EQUIVARIANT LOCALIZATION $\Rightarrow$ GENERALIZED DUISTERMAAT-HECKMAN I

COMMON ASSUMPTIONS: M IS A COMPACT, SMOOTH MANIFOLD OF DIMENSION $n = 2k$. G IS A COMPACT LIE GROUP (LIE ALGEBRA $\mathfrak{g}$) ACTING ON M ON THE LEFT. $\xi \in \mathfrak{g}$ IS SUCH THAT $\xi^{\#} \in \Gamma(TM)$ HAS A FINITE ZERO SET $Z(\xi^{\#})$.

GENERALIZED DUISTERMAAT-HECKMAN: σ IS A SYMPLECTIC FORM ON M, M IS ORIENTED BY THE LIOUVILLE FORM $dV_\sigma = \frac{1}{k!}\,\sigma \wedge \overset{k}{\cdots} \wedge \sigma$, AND THE G-ACTION IS HAMILTONIAN WITH SYMPLECTIC MOMENTS GIVEN BY $\mu : \mathfrak{g} \to C^\infty(M)$. THEN

$$\int_M e^{i\mu(\xi)}\, dV_\sigma = \sum_{p \in Z(\xi^{\#})} (2\pi i)^k \frac{e^{i\mu(\xi)(p)}}{\mathrm{Pf}(L_p(\xi))}$$

EQUIVARIANT LOCALIZATION: M IS ORIENTED AND THE G-ACTION IS ORIENTATION PRESERVING. $\alpha \in \Omega_G^*(M)$ IS G-EQUIVARIANTLY CLOSED ($d_G \alpha = 0$). THEN

$$\int_M \alpha(\xi) = \sum_{p \in Z(\xi^{\#})} (-2\pi)^k \frac{\alpha(\xi)_{[0]}(p)}{\mathrm{Pf}(L_p(\xi))}$$

IDEALLY, WE WOULD LIKE (GIVEN THE DATA IN GENERALIZED DUISTERMAAT-HECKMAN) TO CONJURE UP AN $\alpha \in \Omega_G^*(M)$ FOR WHICH THE SECOND INTEGRAL REDUCES TO THE FIRST.

WE CAN "ALMOST" DO THIS, BUT NOT QUITE. FORTUNATELY, THE PROOF OF EQUIVARIANT LOCALIZATION (WHICH WE DESCRIBE IN THE NEXT SECTION) DOES NOT USE THE FULL STRENGTH OF THE HYPOTHESES AND ESTABLISHES A RESULT THAT IS ADEQUATE FOR OUR PURPOSES.

AN $\alpha \in \Omega_G^*(M)$ IS AN <u>EQUIVARIANT</u>, <u>POLYNOMIAL</u> MAP

$$\alpha : \mathfrak{g} \longrightarrow \Omega^*(M)$$

AND $d_G \alpha = 0$ MEANS

$$(d - \iota_{\xi^\#})(\alpha(\xi)) = 0$$

FOR <u>EVERY</u> $\xi \in \mathfrak{g}$.

THE INTEGRAL FORMULA IN EQUIVARIANT LOCALIZATION WILL BE PROVED FOR <u>ANY</u> MAP

$$\alpha : \mathfrak{g} \longrightarrow \Omega^*(M)$$

PROVIDED ONLY THAT

$$\mathcal{L}_{\xi^\#} \alpha(\xi) = 0 \qquad \text{AND} \qquad (d - \iota_{\xi^\#})(\alpha(\xi)) = 0$$

FOR THE <u>SPECIFIC</u> $\xi \in \mathfrak{g}$ UNDER CONSIDERATION.

NOW WE PROCEED WITH THE PROOF. ASSUME σ IS A SYMPLECTIC FORM ON M, M IS ORIENTED BY dV_σ AND THE G-ACTION IS HAMILTONIAN WITH SYMPLECTIC MOMENTS $\mu : \mathfrak{g} \to C^\infty(M)$.

DEFINE THE <u>EQUIVARIANT</u> <u>SYMPLECTIC</u> <u>FORM</u>

$$\sigma_G : \mathfrak{g} \to \Omega^*(M)$$

$$\sigma_G = \mu + \sigma$$

$$\sigma_G(\xi) = \mu(\xi) + \sigma \in \Omega^*(M)$$

NOTE THAT

$$\begin{aligned}
\mathcal{L}_{\xi^\#}(\sigma_G(\xi)) &= (d \circ \iota_{\xi^\#} + \iota_{\xi^\#} \circ d)(\sigma_G(\xi)) \\
&= \underbrace{d(\iota_{\xi^\#}(\mu(\xi)))}_{0} + d(\iota_{\xi^\#}\sigma) + \\
&\qquad \iota_{\xi^\#}(d\mu(\xi)) + \iota_{\xi^\#}(\underbrace{d\sigma}_{0}) \\
&= d(d\mu(\xi)) + \iota_{\xi^\#}(\iota_{\xi^\#}\sigma) \\
&= 0 + \sigma(\xi^\#, \xi^\#) \\
&= 0
\end{aligned}$$

AND

$$\begin{aligned}
(d - \iota_{\xi^\#})(\sigma_G(\xi)) &= d\mu(\xi) + d\sigma - \iota_{\xi^\#}(\mu(\xi)) - \iota_{\xi^\#}\sigma \\
&= \iota_{\xi^\#}\sigma + 0 - 0 - \iota_{\xi^\#}\sigma \\
&= 0
\end{aligned}$$

NEVERTHELESS, σ_G IS NOT QUITE THE α WE WANT. DEFINE

$$e^{i\sigma_G} : \mathfrak{g} \longrightarrow \Omega^*(M)$$

$$e^{i\sigma_G(\xi)} = e^{i(\mu(\xi)+\sigma)} := e^{i\mu(\xi)}e^{i\sigma}$$

$$= e^{i\mu(\xi)}\left(1 + i\sigma + \frac{1}{2!}i^2\sigma^2 + \cdots + \frac{1}{k!}i^k\sigma^k\right)$$

WHERE $\sigma^i = \sigma \wedge \overset{i}{\cdots} \wedge \sigma$. AS ABOVE ONE CHECKS THAT

$$\mathcal{L}_{\xi^\#}(e^{i\sigma_G(\xi)}) = 0$$

AND

$$(d - \iota_{\xi^\#})(e^{i\sigma_G(\xi)}) = 0$$

SO WE CAN APPLY EQUIVARIANT LOCALIZATION TO $\alpha(\xi) := e^{i\sigma_G(\xi)}$.

NOTING THAT

$$(e^{i\sigma_G(\xi)})_{[0]} = e^{i\mu(\xi)}$$

WE OBTAIN

$$\sum_{p \in Z(\xi^\#)} (-2\pi)^k \frac{e^{i\mu(\xi)(p)}}{Pf(L_p(\xi))} = \int_M e^{i\sigma_G(\xi)} = \int_M e^{i\mu(\xi)}e^{i\sigma}$$

$$= \int_M e^{i\mu(\xi)}\left(1 + i\sigma + \cdots + \frac{1}{k!}i^k\sigma^k\right)$$

$$= \int_M e^{i\mu(\xi)}\left(\frac{1}{k!}i^k\sigma^k\right)$$

$$= i^k \int_M e^{i\mu(\xi)}\,dV_\sigma$$

AS REQUIRED.