FIBER INTEGRATION, POINCARÉ DUALS AND THOM CLASSES I

$$\begin{array}{c} Y \\ \downarrow \pi \\ X \end{array}$$

X^n AND Y^m SMOOTH, ORIENTED MANIFOLDS AND A LOCALLY TRIVIAL FIBER BUNDLE WITH TYPICAL FIBER F OF DIMENSION $k = m-n > 0$.

FIBER INTEGRATION (PUSHFORWARD): FOR EACH $p \geq k$,

$$\pi_* : \Omega^p_\square(Y) \to \Omega^{p-k}(X)$$

(FILL IN $\square$ WITH c : COMPACT SUPPORT, cv = COMPACT VERTICAL SUPPORT, OR RDV = RAPIDLY DECREASING VERTICALLY, AS APPROPRIATE)

π_* "INTEGRATES OUT THE k FIBER DIMENSIONS" AND $\pi_*\mu$ IS CHARACTERIZED BY

$$\boxed{\int_Y \pi^*\beta \wedge \mu = \int_X \beta \wedge \pi_*\mu}$$

$\forall\ \beta \in \Omega_c^{m-p}(X)$.

PROPERTIES:

1. $\pi_* \circ d = d \circ \pi_*$ (SO π_* DESCENDS TO COHOMOLOGY)

2. $\pi_*(\pi^*\beta \wedge \mu) = \beta \wedge \pi_*\mu$

3. $V \in \Gamma(TY)$ AND $W \in \Gamma(TX)$ π-RELATED (I.E., $(\pi)_{*p}(V(p)) = W(\pi(p))\ \forall p \in Y$) $\Rightarrow$

$$\pi_* \circ \iota_V = \iota_W \circ \pi_*$$

$$\pi_* \circ \mathcal{L}_V = \mathcal{L}_W \circ \pi_* .$$

<u>THOM</u> <u>ISOMORPHISM</u> <u>THEOREM</u> : LET $\pi : E \to X$ BE AN ORIENTABLE REAL VECTOR BUNDLE OF RANK k OVER THE ORIENTABLE n-MANIFOLD X OF FINITE TYPE. THEN

$$\pi_* : H^*_{cv}(E) \to H^{*-k}(X)$$

IS AN ISOMORPHISM.

<u>NOTE</u> : $H^*_{RDV}(E) \cong H^*_{cv}(E)$

THE INVERSE IS THE <u>THOM</u> <u>ISOMORPHISM</u>

$$Th : H^*(X) \to H^{*+k}_{cv}(E)$$

THE <u>THOM</u> <u>CLASS</u> $U(E) = [U]$ OF $\pi : E \to X$ IS THE IMAGE OF $[1] \in H^0(X)$ IN $H^k_{cv}(E)$

$$\boxed{\int_E \pi^*\beta \wedge U = \int_X \beta}$$

$\forall\ \beta \in \Omega^n_c(X)$.

THE THOM CLASS $U(E)$ IS CHARACTERIZED BY THE FACT THAT ANY REPRESENTATIVE U INTEGRATES TO 1 OVER EVERY FIBER.

OUR INTEREST IN THE THOM CLASS STEMS PRIMARILY FROM THE FOLLOWING

THEOREM : $\pi : E \to X$ AN ORIENTED REAL VECTOR BUNDLE OF RANK k OVER THE ORIENTED n-MANIFOLD X WITH $k \leq n$. IF U IS ANY REPRESENTATIVE OF THE THOM CLASS $U(E)$ AND $s : X \to E$ IS ANY SECTION, THEN $s^* U$ IS A REPRESENTATIVE OF THE EULER CLASS $e(E)$ OF $\pi : E \to X$.

$$\boxed{s^* U(E) = e(E)}$$

ANOTHER VIEW OF THE THOM AND EULER CLASSES VIA POINCARÉ DUALITY : RECALL THE

POINCARÉ DUALITY THEOREM : LET N BE A SMOOTH, ORIENTED d-MANIFOLD. FOR EACH $i = 0, \dots, d$, THE BILINEAR MAP

$$PD : H^i(N) \times H_c^{d-i}(N) \to \mathbb{R}$$

$$PD([\alpha], [\beta]) = \int_N \alpha \wedge \beta$$

IS NONDEGENERATE.

<u>CONSEQUENCE</u> : THE MAP

$$[\alpha] \longrightarrow PD([\alpha], \cdot) : H^i(N) \longrightarrow (H_c^{d-i}(N))^*$$

IS AN ISOMORPHISM.

<u>APPLICATION</u> : LET M BE A (TOPOLOGICALLY) CLOSED, ORIENTED SUBMANIFOLD OF DIMENSION n IN AN ORIENTED MANIFOLD N OF DIMENSION $d = n + k$. INTEGRATION OVER M GIVES A LINEAR FUNCTIONAL ON $H_c^n(N)$:

$$[\beta] \in H_c^n(N) \longrightarrow \int_M \iota^* \beta$$

$(\iota : M \hookrightarrow N)$ SO $\exists !$ $[\tau_M] \in H^k(N)$ SUCH THAT

$$\boxed{\int_M \iota^* \beta = \int_N \beta \wedge \tau_M}$$

$\forall$ $[\beta] \in H_c^n(N)$.

$[\tau_M] \in H^k(N)$ IS CALLED THE <u>POINCARÉ DUAL</u> OF M IN N.

THINK OF IT AS SORT OF A "DELTA FUNCTION CONCENTRATED AT M".

THE RELATIONSHIP BETWEEN POINCARÉ DUALS AND THOM CLASSES CAN BE ROUGHLY STATED AS

" THE POINCARÉ DUAL OF M IN N IS THE THOM CLASS OF ITS NORMAL BUNDLE. "

OR

" THE THOM CLASS OF A VECTOR BUNDLE IS THE POINCARÉ DUAL OF ITS 0-SECTION. "

TO MAKE THESE MORE PRECISE WE RECALL A FEW RESULTS FROM TOPOLOGY.

LET $\iota : M \hookrightarrow N$ BE THE INCLUSION. IDENTIFY $\iota_{*p}(T_p(M)) \subseteq T_p(N)$ WITH $T_p(M)$. CHOOSE A RIEMANNIAN METRIC $\langle\ ,\ \rangle$ ON N AND LET $N_p(M) = T_p(M)^{\perp} \subseteq T_p(N)$. THEN

$$N(M) = \bigcup_{p \in M} N_p(M)$$

IS A SUBBUNDLE OF TN CALLED THE <u>NORMAL BUNDLE</u> OF M IN N (DIFFERENT CHOICES OF $\langle\ ,\ \rangle$ GIVE EQUIVALENT BUNDLES).

A <u>TUBULAR NEIGHBORHOOD</u> OF M IN N IS AN OPEN NEIGHBORHOOD W OF M IN N THAT IS ALSO THE TOTAL SPACE OF A VECTOR BUNDLE $\pi : W \to M$ OF RANK k OVER M FOR WHICH THE 0-SECTION IS JUST THE INCLUSION $M \hookrightarrow W$.

THE <u>TUBULAR</u> <u>NEIGHBORHOOD</u> <u>THEOREM</u> ASSERTS THAT M HAS A TUBULAR NEIGHBORHOOD IN N AND, IN FACT, HAS ONE THAT IS EQUIVALENT TO THE NORMAL BUNDLE OF M IN N.

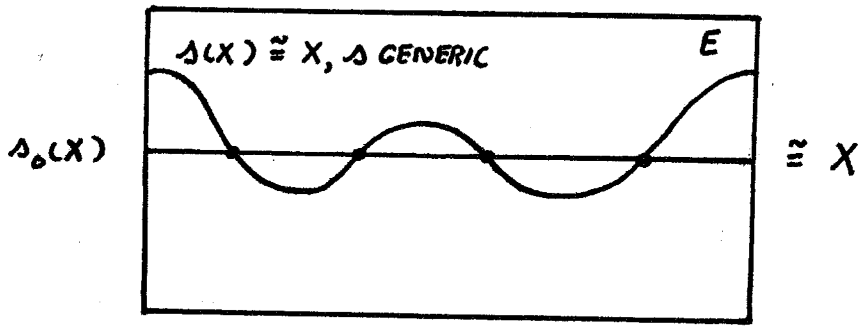
NOW ONE CAN PROVE THE FOLLOWING.

> IF M IS A (TOPOLOGICALLY) CLOSED, ORIENTED SUBMANIFOLD OF THE ORIENTED MANIFOLD N AND W IS A TUBULAR NEIGHBORHOOD OF M IN N (IDENTIFIED WITH THE NORMAL BUNDLE), THEN THE POINCARÉ DUAL OF M IN N AND THE THOM CLASS OF W CAN BE REPRESENTED BY THE SAME FORM ON N WITH SUPPORT CONTAINED IN W.

IT FOLLOWS FROM THIS THAT

> IF $\pi : E \to X$ IS AN ORIENTED, REAL VECTOR BUNDLE OVER THE ORIENTED MANIFOLD X AND IF X IS IDENTIFIED WITH THE IMAGE OF THE 0-SECTION, THEN THE THOM CLASS OF E AND THE POINCARÉ DUAL OF THE 0-SECTION CAN BE REPRESENTED BY THE SAME FORM ON E.

SIMILARLY, ONE ARRIVES AT THE FOLLOWING GEOMETRICAL INTERPRETATION OF THE EULER CLASS.



$$Z(s) = s(X) \cap s_0(X)$$

THOM CLASS OF E = POINCARÉ DUAL OF $s_0(X)$ IN E

EULER CLASS OF E = POINCARÉ DUAL OF $Z(s)$ IN X
FOR ANY GENERIC SECTION s
(GENERIC MEANS $s(X) \pitchfork s_0(X)$)

IN PARTICULAR, SINCE $e(E) = s^* U(E)$,

$$\boxed{\int_X \theta \wedge s^* U = \int_{Z(s)} \iota^* \theta}$$

$\forall\, \theta \in \Omega_c^{n-k}(X)$.