EQUIVARIANT LOCALIZATION ⇒ GENERALIZED DUISTERMAAT-HECKMAN II

EQUIVARIANT LOCALIZATION THEOREM :

M = COMPACT, ORIENTED, SMOOTH $2k$-MANIFOLD

G = COMPACT LIE GROUP (LIE ALGEBRA $\mathfrak{g}$)

ACTING ON M ON THE LEFT & ORIENTATION PRESERVING

$\xi \in \mathfrak{g}$ SUCH THAT $Z(\xi^{\#})$ IS DISCRETE

$\alpha : \mathfrak{g} \longrightarrow \Omega^*(M)$ SUCH THAT $d_{\xi^{\#}}(\alpha(\xi)) = 0$ AND $d_{\xi^{\#}}(\alpha(\xi)) = 0$

$$\int_M \alpha(\xi) = \sum_{p \in Z(\xi^{\#})} (-2\pi)^k \frac{\alpha(\xi)_{[0]}(p)}{Pf(L_p(\xi))}$$

GENERALIZED DUISTERMAAT-HECKMAN THEOREM :

M : COMPACT, SMOOTH $2k$-MANIFOLD

σ : SYMPLECTIC FORM ON M WITH LIOUVILLE FORM

$$dV_\sigma = \frac{1}{k!} \sigma \wedge \overset{k}{\cdots} \wedge \sigma$$

(M IS ORIENTED BY dV_σ)

G : COMPACT LIE GROUP (LIE ALGEBRA $\mathfrak{g}$).

THERE IS A HAMILTONIAN ACTION OF G ON M

WITH SYMPLECTIC MOMENTS GIVEN BY

$$\mu : \mathfrak{g} \rightarrow C^\infty(M)$$

$\xi \in \mathfrak{g}$ SUCH THAT $Z(\xi^{\#})$ IS DISCRETE

$$\int_M e^{i\mu(\xi)} dV_\sigma = \sum_{p \in Z(\xi^{\#})} (2\pi i)^k \frac{e^{i\mu(\xi)(p)}}{Pf(L_p(\xi))}$$

PROOF THAT EL $\Rightarrow$ GD-H : THE TRICK IS TO FIND THE RIGHT $\alpha(\xi)$ TO WHICH TO APPLY EL.

CONSIDER THE MAP $\mathfrak{g} \to C^\infty(M)$ CALLED THE <u>EQUIVARIANT</u> <u>SYMPLECTIC</u> <u>FORM</u>, DENOTED σ_G AND DEFINED BY

$$\sigma_G = \mu + \sigma$$

$$\sigma_G(\xi) = \mu(\xi) + \sigma \quad \in \Omega^*(M)$$

NOTE THAT

$$\begin{aligned}
\mathcal{L}_{\xi^\#}(\sigma_G(\xi)) &= (d \circ \iota_{\xi^\#} + \iota_{\xi^\#} \circ d)(\mu(\xi) + \sigma) \\
&= d(\underbrace{\iota_{\xi^\#}(\mu(\xi))}_{0}) + \iota_{\xi^\#}(d\mu(\xi)) + \\
&\qquad d(\iota_{\xi^\#}\sigma) + \iota_{\xi^\#}(\underbrace{d\sigma}_{0}) \\
&= \iota_{\xi^\#}(d\mu(\xi)) + d(\iota_{\xi^\#}\sigma) \\
&= \iota_{\xi^\#}(\iota_{\xi^\#}\sigma) + \underbrace{d(d\mu(\xi))}_{0} \\
&= \sigma(\xi^\#, \xi^\#) \\
&= 0 \qquad \text{BY SKEW-SYMMETRY}
\end{aligned}$$

SO $\sigma_G(\xi) \in \Omega^*_{\xi^\#}(M)$ AND

$$\begin{aligned}
d_{\xi^\#}(\sigma_G(\xi)) &= (d - \iota_{\xi^\#})(\mu(\xi) + \sigma) \\
&= d\mu(\xi) + d\sigma - \iota_{\xi^\#}(\mu(\xi)) - \iota_{\xi^\#}\sigma \\
&= \iota_{\xi^\#}\sigma + 0 - 0 - \iota_{\xi^\#}\sigma \\
&= 0
\end{aligned}$$

$\sigma_G(\xi)$ IS, HOWEVER, STILL NOT THE $\alpha(\xi)$ WE ARE AFTER. WE CONSIDER $e^{i\sigma_G(\xi)} \in \Omega^*(M)$ DEFINED BY

$$e^{i\sigma_G(\xi)} = e^{i(\mu(\xi)+\sigma)} = e^{i\mu(\xi)} e^{i\sigma}$$
$$= e^{i\mu(\xi)}\left(1 + i\sigma + \frac{1}{2!} i^2\sigma^2 + \frac{1}{3!} i^3\sigma^3 + \dots + \frac{1}{k!} i^k \sigma^k\right)$$

WHERE, OF COURSE, $\sigma^i = \sigma \wedge \dots \wedge \sigma$. WE CLAIM THAT $e^{i\sigma_G(\xi)}$ SATISFIES

(1) $$d_{\xi^\#} e^{i\sigma_G(\xi)} = 0$$

TO SEE THIS WE COMPUTE

$$d_{\xi^\#} e^{i\sigma_G(\xi)} = d_{\xi^\#}\left(e^{i\mu(\xi)} e^{i\sigma}\right)$$
$$= d_{\xi^\#} e^{i\mu(\xi)} \wedge e^{i\sigma} + e^{i\mu(\xi)} d_{\xi^\#} e^{i\sigma}$$

NOW,

$$d_{\xi^\#} e^{i\mu(\xi)} = de^{i\mu(\xi)} - \iota_{\xi^\#} e^{i\mu(\xi)} = de^{i\mu(\xi)} - 0$$
$$= i e^{i\mu(\xi)} d\mu(\xi)$$

SO

$$d_{\xi^\#} e^{i\mu(\xi)} \wedge e^{i\sigma} = e^{i\mu(\xi)}\left(i d\mu(\xi) + i^2 \sigma \wedge d\mu(\xi) + \frac{1}{2!} i^3 \sigma^2 \wedge d\mu(\xi) + \dots + \frac{1}{(k-1)!} i^k \sigma^{k-1} \wedge d\mu(\xi) + 0\right)$$

MOREOVER,

$$e^{i\mu(\xi)} d_{\xi^\#} e^{i\sigma} = e^{i\mu(\xi)}\left(0 + i d_{\xi^\#}\sigma + \frac{1}{2!} i^2 d_{\xi^\#}\sigma^2 + \frac{1}{3!} i^3 d_{\xi^\#}\sigma^3 + \dots + \frac{1}{k!} i^k d_{\xi^\#}\sigma^k\right)$$

SO IT WILL BE ENOUGH TO SHOW THAT

$d_{\xi*}\sigma = -d\mu(\xi)$, $d_{\xi*}\sigma^2 = -2\sigma\wedge d\mu(\xi)$, $d_{\xi*}\sigma^3 = -3\sigma^2\wedge d\mu(\xi)$, ...,

$d_{\xi*}\sigma^k = -k\sigma^{k-1}\wedge d\mu(\xi)$. BUT

$$d_{\xi*}\sigma = d\sigma - \iota_{\xi*}\sigma = 0 - \iota_{\xi*}\sigma = -d\mu(\xi)$$

$$\begin{aligned} d_{\xi*}\sigma^2 &= (d_{\xi*}\sigma)\wedge\sigma + \sigma\wedge d_{\xi*}\sigma \\ &= (-d\mu(\xi))\wedge\sigma + \sigma\wedge(-d\mu(\xi)) \\ &= -2\sigma\wedge d\mu(\xi) \end{aligned}$$

$$\begin{aligned} d_{\xi*}\sigma^3 &= (d_{\xi*}\sigma)\wedge\sigma^2 + \sigma\wedge d_{\xi*}\sigma^2 \\ &= (-d\mu(\xi))\wedge\sigma^2 + \sigma\wedge(-2\sigma\wedge d\mu(\xi)) \\ &= -\sigma^2\wedge d\mu(\xi) - 2\sigma^2\wedge d\mu(\xi) \\ &= -3\sigma^2\wedge d\mu(\xi) \end{aligned}$$

AND SO ON BY INDUCTION. THIS PROVES (1). SIMILAR CALCULATIONS SHOW THAT

(2) $$d_{\xi*}\, e^{i\sigma_G(\xi)} = 0$$

MOREOVER,

$$(e^{i\sigma_G(\xi)})_{[0]} = e^{i\mu(\xi)}$$

IS CLEAR FROM THE DEFINITION.

NOW WE APPLY THE EQUIVARIANT LOCALIZATION THEOREM TO $\alpha(\xi) = e^{i\sigma_G(\xi)}$.

$$\sum_{p \in Z(\xi^{\#})} (-2\pi)^{k} \frac{e^{i\mu(\xi)(p)}}{Pf(L_p(\xi))} = \int_M e^{i\sigma_G(\xi)}$$

$$= \int_M e^{i\mu(\xi)} e^{i\sigma}$$

$$= \int_M e^{i\mu(\xi)} \left(1 + i\sigma + \cdots + \frac{1}{k!} i^{k} \sigma^{k}\right)$$

$$= \int_M e^{i\mu(\xi)} \left(\frac{1}{k!} i^{k} \sigma^{k}\right)$$

$$= i^{k} \int_M e^{i\mu(\xi)} dV_\sigma$$

SO

$$\int_M e^{i\mu(\xi)} dV_\sigma = \sum_{p \in Z(\xi^{\#})} (2\pi i)^{k} \frac{e^{i\mu(\xi)(p)}}{Pf(L_p(\xi))}$$

AS REQUIRED. ∎