANTLY CLOSED (BIANCHI IDENTITY : $d^{\omega}\Omega = 0$), SO Ω_G IS GENERALLY NOT EQUIVARIANTLY CLOSED ($d_G \Omega_G \neq 0$), BUT DOES SATISFY

$$d_G^{\omega} \Omega_G = 0$$

WHERE, FOR $\alpha \in \Omega_G^{*}(P)$ AND $\xi \in \mathfrak{g}$,

$$(d_G^{\omega} \alpha)(\xi) = d^{\omega}(\alpha(\xi)) - \iota_{\xi^{*}}(\alpha(\xi)).$$

IF $\{J_1, \dots, J_n\}$ IS A BASIS FOR $\mathfrak{h}$, WRITE

$$\Omega_G = \Omega_G^{a} J_a = (\Omega^{a} - J_{\omega^{a}}) J_a$$

$(J_{\omega^{a}}(\xi) = \iota_{\xi^{*}} \omega^{a})$.

<u>EQUIVARIANT</u> <u>CHERN</u>-<u>WEIL</u> : $\forall\, \rho \in \mathbb{C}^{k}[\mathfrak{h}]^{H} \subseteq \mathbb{C}[\mathfrak{h}]^{H}$

DEFINE

$$\begin{aligned} \rho(\Omega_G) &= \rho(\Omega_G^{a_1} J_{a_1}, \dots, \Omega_G^{a_k} J_{a_k}) \\ &= \rho(J_{a_1}, \dots, J_{a_k}) \Omega_G^{a_1} \cdots \Omega_G^{a_k} \\ &= \rho(J_{a_1}, \dots, J_{a_k})(\Omega^{a_1} - J_{\omega^{a_1}}) \cdots (\Omega^{a_k} - J_{\omega^{a_k}}) \end{aligned}$$

WHERE THE PRODUCTS ARE IN $\mathbb{C}[\mathfrak{g}] \otimes \Omega^{*}(P)$.

THEN

$$\rho(\Omega_G) \in \Omega_G^*(P)_{H\text{-BASIC}}$$

SO $\exists\ \bar{\rho}(\Omega_G) \in \Omega_G^*(X)$ SUCH THAT

$$\rho(\Omega_G) = \pi_P^*(\bar{\rho}(\Omega_G)).$$

MOREOVER

$$d_G(\bar{\rho}(\Omega_G)) = 0$$

AND THE COHOMOLOGY CLASS

$$[\bar{\rho}(\Omega_G)] \in H_G^*(X)$$

IS INDEPENDENT OF THE CHOICE OF ω AND $\{J_1, \dots, J_n\}$. IT IS A <u>G-EQUIVARIANT</u> <u>CHARACTERISTIC</u> <u>CLASS</u> OF $H \hookrightarrow P \xrightarrow{\pi_P} X$.

<u>EQUIVARIANT</u> <u>EULER</u> <u>CLASS</u> : $\pi_E : E \to X$ A G-EQUIVARIANT, ORIENTED, REAL VECTOR BUNDLE OF FIBER DIMENSION $2k$. CHOOSE A G-INVARIANT FIBER METRIC ON E AND CONSIDER THE CORRESPONDING G-EQUIVARIANT PRINCIPAL $SO(2k)$-BUNDLE

$$SO(2k) \hookrightarrow F_{SO}(E) \xrightarrow{\pi_{SO}} X.$$

IN THE CONSTRUCTION ABOVE CHOOSE $\rho = Pf \in \mathbb{C}^k[so(2k)]^{SO(2k)}$ TO OBTAIN

$$e_G(E) = [\overline{Pf}(\Omega_G)]$$

EXAMPLE : EQUIVARIANT EULER CLASS OF THE S^1-EQUIVARIANT VECTOR BUNDLE

$$\begin{array}{ccc} \overbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}^{k} & = & E \\ \Big\downarrow \pi_E & & \\ \{p\} & = & X \end{array}$$

WHERE THE S^1-ACTION ON E IS GIVEN BY

$$e^{ti} \cdot (z_1, \dots, z_{r_p}) = (e^{n_1 ti} z_1, \dots, e^{n_k ti} z_{r_p})$$

FOR SOME SEQUENCE

$$0 < n_1 \leq n_2 \leq \cdots \leq n_k$$

OF POSITIVE INTEGERS AND THE S^1-ACTION ON $\{p\}$ IS THE ONLY POSSIBLE ONE ($e^{ti} \cdot p = p$).

$$\boxed{e_{S^1}(E) = n_1 \cdots n_k \, y^k}$$

WHERE $\{y\}$ IS THE BASIS FOR $\mathrm{Lie}(S^1)^*$ DUAL TO THE BASIS $\{i\}$ FOR $\mathrm{Lie}(S^1) = i\mathbb{R}$. THUS,

$$\xi = ia \Rightarrow e_{S^1}(E)(\xi) = n_1 \cdots n_k \, a^k .$$

DERIVATION FOR $k = 1$:

$$\begin{array}{ccc} \mathbb{C} & = E & e^{ti} \cdot z = e^{n_1 ti} z \\ \downarrow \pi_E & & \\ \{p\} & = X & e^{ti} \cdot p = p \end{array}$$

$\mathbb{C}$ HAS ITS USUAL ORIENTATION AS $\mathbb{R}^2$.

CHOOSE THE USUAL INNER PRODUCT ON $\mathbb{C} = \mathbb{R}^2$. THIS CAN BE WRITTEN

$$\langle z, w \rangle = \mathrm{Re}(z\bar{w})$$

AND IS INVARIANT UNDER THE S^1-ACTION ON $\mathbb{C}$.

THE CORRESPONDING ORIENTED, ORTHONORMAL FRAME BUNDLE

$$SO(2) \hookrightarrow F_{SO}(E) \xrightarrow{\pi_{SO}} \{p\}$$

HAS A SINGLE FIBER CONSISTING OF ALL ORIENTED, ORTHONORMAL BASES FOR $\mathbb{R}^2$ AND CAN BE IDENTIFIED WITH

$$S^1 \hookrightarrow S^1 \longrightarrow \{p\}$$

$$\xi \cdot e^{ti} = \xi e^{ti}$$

IDENTIFICATIONS :

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2) \quad \leftrightarrow \quad e^{ti} \in S^1$$

$$\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \in so(2) \quad \leftrightarrow \quad \lambda i \in Lie(S^1) = i\mathbb{R}$$

$$\xi \in S^1 \quad \leftrightarrow \quad \text{ORIENTED, ORTHONORMAL BASIS } \{\xi, e^{\pi i/2}\xi\}$$

A CONNECTION ON $S^1 \hookrightarrow S^1 \to \{p\}$ IS AN $i\mathbb{R}$-VALUED 1-FORM ω ON S^1 SUCH THAT

$$(1) \quad R_g^{*}\omega = g^{-1}\omega g \quad (= \omega) \quad \forall g \in S^1$$

$$(2) \quad \omega(A^{\#}) = A \quad \forall A = ai \in i\mathbb{R}$$ (AND $\#$ REFERS TO THE RIGHT S^1-ACTION IN $S^1 \hookrightarrow S^1 \to \{p\}$)

SINCE THE LEFT ACTION OF S^1 ON $F_{SO}(E) = S^1$ IS MULTIPLICATION BY $e^{n_1 t i}$, (1) $\Rightarrow$ ANY CONNECTION ON $F_{SO}(E)$ IS S^1-INVARIANT.

$\dim \Omega^1(S^1, i\mathbb{R}) = 1$ SO ANY CONNECTION ω IS A MULTIPLE OF THE STANDARD (METRIC) VOLUME FORM $d\theta$ ON S^1.

ANY MULTIPLE OF $d\theta$ SATISFIES (1) BECAUSE R_g IS AN ISOMETRY. WE SHOW THAT (2) REQUIRES THAT $\omega = i\,d\theta$:

COMPUTE $d\theta(A^{\#})$ FOR $A = ai \in i\mathbb{R}$: AT ANY $\xi \in S^1$,

$$
\begin{aligned}
(d\theta(A^{\#}))(\xi) &= A^{\#}(\xi)(\theta) \\
&= \frac{d}{dt}\left(\theta(\xi \cdot \exp(tA))\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\theta(\xi e^{iat})\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\theta(e^{i\phi} e^{iat})\right)\Big|_{t=0} \qquad (\xi = e^{i\phi}) \\
&= \frac{d}{dt}\left(\theta(e^{i(\phi + at)})\right)\Big|_{t=0} \\
&= \frac{d}{dt}(\phi + at)\Big|_{t=0} \\
&= a \qquad (\text{INDEPENDENT OF } \xi)
\end{aligned}
$$

THUS,

$$(i\,d\theta)(A^{\#}) = ia = A$$

WE CAN SATISFY (2) ONLY IF

$$\omega = i\,d\theta$$

WHICH IS OUR (S^1-INVARIANT) CONNECTION ON THE FRAME BUNDLE.

FOR THE CURVATURE WE HAVE

$$\Omega = 0$$

(CAREFUL : THIS DOES <u>NOT</u> FOLLOW FROM $d^2 = 0$!)

THUS,

$$\Omega_{S^1} = \Omega \cdot \mathcal{J}_\omega = -\mathcal{J}_\omega = -\mathcal{J}_{id\theta} .$$

THE IDENTIFICATION $\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \in so(2) \leftrightarrow \lambda i \in i\mathbb{R}$ GIVES

$Pf(\lambda i) = \lambda = -i(\lambda i)$ SO

$$Pf(\Omega_{S^1}) = -i\,\Omega_{S^1} = i\,\mathcal{J}_{id\theta}$$

FOR ANY $\xi = ia \in Lie(S^1)$,

$$Pf(\Omega_{S^1})(\xi) = i\,\mathcal{J}_{id\theta}(\xi) = i\,(id\theta)(\xi^{\#}) = -\xi^{\#}(\theta)$$

($\#$ NOW REFERS TO THE LEFT ACTION OF S^1 ON $F_{SO}(E) = S^1$ INDUCED BY THE S^1-ACTION ON E).

FOR ANY $\mathfrak{f} \in F_{SO}(E)$,

$$\begin{aligned} \xi^{\#}(\mathfrak{f}) &= \frac{d}{dt}(\exp(-t\xi)\cdot \mathfrak{f})\Big|_{t=0} \\ &= \frac{d}{dt}(e^{-tai}\cdot \mathfrak{f})\Big|_{t=0} \\ &= \frac{d}{dt}(e^{-tn_1 ai}\,\mathfrak{f})\Big|_{t=0} \end{aligned}$$

SO

$$\begin{aligned} (Pf(\Omega_{S^1})(\xi))(\mathfrak{f}) &= -\xi^{\#}(\theta)(\mathfrak{f}) \\ &= -\xi^{\#}(\mathfrak{f})(\theta) \end{aligned}$$

$$(Pf(\Omega_{S^1})(\xi))(\xi) = -\frac{d}{dt}(\theta(e^{-tn_1 a i}\xi))\Big|_{t=0}$$

$$= -\frac{d}{dt}(\theta(e^{(\phi - tn_1 a)i}))\Big|_{t=0}$$

$$(\xi = e^{\phi i})$$

$$= -\frac{d}{dt}(\phi - tn_1 a)\Big|_{t=0}$$

$$= n_1 a$$

$$= n_1 y(\xi) \qquad \text{(INDEPENDENT OF } \xi \text{)}$$

WHERE $\{y\}$ IS THE BASIS FOR $\mathrm{Lie}(S')^*$ DUAL TO THE BASIS $\{i\}$ FOR $\mathrm{Lie}(S')$.

$$Pf(\Omega_{S^1}) = n_1 y$$

SINCE THIS HAS NO FORM PARTS, $\overline{Pf}(\Omega_{S^1}) = n_1 y$ ALSO. SINCE THE BASE SPACE $\{p\}$ IS 0-DIMENSIONAL, $d_{S^1} = 0$ SO THIS IS ALSO A COHOMOLOGY CLASS. THUS,

$$\begin{array}{lll} E = \mathbb{C} & e^{ti}\cdot z = e^{n_1 ti} z & \\ \big\downarrow \pi_E & & e_{S^1}(E) = n_1 y \\ X = \{p\} & e^{ti}\cdot p = p & \end{array}$$

THE GENERAL CASE INVOLVES A BIT MORE WORK (DUE TO THE FACT THAT THE STRUCTURE GROUP OF THE FRAME BUNDLE IS NOT $SO(2) \cong S^1$), BUT ESSENTIALLY THE SAME IDEAS.

RETURN TO THE SITUATION OF INTEREST IN DISCRETE S^1-LOCALIZATION:

$$N_p(Z(\xi^\#)) = N_{p,1}(Z(\xi^\#)) \oplus \cdots \oplus N_{p,\ell_p}(Z(\xi^\#))$$

$$\cong (\mathbb{C} \oplus \overset{r_{p,1}}{\cdots} \oplus \mathbb{C}) \oplus \cdots \oplus (\mathbb{C} \oplus \overset{r_{p,\ell_p}}{\cdots} \oplus \mathbb{C})$$

$$\begin{array}{cccccccc} \uparrow & & \uparrow & & \uparrow & & \uparrow \\ e^{m_{p,1}ti} & \cdots & e^{m_{p,1}ti} & \cdots & e^{m_{p,\ell_p}ti} & \cdots & e^{m_{p,\ell_p}ti} \\ e^{n_{p,1}ti} & & & \cdots & & & e^{n_{p,k}ti} \end{array}$$

$$\begin{array}{c} N_p(Z(\xi^\#)) \\ \downarrow \\ \{p\} \end{array}$$

$$e_{S^1}(N_p(Z(\xi^\#))) = n_{p,1} \cdots n_{p,k}\, y^k$$

$$\xi = ia \Rightarrow$$

$$\boxed{e_{S^1}(N_p(Z(\xi^\#)))(\xi) = n_{p,1} \cdots n_{p,k}\, a^k}$$

ALSO RECALL THAT

$$L_p(\xi) = L_p(ia) = a\,L_p(i)$$

SO

$$Pf(L_p(\xi)) = Pf(L_p(i))\,a^{k} .$$

BUT $L_p(i) \,\big|\, N_{p,j}(Z(\xi''))= m_{p,j}\, J_{p,j}$ SO THE MATRIX OF $L_p(i)$ RELATIVE TO THE BASIS GIVING THE DECOMPOSITION ABOVE CONSISTS OF DIAGONAL BLOCKS OF THE FORM

$$\begin{pmatrix} 0 & m_{p,j} & \cdots & 0 & 0 \\ -m_{p,j} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & m_{p,j} \\ 0 & 0 & \cdots & -m_{p,j} & 0 \end{pmatrix}$$

SO $Pf(L_p(i)) = n_{p,1} \cdots n_{p,k}$. THUS,

$$Pf(L_p(\xi)) = e_{S'}(N_p(Z(\xi'')))(\xi)$$

AND WE HAVE ACCOMPLISHED OUR OBJECTIVE OF SHOWING THAT (1) CAN BE WRITTEN IN THE FORM (2).