DUISTERMAAT - HECKMAN THEOREM AND EXACT STATIONARY PHASE APPROXIMATION II

M = COMPACT, ORIENTED, SMOOTH MANIFOLD OF DIMENSION $n = 2k$

dV = VOLUME (I.E., ORIENTATION) FORM ON M

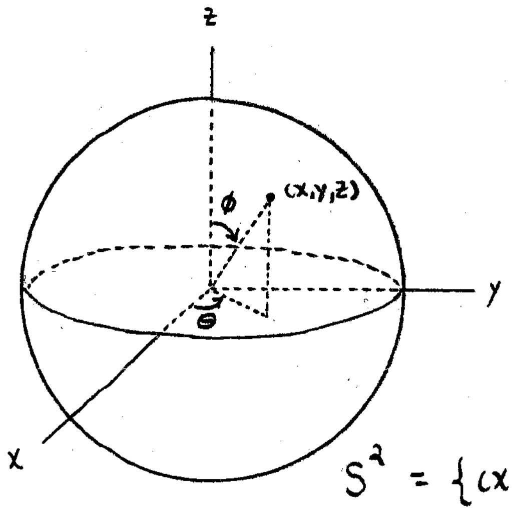
$H : M \longrightarrow \mathbb{R}$ A MORSE FUNCTION ON M, I.E., A SMOOTH FUNCTION WHOSE CRITICAL POINTS ($dH(p) = 0$) ARE ALL NONDEGENERATE (HESSIAN $\mathcal{H}_H(p) : T_p(M) \times T_p(M) \longrightarrow \mathbb{R}$ DEFINED BY $\mathcal{H}_H(p)(V_p, W_p) = V_p(W(H))$ IS A NONDEGENERATE BILINEAR FORM)

T = REAL PARAMETER

WE WILL CONSIDER INTEGRALS OF THE FORM
$$\int_M e^{iTH}\, dV$$
AND ESPECIALLY THEIR ASYMPTOTIC BEHAVIOR AS $T \longrightarrow \infty$.

<u>EXAMPLE</u> : $M = S^2$, dV = METRIC VOLUME FORM FOR THE USUAL RIEMANNIAN METRIC ON S^2

H = "HEIGHT FUNCTION" ON S^2 ($H(x,y,z) = z$)



$$\int_{S^2} e^{iTz}\, dV = \int_{S^2} e^{iT\cos\phi} \sin\phi\, d\phi \wedge d\theta$$

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

STATIONARY PHASE APPROXIMATION ASSERTS ROUGHLY THAT, FOR LARGE T, THE DOMINANT CONTRIBUTIONS TO SUCH AN INTEGRAL COME FROM THE CRITICAL POINTS OF H.

MORE PRECISELY,

$$\int_M e^{iTH}\,dV = \boxed{\sum_{\substack{p\in M\\ dH(p)=0}} \left(\frac{2\pi}{T}\right)^{k} e^{\pi i\,(\mathrm{SGN}\,\mathcal{H}_H(p))/4}\,\left|\det \mathcal{H}_H(p)(e_i,e_j)\right|^{-\frac{1}{2}} e^{iTH(p)}} + O(T^{-(k+1)}),$$

WHERE $\mathrm{SGN}\,\mathcal{H}_H(p)$ IS THE SIGNATURE OF $\mathcal{H}_H(p)$ (# POSITIVE EIGENVALUES MINUS # NEGATIVE EIGENVALUES) AND $\{e_1,\ldots,e_{2k}\}$ IS A BASIS FOR $T_p(M)$ WITH $dV_p(e_1,\ldots,e_{2k}) = 1$.

THE $\boxed{\text{SUM}}$ IS THE <u>STATIONARY</u> <u>PHASE</u> <u>APPROXIMATION</u> OF THE INTEGRAL.

THE TERMS IN THE SUM ARISE FROM WRITING H NEAR p AS A QUADRATIC FUNCTION (MORSE LEMMA) AND COMPUTING DIRECTLY THE RESULTING GAUSSIAN INTEGRAL (FOR A DISCUSSION AND PROOF SEE CHAPTER I OF [12]).

<u>EXAMPLE</u> : $\displaystyle\int_{S^2} e^{iTz}\,dV$

THE CRITICAL POINTS OF $H(z) = z$ ON S^2 ARE $N = (0,0,1)$ AND $S = (0,0,-1)$ FOR EXAMPLE, ON $z > 0$, $(x,y,z) \longrightarrow (x,y)$ IS A CHART WITH

INVERSE $(x,y) \longrightarrow (x,y,\sqrt{1-x^2-y^2})$. IN THESE COORDINATES,

$$H(x,y) = \sqrt{1-x^2-y^2}$$

$$dH(x,y) = -(1-x^2-y^2)^{-\frac{1}{2}}(x\,dx + y\,dy)$$

SO THE ONLY CRITICAL POINT OCCURS WHEN $(x,y) = (0,0)$, I.E., AT $N = (0,0,1)$.

IN THESE COORDINATES THE HESSIAN AT ANY (x,y) IS

$$\begin{pmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial y\,\partial x} \\ \frac{\partial^2 H}{\partial x\,\partial y} & \frac{\partial^2 H}{\partial y^2} \end{pmatrix} = -(1-x^2-y^2)^{-3/2}\begin{pmatrix} 1-y^2 & xy \\ xy & 1-x^2 \end{pmatrix}$$

WHICH, AT $(x,y) = (0,0)$ IS $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ SO

$$\text{SGN}\,\mathcal{H}_H(N) = 0 - 2 = -2.$$

NEXT NOTE THAT dV IS THE RESTRICTION TO S^2 OF

$$x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy$$

AND EVALUATING THIS AT (∂_x, ∂_y) GIVES z. THUS, AT N,

$\omega_N(\partial_x(N), \partial_y(N)) = 1$ SO $\{e_1, e_2\} = \{\partial_x(N), \partial_y(N)\}$ IS A BASIS FOR $T_N(S^2)$ OF THE REQUIRED TYPE AND

$$|\det \mathcal{H}_H(N)(e_i, e_j)|^{-\frac{1}{2}} = \left|\det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right|^{-\frac{1}{2}} = 1$$

SIMILARLY,

$$\text{SGN}\,\mathcal{H}_H(S) = 2$$

$$|\det \mathcal{H}_H(S)(e_i, e_j)|^{-\frac{1}{2}} = \left|\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right|^{-\frac{1}{2}} = 1$$

THE STATIONARY PHASE APPROXIMATION OF $\int_{S^2} e^{iTz}\,dV$ IS THEREFORE

$$\sum_{\substack{p\in S^2\\ dH(p)=0}} \left(\frac{2\pi}{T}\right)^1 e^{\pi i(\operatorname{SGN}\mathcal{H}_H(p))/4} \left|\det \mathcal{H}_H(p)(e_i,e_j)\right|^{-\frac{1}{2}} e^{iTH(p)} =$$

$$\left(\frac{2\pi}{T}\right) e^{\pi i(\operatorname{SGN}\mathcal{H}_H(N))/4} \left|\det \mathcal{H}_H(N)(e_i,e_j)\right|^{-\frac{1}{2}} e^{iTH(N)} +$$

$$\left(\frac{2\pi}{T}\right) e^{\pi i(\operatorname{SGN}\mathcal{H}_H(S))/4} \left|\det \mathcal{H}_H(S)(e_i,e_j)\right|^{-\frac{1}{2}} e^{iTH(S)} =$$

$$\left(\frac{2\pi}{T}\right) e^{-\pi i/2}(1)e^{iT} + \left(\frac{2\pi}{T}\right) e^{\pi i/2}(1)e^{-iT} =$$

$$\left(\frac{2\pi}{T}\right)(-i)e^{iT} + \left(\frac{2\pi}{T}\right)(i)e^{-iT} = \frac{2\pi i}{T}\left(e^{-iT}-e^{iT}\right)$$

$$= \boxed{4\pi\left(\frac{\sin T}{T}\right)}$$

NEXT WE <u>COMPUTE THE INTEGRAL</u> $\int_{S^2} e^{iTz}\,dV$ <u>EXACTLY</u>.

LET $\iota : S^2 \hookrightarrow \mathbb{R}^3$ BE THE INCLUSION MAP. THEN THE STANDARD VOLUME FORM dV ON S^2 IS GIVEN BY

$$dV = \iota^*(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy)$$

DEFINE AN ORIENTATION PRESERVING DIFFEOMORPHISM $\mathcal{C}$ OF $(0,\pi)\times(-\pi,\pi)$ INTO S^2 BY

$$(\iota \circ \varphi)(\phi, \theta) = (\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi).$$

THE IMAGE OF THIS MAP COVERS ALL OF S^2 EXCEPT A SET OF MEASURE ZERO (THE SEMICIRCLE $\{(-\sin\phi,\ 0,\ \cos\phi) : 0 \le \phi \le \pi\}$ ON S^2).

A COMPUTATION SHOWS THAT

$$\begin{aligned}\varphi^*(dV) &= \varphi^*(\iota^*(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy)) \\ &= (\iota\circ\varphi)^*(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy) \\ &= \sin\phi\ d\phi\wedge d\theta\end{aligned}$$

(E.G., $(\iota\circ\varphi)^*(x\,dy\wedge dz) = (\sin\phi\cos\theta)\,[d(\sin\phi\sin\theta)\wedge d(\cos\phi)] =$ $(\sin\phi\cos\theta)\,[(\cos\phi\sin\theta\,d\phi + \sin\phi\cos\theta\,d\theta)\wedge(-\sin\phi\,d\phi)] =$ $(\sin\phi\cos\theta)\,[\quad 0 \quad - \sin^2\phi\cos\theta\,d\theta\wedge d\phi] = \sin^3\phi\cos^2\theta\,d\phi\wedge d\theta$, ETC.

THUS,

$$\varphi^*(e^{iTz}dV) = e^{iT\cos\phi}\sin\phi\,d\phi\wedge d\theta$$

AND SO

$$\boxed{\int_{S^2} e^{iTz}dV} = \int_{(0,\pi)\times(-\pi,\pi)} e^{iT\cos\phi}\sin\phi\,d\phi\wedge d\theta$$

$$= \int_{[0,\pi]\times[-\pi,\pi]} e^{iT\cos\phi}\sin\phi\,dm \qquad (dm = \text{LEBESGUE MEASURE})$$

$$= \int_{-\pi}^{\pi}\int_0^{\pi} e^{iT\cos\phi}\sin\phi\,d\phi\,d\theta = -\frac{2\pi}{iT}\left[e^{iT\cos\phi}\right]_0^{\pi}$$

$$= \frac{2\pi i}{T}\left[e^{-iT} - e^{iT}\right] \boxed{= 4\pi\left(\frac{\sin T}{T}\right)}$$

THUS, IN THIS CASE THE STATIONARY PHASE APPROXIMATION GIVES THE EXACT VALUE OF THE INTEGRAL. THE QUESTION IS "WHY?"

<u>SOME OBSERVATIONS ABOUT THIS EXAMPLE</u>:

THE VOLUME FORM dV ON S^2 IS ALSO A <u>SYMPLECTIC FORM</u>, I.E., A CLOSED, NONDEGENERATE 2-FORM (ANY VOLUME FORM dV ON ANY ORIENTABLE SURFACE IS A SYMPLECTIC FORM: CLOSED BECAUSE IT'S A 2-FORM ON A 2-MANIFOLD. NONDEGENERATE BECAUSE, AT EACH POINT, AN ORIENTED BASIS $\{e_1, e_2\}$ SATISFIES $dV(e_1, e_2) > 0$ SO, IF $v = v^1 e_1 + v^2 e_2 \neq 0$ (SAY, $v^1 \neq 0$), THEN $dV(v, e_2) = v^1 dV(e_1, e_2) \neq 0$.)

> <u>NOTE</u>: WE WILL DENOTE THE FORM σ WHEN THINKING OF IT AS A SYMPLECTIC FORM.

THE HEIGHT FUNCTION H (LIKE ANY SMOOTH, REAL-VALUED FUNCTION ON (S^2, σ)) DETERMINES A CORRESPONDING <u>HAMILTONIAN VECTOR FIELD</u> V_H: THE UNIQUE VECTOR FIELD ON S^2 SATISFYING

$$dH = \iota_{V_H} \sigma \qquad (= \sigma(V_H, \cdot))$$

I.E.,

$$dH(w) = \sigma(V_H, w) \qquad \forall\, w \in T(TS^2)$$

<u>CLAIM</u>: $\quad V_H = \partial_\theta \quad$ (TAKEN TO BE ZERO AT N AND S)

PROOF : IN SPHERICAL COORDINATES, $H(\phi, \theta) = \cos\phi$ SO

$$dH = -\sin\phi \, d\phi$$

AND $\sigma = \sin\phi \, d\phi \wedge d\theta$. NOW,

$$\begin{aligned}
\iota_{\partial_\theta} \sigma &= \iota_{\partial_\theta} (\sin\phi \, d\phi \otimes d\theta - \sin\phi \, d\theta \otimes d\phi) \\
&= \sin\phi \, (d\phi(\partial_\theta)) d\theta - \sin\phi \, (d\theta(\partial_\theta)) d\phi \\
&= -\sin\phi \, d\phi \\
&= dH .
\end{aligned}$$

$\square$

THE <u>INTEGRAL CURVES OF</u> $V_H = \partial_\theta$ ARE EASILY FOUND ("HORIZONTAL" CIRCLES TRAVERSED ONCE ON $[0, 2\pi]$). THE UNIQUE ONE THROUGH $p = \mathcal{C}(\phi, \theta)$ AT $t = 0$ IS

$$\alpha_p(t) = (\sin\phi \cos(\theta + t), \sin\phi \sin(\theta + t), \cos\phi)$$

AND IS PERIODIC WITH PERIOD 2π. THE FLOW

$$\alpha : S^2 \times \mathbb{R} \longrightarrow S^2$$

$$\alpha(p, t) = \alpha_p(t)$$

IS THEREFORE ALSO PERIODIC IN t (ALL NONTRIVIAL ORBITS HAVE THE SAME MINIMAL PERIOD).

FINALLY, RECALL THAT ANY SYMPLECTIC MANIFOLD (M^{2k}, σ) HAS A CANONICAL ORIENTATION (VOLUME FORM) CALLED THE <u>LIOUVILLE FORM</u>

$$dV_\sigma = \frac{1}{k!}\sigma^k = \frac{1}{k!}\,\sigma\wedge\overset{k}{\cdots}\wedge\sigma .$$

FOR $k=1$ THIS IS JUST σ SO, IN THE CASE OF S^2, dV, σ AND dV_σ ARE ALL THE SAME.

<u>DUISTERMAAT - HECKMAN THEOREM</u> : LET M BE A COMPACT MANIFOLD OF DIMENSION $n = 2k$ WITH SYMPLECTIC FORM σ AND ORIENTED BY THE CORRESPONDING LIOUVILLE FORM $dV_\sigma = \frac{1}{k!}\sigma^k$. LET $H \in C^\infty(M)$ BE A MORSE FUNCTION AND V_H ITS HAMILTONIAN VECTOR FIELD ($dH = \iota_{V_H}\sigma$). ASSUME THAT THE FLOW OF V_H IS PERIODIC. THEN

$$\int_M e^{iTH}dV_\sigma = \sum_{\substack{p\in M\\ dH(p)=0}} \left(\frac{2\pi}{T}\right)^k e^{\pi i\,(\mathrm{SGN}\,\mathcal{H}_H(p))/4}\,\left|\det(\mathcal{H}_H(p)(e_i,e_j))\right|^{-\frac{1}{2}} e^{iTH(p)}$$

FOR ANY REAL $T>0$, WHERE $\mathcal{H}_H(p) : T_p(M)\times T_p(M) \to \mathbb{R}$ IS THE HESSIAN OF H AT p AND $\{e_1,\dots,e_{2k}\}$ IS A BASIS FOR $T_p(M)$ WITH $(dV_\sigma)_p(e_1,\dots,e_{2k}) = 1$.

ALTHOUGH NOT APPARENT AT FIRST SIGHT, THIS IS REALLY A STATEMENT ("LOCALIZATION THEOREM") ABOUT INTEGRALS OF "EQUIVARIANT DIFFERENTIAL FORMS" (WHICH WE WILL INTRODUCE SHORTLY).

WE WILL NOW DESCRIBE A <u>GROUP</u> <u>ACTION</u> UNDERLYING OUR EXAMPLE ON S^2 AND USE IT TO MOTIVATE A "GENERALIZED DUISTERMAAT - HECKMAN THEOREM" FROM WHICH THE ONE ABOVE FOLLOWS.