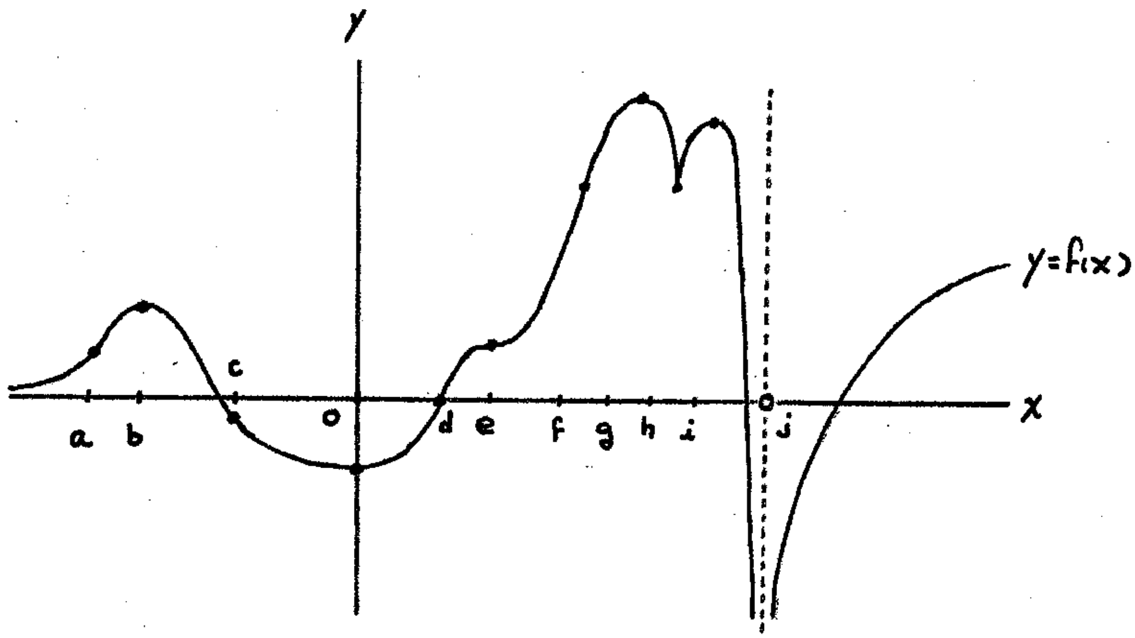
EXTREMA AND GRAPHING



RECALL :

RELATIVE MAXIMUM : FUNCTION CHANGES FROM INCREASING TO DECREASING

RELATIVE MINIMUM : FUNCTION CHANGES FROM DECREASING TO INCREASING

THE TERM RELATIVE EXTREMUM IS USED TO MEAN EITHER OF THESE TWO (PLURAL IS EXTREMA)

POINTS WHERE THE DERIVATIVE $f'(x)$ IS EITHER 0 OR UNDEFINED ARE CALLED CRITICAL POINTS

EXTREMA OCCUR AT CRITICAL POINTS,
BUT NOT EVERY CRITICAL POINT IS AN EXTREMUM
(SEE $x = e$ ABOVE).

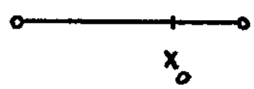
TO DETERMINE THE EXTREMA WE MUST THEREFORE DO TWO THINGS:

1. FIND THE CRITICAL POINTS (COMPUTE $f'(x)$ AND FIND OUT WHERE IT IS EITHER 0 OR UNDEFINED)
2. "TEST" EACH CRITICAL POINT TO DETERMINE IF IT IS A RELATIVE MAXIMUM, A RELATIVE MINIMUM, OR NEITHER.

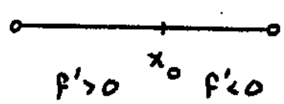
THE FIRST OF THESE IS EASY AND WE'LL SEE MANY EXAMPLES SHORTLY.

FOR THE SECOND, THERE ARE TWO "TESTS" AVAILABLE, THE FIRST OF WHICH WE HAVE REALLY ALREADY USED, ALTHOUGH WE DIDN'T GIVE IT A NAME.

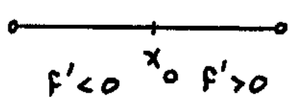
<u>THE</u> <u>FIRST</u> <u>DERIVATIVE</u> <u>TEST</u>: SUPPOSE $f(x)$ HAS A CRITICAL POINT AT x_0 (EITHER $f'(x_0) = 0$ OR f' IS NOT DEFINED AT x_0). IF



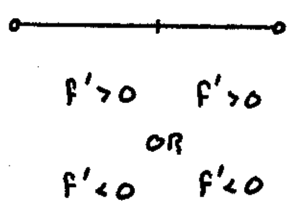
IS AN INTERVAL THAT CONTAINS NO OTHER CRITICAL POINTS OF f, THEN

(a) 

$\Rightarrow$ f HAS A RELATIVE <u>MAXIMUM</u> AT x_0

(b) 

$\Rightarrow$ f HAS A RELATIVE <u>MINIMUM</u> AT x_0

(c) 

$\Rightarrow$ f HAS <u>NEITHER</u> A RELATIVE MAXIMUM NOR A RELATIVE MINIMUM AT x_0

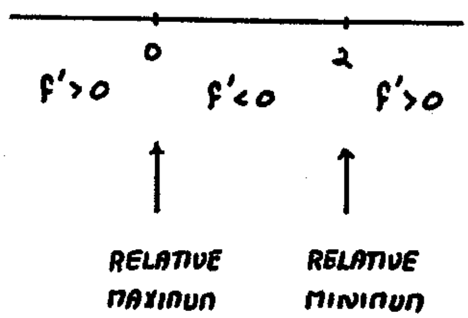
EXAMPLES: FOR EACH OF THE FOLLOWING FUNCTIONS $f(x)$ FIND ALL OF THE CRITICAL POINTS AND CLASSIFY EACH AS A RELATIVE MAXIMUM, RELATIVE MINIMUM, OR NEITHER.

1. $f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$

CRITICAL POINTS :

$$f'(x) = 5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}}$$
$$= 5x^{-\frac{1}{3}}(x-2)$$
$$= \frac{5(x-2)}{x^{\frac{1}{3}}}$$

WHICH IS 0 WHEN $x = 2$ AND UNDEFINED WHEN $x = 0$

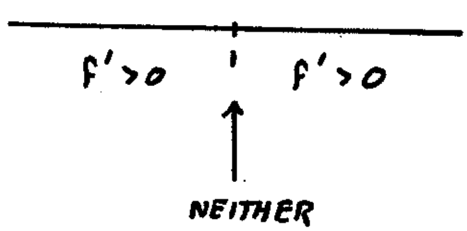


2. $f(x) = x^3 - 3x^2 + 3x - 1$

CRITICAL POINTS :

$$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1)$$
$$= 3(x-1)^2$$

WHICH IS DEFINED EVERYWHERE AND 0 WHEN $x = 1$.

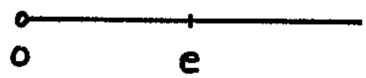


3. $f(x) = \dfrac{\ln x}{x}$ <u>NOTE</u> : DOMAIN IS $x > 0$

CRITICAL POINTS : $$f'(x) = \frac{x\left(\frac{1}{x}\right) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

WHICH IS ALWAYS DEFINED (ON $x > 0$) AND IS 0 WHEN

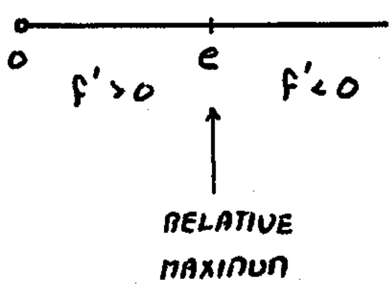
$$\frac{1 - \ln x}{x^2} = 0$$
$$1 - \ln x = 0$$
$$\ln x = 1$$
$$e^{\ln x} = e^1$$
$$x = e$$



CHECK THE SIGN OF $f'(x)$ AT

$$x = \frac{1}{e} = e^{-1} \; : \; f'\left(\tfrac{1}{e}\right) = \frac{1 - \ln(e^{-1})}{\left(\frac{1}{e}\right)^2} = \frac{1 + 1}{\left(\frac{1}{e}\right)^2} > 0$$

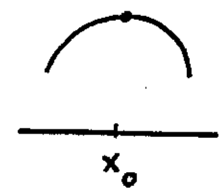
$$x = e^2 \; : \; f'(e^2) = \frac{1 - \ln(e^2)}{(e^2)^2} = \frac{1 - 2}{e^4} < 0$$



<u>SOMETIMES</u> THERE IS AN EASIER WAY TO " TEST " A CRITICAL POINT.

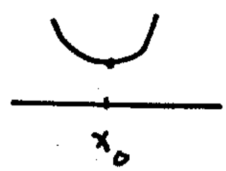
SUPPOSE $f'(x_0) = 0$ (A POINT WHERE THE DERIVATIVE IS 0 IS SOMETIMES CALLED A <u>STATIONARY</u> <u>POINT</u>).

THEN



$f''(x_0) < 0 \Rightarrow$ CONCAVITY IS DOWN AT x_0

$\Rightarrow$ RELATIVE MAXIMUM AT x_0



$f''(x_0) > 0 \Rightarrow$ CONCAVITY IS UP AT x_0

$\Rightarrow$ RELATIVE MINIMUM AT x_0

HOWEVER, IF $f''(x_0) = 0$ THEN NOTHING CAN BE CONCLUDED ABOUT THE NATURE OF THE CRITICAL POINT, I.E., IT MIGHT BE A MINIMUM (E.G., $f(x) = x^4$ AT $x_0 = 0$), A MAXIMUM (E.G., $f(x) = -x^4$ AT $x_0 = 0$), OR NEITHER (E.G., $f(x) = x^3$ AT $x_0 = 0$).

<u>THE</u> <u>SECOND</u> <u>DERIVATIVE</u> <u>TEST</u> : IF $f'(x_0) = 0$, THEN

1. $f''(x_0) < 0 \Rightarrow$ RELATIVE MAXIMUM AT x_0
2. $f''(x_0) > 0 \Rightarrow$ RELATIVE MINIMUM AT x_0
3. $f''(x_0) = 0 \Rightarrow$ NOTHING (TEST FAILS!)

<u>EXAMPLES</u>: FOR EACH OF THE FOLLOWING FUNCTIONS $f(x)$ FIND ALL OF THE CRITICAL POINTS AND CLASSIFY EACH AS A RELATIVE MAXIMUM, RELATIVE MINIMUM, OR NEITHER.

1. $f(x) = x^3 - 3x + 2$

   $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$

   WHICH IS DEFINED EVERYWHERE AND 0 AT $x = -1$ AND $x = 1$.

   $f''(x) = 6x$

   $f''(-1) = -6 < 0 \Rightarrow$ RELATIVE MAXIMUM AT $x = -1$

   $f''(1) = 6 > 0 \Rightarrow$ RELATIVE MINIMUM AT $x = 1$

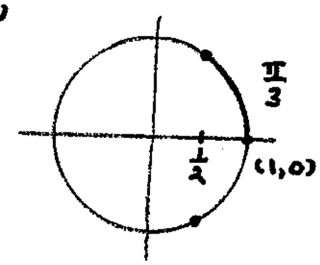
2. $f(x) = \frac{1}{2}x - \sin x$ ON $0 < x < 2\pi$

   $f'(x) = \frac{1}{2} - \cos x$

   WHICH IS DEFINED EVERYWHERE AND 0 WHEN

   $$\cos x = \frac{1}{2}$$

   $$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

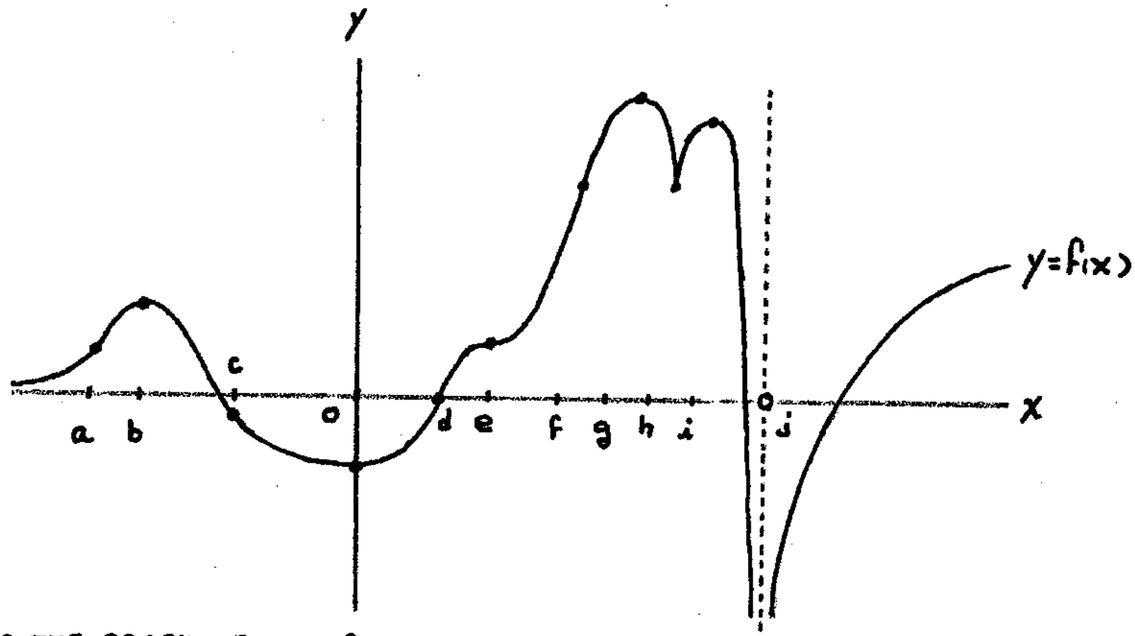


   $f''(x) = \sin x$

   $f''\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} > 0 \Rightarrow$ RELATIVE MINIMUM AT $x = \frac{\pi}{3}$

   $f''\left(\frac{5\pi}{3}\right) = \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2} < 0 \Rightarrow$ RELATIVE MAXIMUM AT $x = \frac{5\pi}{3}$

<u>NOTE</u>: IF <u>EITHER</u> f' IS UNDEFINED AT x_0 <u>OR</u> $f''(x_0) = 0$, THEN THE SECOND DERIVATIVE TEST IS OF NO USE. USE THE FIRST DERIVATIVE TEST INSTEAD.



<u>SKETCHING</u> <u>THE</u> <u>GRAPH</u> <u>OF</u> $y = f(x)$:

1. FIND ALL POINTS AT WHICH THE BEHAVIOR OF THE GRAPH COULD CHANGE.

   (a) POINTS NOT IN THE DOMAIN OF f : $f(x)$ UNDEFINED

   (b) POSSIBLE EXTREME POINTS : $f'(x)$ ZERO OR UNDEFINED

   (c) POSSIBLE INFLECTION POINTS : $f''(x)$ ZERO OR UNDEFINED

2. THE X - COORDINATES OF THE POINTS FROM #1 DIVIDE THE X-AXIS UP INTO INTERVALS ON WHICH THE BEHAVIOR CANNOT CHANGE, I.E., ON WHICH THE GRAPH IS ONE OF THE FOLLOWING TYPES :

   (a) $f' > 0$ AND $f'' > 0$: 

   (b) $f' > 0$ AND $f'' < 0$: 

   (c) $f' < 0$ AND $f'' > 0$: 

   (d) $f' < 0$ AND $f'' < 0$: 

DETERMINE THE SIGN OF f' AND f'' ON EACH INTERVAL BY CHECKING ONE POINT IN EACH.

PIECE THESE CURVES TOGETHER, BEING CAREFUL TO INDICATE ANY HORIZONTAL ($f' = 0$) OR VERTICAL (f' UNDEFINED) TANGENTS.

3. WHEN NECESSARY (AND FEASIBLE) ADD INTERCEPTS ($x = 0$ AND $y = 0$) AND HORIZONTAL ASYMPTOTES ($\lim_{x \to \pm\infty} f(x)$)

<u>EXAMPLES</u> :

1. $y = f(x) = x^4 - 2x^3$ (DEFINED EVERYWHERE)

$= x^3(x-2)$ (X-INTERCEPTS AT $x = 0$, $x = 2$)

$f'(x) = 4x^3 - 6x^2$ (DEFINED EVERYWHERE)

$= 2x^2(2x-3)$

$f'(x) = 0$ AT $x = 0$ AND $x = \frac{3}{2}$

Y-COORDINATES : $f(0) = 0$

$f(\frac{3}{2}) = -\frac{27}{16}$

$(0,0)$ $(\frac{3}{2}, -\frac{27}{16})$

$f''(x) = 12x^2 - 12x$ (DEFINED EVERYWHERE)

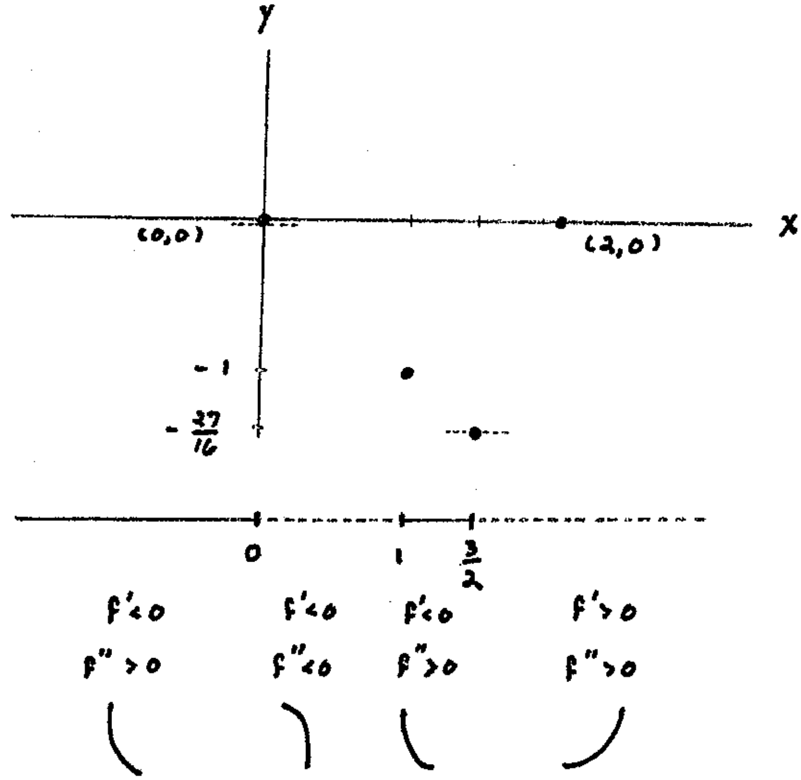
$= 12x(x-1)$

$f''(x) = 0$ AT $x = 0$ AND $x = 1$

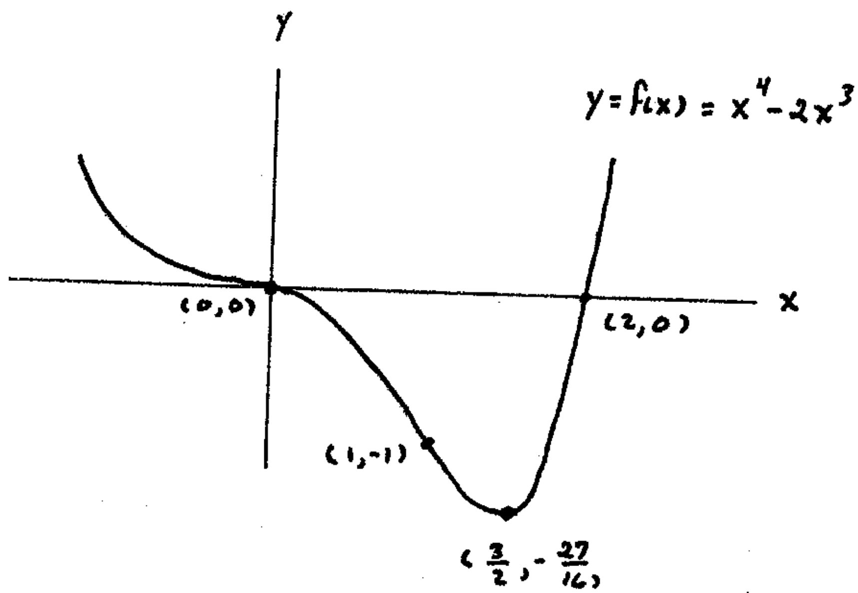
Y-COORDINATES : $f(0) = 0$

$f(1) = -1$

$(0,0)$ $(1,-1)$



NOW PIECE THESE TOGETHER, NOTING THAT THERE ARE NO HORIZONTAL ASYMPTOTES $\left(\lim_{x \to \pm\infty} (x^4 - 2x^3) = \infty \right)$, AND THAT THE GRAPH PASSES THROUGH $(0,0)$ AND $\left(\frac{3}{2}, -\frac{27}{16}\right)$ HORIZONTALLY.



2. $y = f(x) = \dfrac{x^2+1}{x^2-1}$

UNDEFINED AT $x = -1$ AND $x = 1$

NEVER 0 SO NO X-INTERCEPTS.

$$\lim_{x \to \pm\infty} \frac{x^2+1}{x^2-1} = \lim_{x \to \pm\infty} \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = 1$$

SO THERE IS A HORIZONTAL ASYMPTOTE AT HEIGHT 1 IN BOTH DIRECTIONS.

$f'(x) = \dfrac{-4x}{(x^2-1)^2}$ (COMPUTE THIS YOURSELF)

DEFINED EVERYWHERE (ON THE DOMAIN OF f)

$f'(x) = 0$ AT $x = 0$

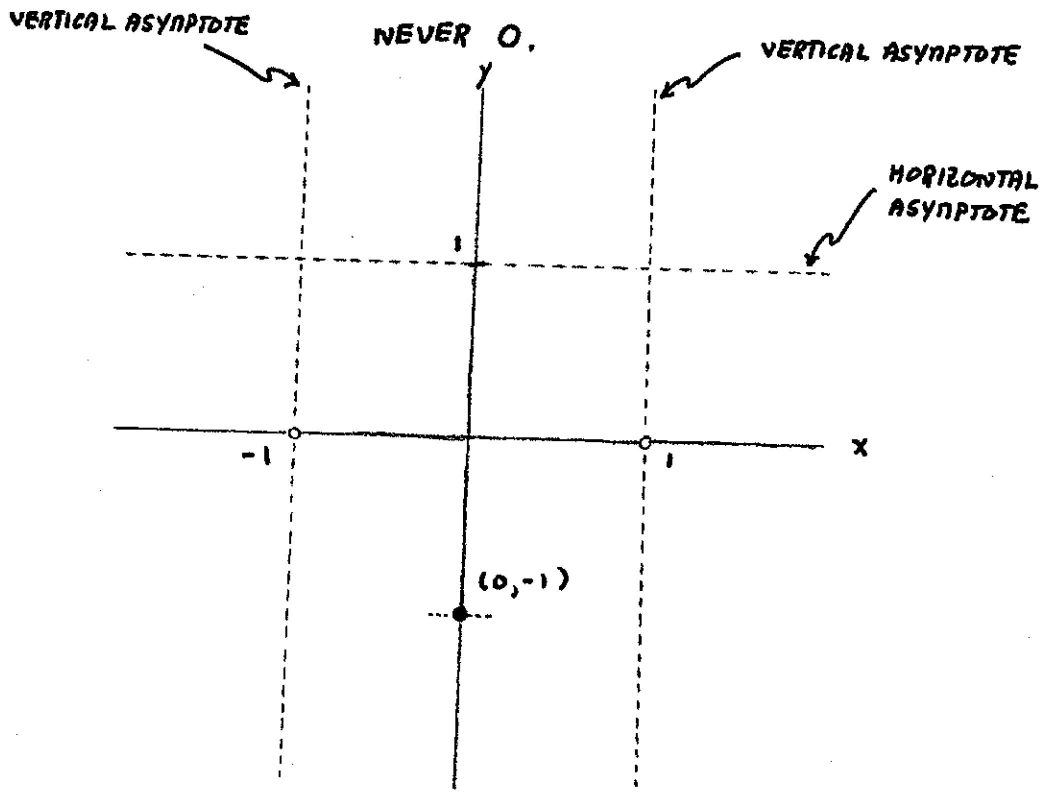
y-COORDINATE : $f(0) = \dfrac{0^2+1}{0^2-1} = -1$

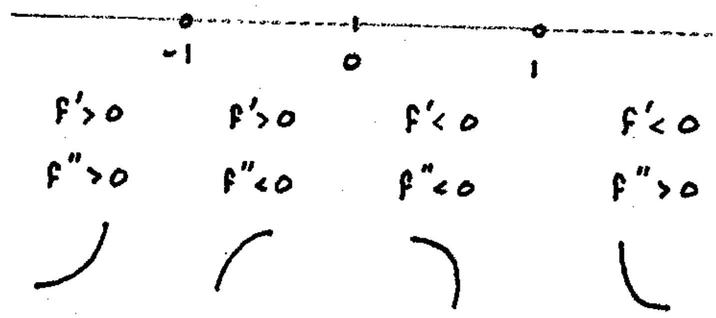
$(0,-1)$

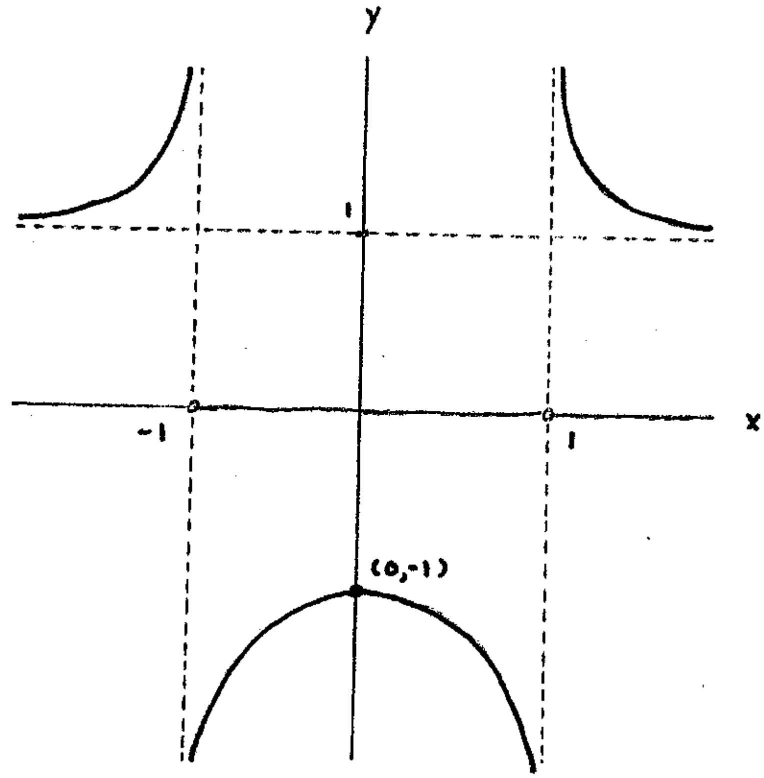
$f''(x) = \dfrac{4(3x^2+1)}{(x^2-1)^3}$ (COMPUTE THIS YOURSELF)

DEFINED EVERYWHERE (ON THE DOMAIN OF f)

NEVER 0.







3. $y = f(x) = x^{\frac{1}{3}}(x+3)^{\frac{2}{3}}$ (DEFINED EVERYWHERE)

(X-INTERCEPTS AT $x = 0$, $x = -3$)

$$f'(x) = \frac{x+1}{x^{2/3}(x+3)^{1/3}}$$ (COMPUTE THIS YOURSELF)

$f'(x) = 0$ AT $x = -1$

$f'(x)$ UNDEFINED AT $x = 0$ AND $x = -3$

Y-COORDINATES: $f(-1) = -\sqrt[3]{4}$

$f(0) = 0$

$f(-3) = 0$

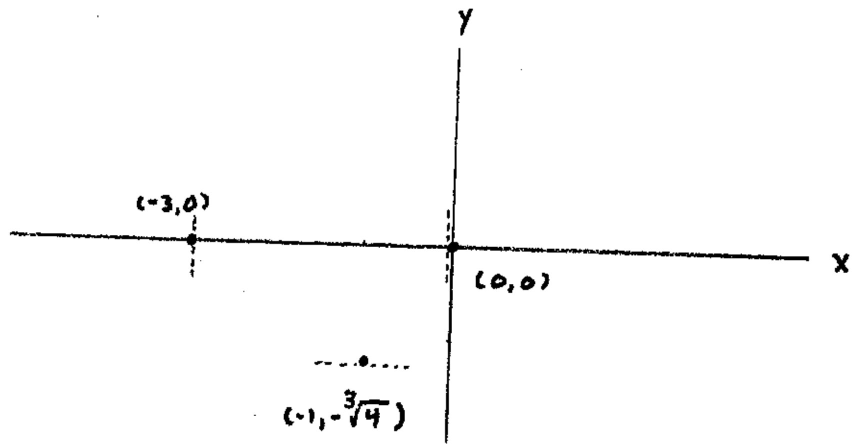
$(-1, -\sqrt[3]{4})$, $(0,0)$, $(-3,0)$

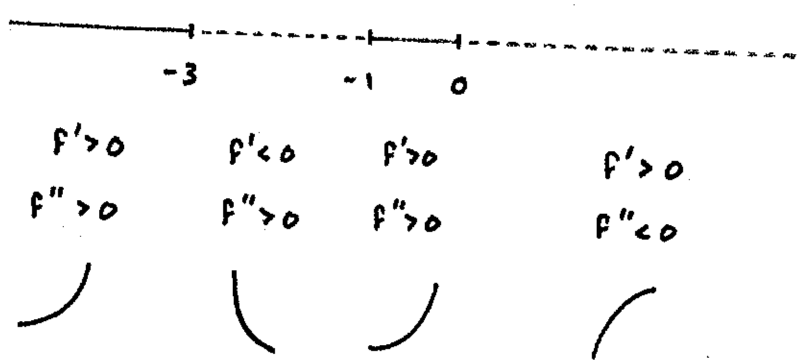
$$f''(x) = \frac{-2}{x^{5/3}(x+3)^{4/3}}$$

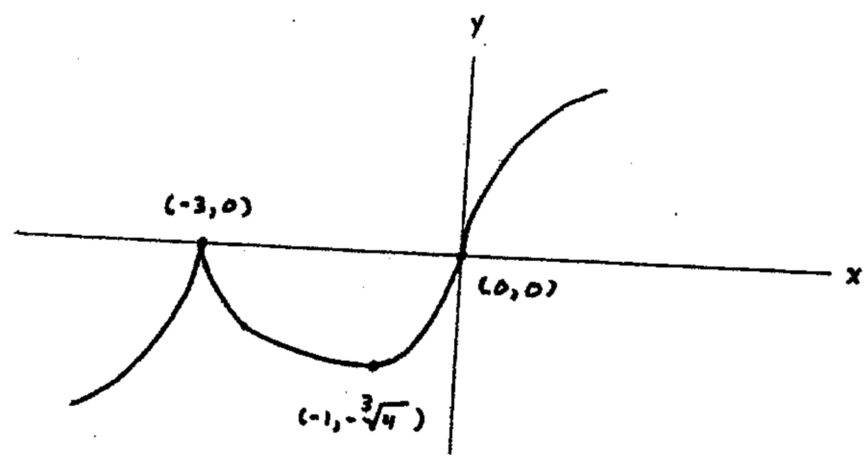
NEVER 0.

$f''(x)$ UNDEFINED AT $x = 0$ AND $x = -3$

(ALREADY HAVE THESE POINTS)







4. $y = f(x) = xe^{-x}$: DEFINED EVERYWHERE. X-INTERCEPT AT $x = 0$

$\lim_{x\to\infty} f(x) = 0$. $\lim_{x\to-\infty} f(x) = -\infty$

$f'(x) = (1-x)e^{-x}$: HORIZONTAL TANGENT AT $(1, \frac{1}{e})$

INCREASING $x < 1$. DECREASING $x > 1$.

$f''(x) = (x-2)e^{-x}$: CONCAVE UP $x > 2$. CONCAVE DOWN $x < 2$

INFLECTION POINT AT $(2, \frac{2}{e^2})$