DUISTERMAAT - HECKMAN THEOREM
AND
EXACT STATIONARY PHASE APPROXIMATION I

M : COMPACT, ORIENTED, SMOOTH MANIFOLD OF DIMENSION $n = 2k$

dV : VOLUME (ORIENTATION) FORM ON M

H : MORSE FUNCTION ON M

I.E., A SMOOTH, REAL-VALUED FUNCTION ON M WHOSE CRITICAL POINTS ($dH(p) = 0$) ARE ALL NONDEGENERATE (HESSIAN $\mathcal{H}_H(p) : T_p(M) \times T_p(M) \longrightarrow \mathbb{R}$ DEFINED BY

$$\mathcal{H}_H(p)(v_p, w_p) = v_p(W(H))$$

IS A NONDEGENERATE BILINEAR FORM)

T : POSITIVE REAL PARAMETER

WE WILL CONSIDER INTEGRALS OF THE FORM

$$\int_M e^{iTH} dV$$

AND ESPECIALLY THEIR ASYMPTOTIC BEHAVIOR AS $T \to \infty$

<u>EXAMPLE</u> : $M = S^2$, dV = STANDARD METRIC VOLUME (AREA) FORM, H = " HEIGHT FUNCTION " ($H(x,y,z) = z$)

$$\int_{S^2} e^{iTz} dV$$

" STATIONARY PHASE APPROXIMATION " ASSERTS ROUGHLY THAT, FOR LARGE T, THE DOMINANT CONTRIBUTIONS TO SUCH AN INTEGRAL COME FROM THE CRITICAL POINTS OF H. MORE PRECISELY,

$$\int_M e^{iTH}\,dV = \sum_{dH(p)=0} \left(\frac{2\pi}{T}\right)^k \frac{e^{\pi i\,(\operatorname{sgn}\mathcal{H}_H(p))/4}\; e^{iTH(p)}}{\sqrt{|\det \mathcal{H}_H(p)(e_i,e_j)|}} + O(T^{-(k+1)})$$

WHERE $\operatorname{SGN}\mathcal{H}_H(p)$ IS THE SIGNATURE OF $\mathcal{H}_H(p)$ AND $\{e_1,\dots,e_{2k}\}$ IS A BASIS FOR $T_p(M)$ WITH $dV_p(e_1,\dots,e_{2k}) = 1$.

THE SUM IS THE <u>STATIONARY PHASE APPROXIMATION</u> TO THE INTEGRAL.

THE TERMS IN THE SUM ARISE FROM WRITING H NEAR p AS A QUADRATIC FUNCTION (<u>MORSE LEMMA</u>) AND COMPUTING DIRECTLY THE RESULTING <u>GAUSSIAN INTEGRAL</u> (SEE CHAPTER I OF [12]).

<u>EXAMPLE</u> : $\int_{S^2} e^{iTz}\,dV$

CRITICAL POINTS OF $H(x,y,z) = z$ ON S^2 ARE $N = (0,0,1)$ AND $S = (0,0,-1)$.

$\mathcal{H}_H(N) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ AND $\mathcal{H}_H(S) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. SUBSTITUTING INTO THE SUM ABOVE GIVES

$$\left(\frac{2\pi}{T}\right)^1 \frac{e^{-\pi i/2}\, e^{iT}}{1} + \left(\frac{2\pi}{T}\right)^1 \frac{e^{\pi i/2}\, e^{-iT}}{1} = 4\pi\left(\frac{\sin T}{T}\right)$$

THE INTEGRAL IN THIS LAST EXAMPLE IS ACTUALLY EASY TO COMPUTE EXACTLY. IN SPHERICAL COORDINATES,

$$\int_{S^2} e^{iTz}\,dV = \int_{S^2} e^{iT\cos\phi}\sin\phi\, d\phi \wedge d\theta = 4\pi\left(\frac{\sin T}{T}\right)$$

FOR THE HEIGHT FUNCTION ON S^2, THE STATIONARY PHASE APPROXIMATION IS EXACT.

THIS IS <u>NOT</u> THE CASE FOR THE HEIGHT FUNCTION ON THE TORUS (OR ANY OTHER COMPACT SURFACE OF POSITIVE GENUS).

IT IS ALSO <u>NOT</u> THE CASE FOR ALL MORSE FUNCTIONS ON S^2.

SOME OBSERVATIONS ON THIS EXAMPLE :

ANY VOLUME FORM ON AN ORIENTABLE SURFACE IS ALSO A <u>SYMPLECTIC FORM</u> (CLOSED, NONDEGENERATE 2-FORM)

WE WILL DENOTE THE FORM σ
WHEN THINKING OF IT AS A SYMPLECTIC
FORM.

THE HEIGHT FUNCTION H ON (S^2, σ) THEREFORE DETERMINES A CORRESPONDING <u>HAMILTONIAN VECTOR FIELD</u> V_H , CHARACTERIZED BY

$$dH = \iota_{V_H}\sigma$$

I.E.,

$$dH(W) = \sigma(V_H, W)$$

FOR ALL VECTOR FIELDS W.

FOR THE HEIGHT FUNCTION H ON (S^2, σ),

$$V_H = \frac{\partial}{\partial\theta} \qquad \text{(TAKEN TO BE ZERO AT } N \text{ AND } S \text{)}$$

THE INTEGRAL CURVES ARE POINTS AT N AND S AND OTHERWISE " HORIZONTAL " CIRCLES TRAVERSED ONCE ON $[0, 2\pi]$.

V_H HAS A PERIODIC FLOW AND THIS, ACCORDING TO DUISTERMAAT AND HECKMAN, IS " THE SECRET ".

<u>DUISTERMAAT</u> - <u>HECKMAN</u> <u>THEOREM</u> : LET M BE A COMPACT MANIFOLD OF DIMENSION $n = 2k$ WITH SYMPLECTIC FORM σ AND ORIENTED BY THE CORRESPONDING <u>LIOUVILLE</u> <u>FORM</u> $dV_\sigma = \frac{1}{k!}\,\sigma \wedge \overset{k}{\cdots} \wedge \sigma$. LET $H \in C^\infty(M)$ BE A MORSE FUNCTION FOR WHICH THE HAMILTONIAN VECTOR FIELD V_H HAS A PERIODIC FLOW. THEN, FOR ANY $T > 0$,

$$\int_M e^{iTH}\, dV_\sigma = \sum_{dH(p)=0} \left(\frac{2\pi}{T}\right)^k \frac{e^{\pi i(\mathrm{SGN}\,\mathcal{H}_H(p))/4}\, e^{iTH(p)}}{\sqrt{|\det \mathcal{H}_H(p)(e_i, e_j)|}}$$

WHERE $\{e_1, \ldots, e_{2k}\}$ IS A BASIS FOR $T_p(M)$ WITH $(dV_\sigma)_p(e_1, \ldots, e_{2k}) = 1$.