EQUIVARIANT COHOMOLOGY AND LOCALIZATION I

RECALL: ANY SMOOTH MANIFOLD M HAS A DE RHAM COMPLEX $\Omega^*(M)$ (ASSUME COMPLEX COEFFICIENTS):

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M)$$

$$d \circ d = 0$$

EXACT p-FORMS $(d(\Omega^{p-1}(M)))$ $\subseteq$ CLOSED p-FORMS $(\ker d^p)$

DE RHAM COHOMOLOGY $H^*(M)$:

$$H^p(M) = \text{CLOSED p-FORMS} / \text{EXACT p-FORMS}$$

NOW SUPPOSE THERE IS A G-ACTION ON M.

FREE ACTION $\Rightarrow$ ORBIT SPACE M/G IS A MANIFOLD AND SO HAS A DE RHAM COHOMOLOGY $H^*(M/G)$

CARTAN SHOWED HOW TO CALCULATE $H^*(M/G)$ FROM A COMPLEX <u>ON</u> <u>M</u> AND SO PROVIDED A REASONABLE ALTERNATIVE TO THE DE RHAM COHOMOLOGY OF M/G EVEN WHEN THE ACTION IS NOT FREE SO THAT M/G IS GENERALLY NOT A MANIFOLD.

$\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ = ALGEBRA OF $\Omega^*(M)$-VALUED POLYNOMIALS ON $\mathfrak{g}$

= SUMS OF TERMS OF THE FORM

$$\alpha = \rho \otimes \varphi$$

$$\alpha(\xi) = (\rho \otimes \varphi)(\xi) = \rho(\xi)\varphi$$

GRADING :

$$\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M) = \bigoplus_{2j+i=k} \mathbb{C}^j[\mathfrak{g}] \otimes \Omega^i(M)$$

$$\deg(\rho \otimes \varphi) = 2\deg\rho + \deg\varphi$$

INDUCED G-ACTION :

$$(g\cdot\alpha)(\xi) = (g\cdot(\rho\otimes\varphi))(\xi) = \rho(g^{-1}\xi g)L_{g^{-1}}^*\varphi$$

G-INVARIANT ELEMENTS :

$$g\cdot\alpha = \alpha$$

$$\rho(g^{-1}\xi g)L_{g^{-1}}^*\varphi = \rho(\xi)\varphi$$

$$\alpha(g\xi g^{-1}) = L_{g^{-1}}^*(\alpha(\xi))$$

$\Omega_G^*(M)$ = SUBALGEBRA OF G-INVARIANT ELEMENTS OF $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$

= <u>G</u>-<u>EQUIVARIANT</u> <u>DIFFERENTIAL</u> <u>FORMS</u> ON M

G-EQUIVARIANT EXTERIOR DERIVATIVE :

$$d_G : \Omega_G^*(M) \to \Omega_G^*(M)$$

$$(d_G \alpha)(\xi) = d(\alpha(\xi)) - \iota_{\xi^*}(\alpha(\xi)) = (d - \iota_{\xi^*})(\alpha(\xi))$$

MOTIVATION :

$$d\mu(\xi) = \iota_{\xi^*}\sigma \quad \forall \xi \in \mathfrak{g} \qquad \Leftrightarrow \qquad d_G(\mu+\sigma) = 0$$

$(\Omega_G^*(M), d_G)$ IS A COCHAIN COMPLEX $(d_G \circ d_G = 0)$

AND ITS COHOMOLOGY

$$H_G^*(M)$$

IS (THE CARTAN MODEL OF THE) <u>G-EQUIVARIANT</u>

<u>COHOMOLOGY</u> OF M.

<u>EXAMPLE</u> : $M = S^3 = \{ (z^1, z^2) \in \mathbb{C}^2 : |z^1|^2 + |z^2|^2 = 1 \}$

$$G = S^1 = \{ g \in \mathbb{C} : |g| = 1 \}$$

$$g \cdot (z^1, z^2) = (g z^1, g z^2)$$

$$H^0_{S^1}(S^3) \cong \mathbb{C} \ , \quad H^1_{S^1}(S^3) \cong 0 \ , \quad H^2_{S^1}(S^3) \cong \mathbb{C}$$

$$H^k_{S^1}(S^3) \cong 0 \ , \quad k > 2$$

<u>CARTAN'S THEOREM</u> : IF G IS CONNECTED AND THE ACTION OF G ON M IS FREE, THEN $H_G^*(M)$ IS ISOMORPHIC TO $H^*(M/G)$

<u>EXAMPLE</u> : $H^*_{S^1}(S^3) \cong H^*(S^2)$

<u>NOTE</u> : IF THE ACTION IS ONLY LOCALLY FREE (ALL ISOTROPY GROUPS ARE FINITE), THEN CARTAN'S THEOREM IS TRUE FOR THE ORBIFOLD VERSION OF DE RHAM COHOMOLOGY.

INTEGRATION OF EQUIVARIANT DIFFERENTIAL FORMS :

FOR THIS WE ASSUME THAT THE G-ACTION IS ORIENTATION PRESERVING (EACH L_g IS AN ORIENTATION PRESERVING DIFFEOMORPHISM)

FOR $\alpha \in \Omega_G^*(M)$ WRITE $\alpha = \alpha_{[0]} + \alpha_{[1]} + \cdots + \alpha_{[n]}$, WHERE $\alpha_{[i]}$ HAS DEGREE i IN $\Omega_G^*(M)$.

$$\int : \Omega_G^*(M) \to \mathbb{C}[\mathfrak{g}]^G$$

$$\left(\int \alpha\right)(\xi) = \int_M \alpha(\xi) := \int_M \alpha(\xi)_{[n]}$$

α EQUIVARIANTLY EXACT $(\alpha = d_G \beta)$ $\Rightarrow$

$$\alpha(\xi)_{[n]} = d\left(\beta(\xi)_{[n-1]}\right)$$

SINCE $\iota_{\xi^\#}$ CAN YIELD NO TOP RANK PART SO STOKES' THEOREM IMPLIES THAT $\int : \Omega_G^*(M) \to \mathbb{C}[\mathfrak{g}]^G$ DESCENDS TO COHOMOLOGY

$$\int : H_G^*(M) \to \mathbb{C}[\mathfrak{g}]^G$$

<u>EQUIVARIANT</u> <u>LOCALIZATION</u> <u>THEOREM</u> (<u>ISOLATED</u> <u>ZEROS</u>): LET M BE A COMPACT, ORIENTED, SMOOTH n-MANIFOLD AND G A COMPACT LIE GROUP ACTING ON M ON THE LEFT BY ORIENTATION PRESERVING DIFFEOMORPHISMS. LET $\xi \in \mathfrak{g}$ BE SUCH THAT $\xi^\#$ HAS ONLY ISOLATED ZEROS. THEN $n = 2k$ AND, FOR ANY G-EQUIVARIANTLY CLOSED FORM α ON M,

$$\int_M \alpha(\xi) = \sum_{p \in Z(\xi^\#)} (-2\pi)^k \frac{\alpha(\xi)_{[0]}(p)}{\mathrm{Pf}(L_p(\xi))}$$