FIBER INTEGRATION, POINCARÉ DUALS AND THOM CLASSES II

LET X AND Y BE SMOOTH, ORIENTED MANIFOLDS OF DIMENSION n AND m, RESPECTIVELY, WITH $m > n$ AND

$$\pi : Y \to X$$

AN ORIENTED, LOCALLY TRIVIAL FIBER BUNDLE WITH TYPICAL FIBER F OF DIMENSION $k = m - n > 0$.

REMARK: THERE ARE "EQUIVARIANT" VERSIONS OF MANY OF THE RESULTS TO FOLLOW WHICH WE WILL INTRODUCE AS NEEDED

WE WISH TO DEFINE, FOR EACH $p \geq k$, A MAP π_* FROM p-FORMS ON Y TO $(p-k)$-FORMS ON X THAT BASICALLY "INTEGRATES OUT THE k FIBER DIMENSIONS". IT WILL BE UNIQUELY DETERMINED BY THE CONDITION

$$(1) \qquad \boxed{\int_Y \pi^*\beta \wedge \mu = \int_X \beta \wedge \pi_* \mu}$$

FOR (APPROPRIATE) FORMS μ ON Y AND β ON X.

REMARK: THE CONSTRUCTION OF π_* WILL BE VALID IN A NUMBER OF CONTEXTS (EACH OF WHICH ENSURES THAT ALL OF THE INTEGRALS INVOLVED MAKE SENSE). WE WILL NEED TO

OPERATE IN A NUMBER OF THESE CONTEXTS, BUT, IN THE INTEREST OF ECONOMY, WILL DESCRIBE THE CONSTRUCTION IN JUST THE SIMPLEST OF THESE (COMPACT SUPPORT) WITH APPROPRIATE REMARKS ON HOW IT CAN BE VARIED.

LET $\mu \in \Omega^p_c(Y)$. SUPPOSE WE COULD FIND A FORM $\pi_* \mu \in \Omega^{p-k}_c(X)$ SATISFYING (1) FOR EVERY $\beta \in \Omega^{m-p}_c(X)$ $\quad (m-p = n-(p-k))$. THIS PROPERTY WOULD UNIQUELY DETERMINE $\pi_*\mu$ IN $\Omega^{p-k}_c(X)$ SINCE ANY $\nu \in \Omega^{p-k}_c(X)$ SATISFYING $\int_X \beta \wedge \nu = 0 \quad \forall \beta \in \Omega^{m-p}_c(X)$ MUST VANISH.

TO DEFINE

$$\pi_* : \Omega^p_c(Y) \to \Omega^{p-k}_c(X)$$

WE NEED ONLY CONSIDER $\pi : \pi^{-1}(U) \to U$, WHERE U IS A TRIVIALIZING COORDINATE NEIGHBORHOOD IN X FOR THEN A PARTITION OF UNITY ARGUMENT WILL PIECE TOGETHER THE LOCALLY DEFINED FORMS INTO A GLOBALLY DEFINED $\pi_*\mu$ WITH THE SAME PROPERTY (1).

THUS, SUPPOSE $(x^1, \ldots, x^n, t^1, \ldots, t^k)$ ARE COORDINATES ON $\pi^{-1}(U)$ WITH $(x^1, \ldots, x^n)$ COORDINATES ON $U \subseteq X$ AND $(t^1, \ldots, t^k)$ LOCAL COORDINATES FOR F. WRITE

$$\mu = f_I(x,t)\, dx^I \wedge dt^1 \wedge \cdots \wedge dt^k + \cdots$$

WHERE THE REMAINING TERMS CONTAIN FEWER THAN k FACTORS OF dt^i (ALL OF WHICH WILL BE SENT TO ZERO BY π_*). DEFINE $\pi_*\mu$ AT EACH $x_0 \in U$ BY

$$(\pi_* \mu)(x_0) = \left(\int_{\pi^{-1}(x_0)} f_I(x,t)\, dt^1 \wedge \dots \wedge dt^k \right) dx^I$$

(WE SUPPRESS THE RESTRICTION MAP $\iota^*_{\pi^{-1}(x_0)}$).

REMARK: FOR THIS DEFINITION IT IS NOT REALLY NECESSARY THAT μ HAVE COMPACT SUPPORT ON Y. IT WOULD BE ENOUGH TO HAVE $\mu \in \Omega^*_{cv}(Y)$ ("CV" = "COMPACT VERTICAL" AND MEANS THAT THE RESTRICTION OF μ TO EACH FIBER $\pi^{-1}(x_0)$ HAS COMPACT SUPPORT) OR EVEN $\mu \in \Omega^*_{RDV}(Y)$ ("RDV" = "RAPIDLY DECREASING VERTICAL" AND MEANS THAT THE RESTRICTION OF μ TO EACH FIBER IS "RAPIDLY DECREASING", E.G., GAUSSIAN).

TO SEE WHY THIS DEFINITION HAS THE REQUIRED PROPERTY (1) LET $\beta \in \Omega_c^{m-p}(U)$. WE MUST SHOW THAT

$$\int_{\pi^{-1}(U)} \pi^*\beta \wedge \mu = \int_U \beta \wedge \pi_* \mu .$$

NOTE THAT μ IS A LINEAR COMBINATION OF FORMS OF THE FOLLOWING TWO TYPES:

1. $\mu = (\pi^*\varphi) \wedge f(x,t)\, dt^{i_1} \wedge \dots \wedge dt^{i_r}, \quad \varphi \in \Omega_c^{p-r}(U), \quad r < k$

2. $\mu = (\pi^*\varphi) \wedge f(x,t)\, dt^1 \wedge \dots \wedge dt^k, \quad \varphi \in \Omega_c^{p-k}(U)$

BY LINEARITY IT WILL SUFFICE TO CONSIDER EACH CASE SEPARATELY.

IN CASE (1), $\pi^*\beta \wedge \mu = \pi^*(\beta\wedge e) \wedge f(x,t)\, dt^{i_1} \wedge \cdots \wedge dt^{i_r}$ MUST BE ZERO SINCE IT IS A FORM OF RANK $(m-p) + (p-r) + r = m$ WITHOUT ALL OF THE dt^i (AND THEREFORE REPEATED dx^i). ALSO, $\pi_*\mu = 0$ BY DEFINITION SO BOTH INTEGRALS ARE ZERO. IN CASE (2), IDENTIFY $\pi^{-1}(U) = U \times F$. THEN

$$\pi^*\beta \wedge \mu = \pi^*(\beta\wedge e)\, f(x,t)\, dt^1 \wedge \cdots \wedge dt^k$$

AND

$$\begin{aligned}\beta \wedge \pi_*\mu &= \beta \wedge \pi_*\left((\pi^* e) \wedge f(x,t)\, dt^1 \wedge \cdots \wedge dt^k\right)\\ &= \beta \wedge \left(\left(\int_F f(x,t)\, dt^1 \wedge \cdots \wedge dt^k\right) e\right)\\ &= \left(\int_F f(x,t)\, dt^1 \wedge \cdots \wedge dt^k\right) \beta \wedge e\end{aligned}$$

SO $$\int_{\pi^{-1}(U)} \pi^*\beta \wedge \mu = \int_{\pi^{-1}(U)} \pi^*(\beta\wedge e)\, f(x,t)\, dt^1 \wedge \cdots \wedge dt^k = \int_U \beta \wedge \pi_*\mu$$

BY FUBINI'S THEOREM.

SOME PROPERTIES OF FIBER INTEGRATION :

1. $\pi_* \circ d_Y = d_X \circ \pi_*$ (IN PARTICULAR, π_* CARRIES CLOSED / EXACT FORMS ON Y TO CLOSED/EXACT FORMS ON X AND SO DESCENDS TO COHOMOLOGY)

2. $\pi_*(\pi^*\beta \wedge \mu) = \beta \wedge \pi_*\mu$

3. IF V IS A VECTOR FIELD ON Y AND W IS A VECTOR FIELD ON X AND IF V AND W ARE π - RELATED ($(\pi)_{*p}(V(p)) = W(\pi(p))\ \forall p \in Y$), THEN

$$\pi_* \circ \iota_V = \iota_W \circ \pi_* \qquad (\iota = \text{CONTRACTION})$$

AND

$$\pi_* \circ \mathcal{L}_V = \mathcal{L}_W \circ \pi_* \qquad (\mathcal{L} = \text{LIE DERIVATIVE})$$

MOST OF THESE ARE PROVED IN THE SAME WAY. AS AN ILLUSTRATION WE PROVE #1:

$$\pi_* \circ d_Y = d_X \circ \pi_*$$

(PROOFS OF THE REST ARE AVAILABLE IN [6] AND [14]).

IT WILL SUFFICE TO SHOW THAT, FOR EVERY $\mu \in \Omega_c^{p-1}(Y)$,

$$\int_X \beta \wedge \pi_*(d_Y\mu) = \int_X \beta \wedge d_X(\pi_*\mu)$$

FOR EVERY $\beta \in \Omega_c^{n-p}(X)$ FOR THEN $d_X(\pi_*\mu)$ SATISFIES THE DEFINING CONDITION (1) FOR $\pi_*(d_Y\mu)$, WHICH UNIQUELY DETERMINES IT.

NOTICE FIRST THAT

$$\begin{aligned} d_Y(\pi^*\beta \wedge \mu) &= d_Y(\pi^*\beta)\wedge\mu + (-1)^{n-p}\pi^*\beta \wedge d_Y\mu \\ &= \pi^*(d_Y\beta)\wedge\mu + (-1)^{n-p}\pi^*\beta\wedge d_Y\mu \end{aligned}$$

AND $\int_Y d_Y(\pi^*\beta\wedge\mu) = 0$ (STOKES' THEOREM) SO

$$\int_Y \pi^*\beta\wedge d_Y\mu = (-1)^{n-p-1}\int_Y \pi^*(d_Y\beta)\wedge\mu$$

SIMILARLY, $d_X(\beta\wedge\pi_*\mu) = d_X\beta\wedge\pi_*\mu + (-1)^{n-p}\beta\wedge d_X(\pi_*\mu)$ SO

$$\int_X \beta\wedge d_X(\pi_*\mu) = (-1)^{n-p-1}\int_X d_X\beta\wedge\pi_*\mu .$$

THUS,

$$\int_X \beta \wedge \pi_*(d_Y \mu) = \int_Y \pi^*\beta \wedge d_Y \mu$$

$$= (-1)^{n-p-1} \int_Y \pi^*(d_X \beta) \wedge \mu$$

$$= (-1)^{n-p-1} \int_X d_X \beta \wedge \pi_* \mu$$

$$= \int_X \beta \wedge d_X(\pi_* \mu)$$

AS REQUIRED. $\square$

THE PROOFS FOR # 2 AND THE FIRST PART OF #3 ARE SIMILAR AND THEN WE OBTAIN THE SECOND PART OF #3 FROM THE CARTAN HOMOTOPY FORMULA

$$\pi_* \circ \mathcal{L}_V = \pi_*(d_Y \circ \iota_V + \iota_V \circ d_Y)$$

$$= (\pi_* \circ d_Y) \circ \iota_V + (\pi_* \circ \iota_V) \circ d_Y$$

$$= (d_X \circ \pi_*) \circ \iota_V + (\iota_W \circ \pi_*) \circ d_Y$$

$$= d_X \circ (\pi_* \circ \iota_V) + \iota_W \circ (\pi_* \circ d_Y)$$

$$= (d_X \circ \iota_W) \circ \pi_* + (\iota_W \circ d_Y) \circ \pi_*$$

$$= (d_X \circ \iota_V + \iota_W \circ d_Y) \circ \pi_*$$

$$= \mathcal{L}_W \circ \pi_* \qquad \square$$

LET US NOW FOCUS FOR AWHILE ON THE CASE OF A SMOOTH, ORIENTABLE VECTOR BUNDLE

$$\pi : E \to X$$

OF RANK (FIBER DIMENSION) k OVER THE n-MANIFOLD X. FIBER INTEGRATION IS DEFINED FOR EITHER $\Omega^*_{CV}(E)$ OR $\Omega^*_{RDV}(E)$. WE WILL PHRASE THE FOLLOWING RESULTS IN TERMS OF COMPACT VERTICAL COHOMOLOGY $H^*_{CV}(E)$, BUT $H^*_{RDV}(E)$ WOULD DO AS WELL.

<u>THOM</u> <u>ISOMORPHISM</u> <u>THEOREM</u> : LET $\pi : E \to X$ BE AN ORIENTABLE REAL VECTOR BUNDLE OF RANK k OVER THE ORIENTABLE n-MANIFOLD X OF FINITE TYPE*. THEN

$$\pi_* : H^*_{CV}(E) \longrightarrow H^{*-k}(X)$$

IS AN ISOMORPHISM.

<u>REMARKS</u> : PROOF ON PAGES 63-64 OF [6]. FOR A VERSION OF THE THEOREM THAT DOES NOT REQUIRE "FINITE TYPE" SEE PAGES 130-131 OF [6].

THE INVERSE OF THIS ISOMORPHISM IS DENOTED

$$Th : H^*(X) \longrightarrow H^{*+k}_{CV}(E)$$

IS CALLED THE <u>THOM</u> <u>ISOMORPHISM</u> AND CAN BE DESCRIBED EXPLICITLY AS FOLLOWS.

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* A <u>GOOD</u> <u>COVER</u> FOR AN n-MANIFOLD X IS AN OPEN COVER $\{U_\alpha\}_{\alpha \in A}$ FOR WHICH ALL FINITE INTERSECTIONS $U_{\alpha_1} \cap \cdots \cap U_{\alpha_\ell}$ ARE DIFFEOMORPHIC TO $\mathbb{R}^n$. EVERY SMOOTH MANIFOLD HAS A GOOD COVER (REFINING ANY GIVEN OPEN COVER). X IS OF <u>FINITE</u> <u>TYPE</u> IF IT HAS A FINITE GOOD COVER.

THE IMAGE OF $[1] \in H^0(X)$ IS A CLASS

$$\Upsilon(E) \in H^k_{cv}(E)$$

CALLED THE THOM CLASS OF $\pi : E \to X$. SINCE $\pi_*(\Upsilon(E)) = [1]$, PROPERTY # 2 ON PAGE 4 GIVES

$$\pi_*([\pi^*\beta \wedge \Upsilon]) = [\beta]$$

FOR ANY REPRESENTATIVE Υ OF $\Upsilon(E)$. THUS,

$$Th([\beta]) = [\pi^*\beta \wedge \Upsilon]$$

OR, AT THE LEVEL OF FORMS

$$Th(\beta) = \pi^*\beta \wedge \Upsilon$$

$\forall \beta \in H^*(X)$.

FROM THE DEFINING PROPERTY (1) OF FIBER INTEGRATION WE FIND THAT

> FOR ANY TOP FORM β ON X WITH COMPACT SUPPORT,
>
> $$\int_E \pi^*\beta \wedge \Upsilon = \int_X \beta$$

SINCE $\pi_* \Upsilon = 1$, ANY REPRESENTATIVE OF THE THOM CLASS INTEGRATES TO 1 OVER EACH FIBER AND THIS PROPERTY, IN FACT, CHARACTERIZES $\Upsilon(E)$ IN $H^*_{cv}(E)$.

OUR INTEREST IN THE THOM CLASS STEMS PRIMARILY FROM THE FOLLOWING

THEOREM : LET $\pi : E \to X$ BE AN ORIENTED, REAL VECTOR BUNDLE OF FIBER DIMENSION k OVER THE ORIENTED n-MANIFOLD X WITH $k \le n$. LET U BE ANY REPRESENTATIVE OF THE THOM CLASS $U(E)$ AND $s : X \to E$ ANY SECTION (E.G., THE 0-SECTION). THEN $s^* U$ IS A REPRESENTATIVE OF THE EULER CLASS $e(E)$ OF $\pi : E \to X$,

$$s^* U(E) = e(E)$$

REMARK : ANY TWO SECTIONS ARE HOMOTOPIC AND SO INDUCE THE SAME MAPS IN COHOMOLOGY. OF COURSE, THE REPRESENTATIVES $s^* U$ OF $e(E)$ CAN "LOOK" QUITE DIFFERENT DEPENDING ON THE CHOICE OF U AND s.

THE THEOREM IS GENERALLY PROVED BY SHOWING THAT $[s^* U]$ SATISFIES THE CONDITIONS SPECIFIED IN THE USUAL "AXIOMATIC" DEFINITION OF THE EULER CLASS (SEE, E.G., PAGES 316-318, VOLUME II, OF [17]).

EXAMPLE : $e(E)$ IS SUPPOSED TO BE ZERO WHEN k IS ODD. BUT THE IMAGE OF $s^* U(E)$ UNDER THE THOM ISOMORPHISM IS

$$\begin{aligned} Th(s^* U(E)) &= [\pi^*(s^* U) \wedge U] \\ &= [(s \circ \pi)^* U \wedge U] \\ &= [(id_E)^* U \wedge U] \\ &= [U \wedge U] = [0] \quad \text{WHEN } k \text{ IS ODD} \end{aligned}$$

SO $s^* U(E)$ IS ALSO ZERO.

NOW WE WOULD LIKE TO TAKE MORE GENERAL VIEW OF THIS CONSTRUCTION OF THE THOM CLASS BY INTRODUCING THE NOTION OF THE "POINCARÉ DUAL" OF A SUBMANIFOLD.

LET $\iota : M \hookrightarrow N$ BE INCLUSION OF A (TOPOLOGICALLY) CLOSED, ORIENTED, SUBMANIFOLD M OF DIMENSION n INTO AN ORIENTED MANIFOLD N OF DIMENSION $d = n+k$. IDENTIFY $\iota_{*p}(T_p(M)) \subseteq T_p(N)$ WITH $T_p(M)$ $\forall p \in M$. CHOOSE A RIEMANNIAN METRIC $\langle\ ,\ \rangle$ AND LET $N_p(M) = T_p(M)^{\perp}$ FOR EACH $p \in M$. THEN

$$N(M) = \bigcup_{p \in M} N_p(M)$$

IS A SUBBUNDLE OF THE TANGENT BUNDLE $T(N)$ CALLED THE <u>NORMAL BUNDLE OF</u> M <u>IN</u> N (DIFFERENT CHOICES OF $\langle\ ,\ \rangle$ GIVE EQUIVALENT BUNDLES).

<u>EXAMPLES</u> :

1. THE NORMAL BUNDLE OF S^n IN $\mathbb{R}^{n+1}$ IS THE TRIVIAL REAL LINE BUNDLE OF S^n (IT HAS A NONZERO SECTION CONSISTING OF UNIT OUTWARD NORMAL VECTORS).

2. IF S^1 IS THE CENTRAL CIRCLE IN THE MÖBIUS BAND N, THEN THE NORMAL BUNDLE OF S^1 IN N IS THE USUAL NONTRIVIAL LINE $N \to S^1$.

THE NORMAL BUNDLE OF M IN N IS SOMETHING FOR WHICH WE CAN COMPUTE THE THOM CLASS.

WE WISH TO RELATE THIS THOM CLASS TO WHAT IS CALLED THE "POINCARÉ DUAL" OF M IN N. RECALL THE

<u>POINCARÉ</u> <u>DUALITY</u> <u>THEOREM</u> : LET N BE A SMOOTH, ORIENTED, d-DIMENSIONAL MANIFOLD. THEN, FOR EACH $i = 0, \ldots, d$, THE BILINEAR MAP

$$PD : H^i(N) \times H_c^{d-i}(N) \to \mathbb{R}$$

$$PD([\alpha],[\beta]) = \int_N \alpha \wedge \beta$$

IS NONDEGENERATE.

<u>CONSEQUENCE</u> : THE MAP

$$[\alpha] \longrightarrow PD([\alpha], \cdot) : H^i(N) \longrightarrow (H_c^{d-i}(N))^*$$

IS AN ISOMORPHISM.

<u>NOTE</u> : IF N HAS FINITE-DIMENSIONAL COHOMOLOGY (E.G., IF N IS OF FINITE TYPE), THEN THIS ISOMORPHISM IMPLIES THAT $H_c^{d-i}(N) \cong (H^i(N))^*$ AS WELL.

<u>APPLICATION</u> : CONSIDER AGAIN $\iota : M^n \hookrightarrow N^{n+k}$, TAKE $i = k$ TO GET AN ISOMORPHISM

$$[\alpha] \to PD([\alpha], \cdot) : H^k(N) \to (H_c^n(N))^*$$

INTEGRATION OVER M GIVES A LINEAR FUNCTIONAL ON $H_c^n(N)$:

$$[\beta] \in H_c^n(N) \longrightarrow \int_M \iota^*\beta$$

CONSEQUENTLY, THE ISOMORPHISM ABOVE GIVES A UNIQUE COHOMOLOGY CLASS $[\tau_M] \in H^k(N)$, CALLED THE <u>POINCARÉ DUAL OF</u> M <u>IN</u> N SUCH THAT

$$\boxed{\int_M \iota^*\beta = \int_N \beta \wedge \tau_M}$$

$\forall \ [\beta] \in H_c^n(N)$.

<u>REMARK</u> : INTUITIVELY, τ_M BEHAVES LIKE A "DELTA FUNCTION CONCENTRATED AT M" : INTEGRATING NEXT TO τ_M "EVALUATES AT M".

ALSO NOTE THAT, IF M IS COMPACT, ONE OBTAINS IN THE SAME WAY A LINEAR FUNCTIONAL ON $H^n(N)$ BY INTEGRATING RESTRICTIONS $\iota^*\beta$ OVER M. IF, IN ADDITION, N IS OF FINITE TYPE THIS GIVES A UNIQUE CLASS $[\tau_M'] \in H_c^k(N)$ SUCH THAT

$\int_M \iota^*\beta = \int_N \beta \wedge \tau_M' \quad \forall \ [\beta] \in H^n(N)$. SINCE THIS HOLDS FOR ANY CLOSED k- FORM β ON N IT CERTAINLY HOLDS FOR ANY CLOSED k- FORM WITH COMPACT SUPPORT. THUS, ANY REPRESENTATIVE τ_M' OF THIS $[\tau_M'] \in H_c^k(N)$ IS ALSO A RERESENTATIVE OF

THE POINCARÉ DUAL $[\tau_M] \in H^k(N)$. IN PARTICULAR, WHEN M IS COMPACT AND N IS OF FINITE TYPE, THERE IS A REPRESENTATIVE OF THE POINCARÉ DUAL OF M THAT HAS COMPACT SUPPORT. AS WE WILL SEE IN THE NEXT EXAMPLE, HOWEVER, THE COHOMOLOGY CLASSES $[\tau_M] \in H^k(N)$ AND $[\tau_M'] \in H^k_c(N)$ ARE GENERALLY QUITE DIFFERENT.

<u>EXAMPLE</u> : LET $N = \mathbb{R}^n$ AND $M = \{p\} \xhookrightarrow{\iota} \mathbb{R}^n$ FOR SOME POINT $p \in \mathbb{R}^n$. BY DEFINITION, THE POINCARÉ DUAL OF THE 0-DIMENSIONAL SUBMANIFOLD $\{p\}$ OF $\mathbb{R}^n$ IS A CLASS $[\tau_p] \in H^n(\mathbb{R}^n)$ SUCH THAT

$$\int_{\{p\}} \iota^* \beta = \int_{\mathbb{R}^n} \beta \wedge \tau_p$$

$\forall\ [\beta] \in H^0_c(\mathbb{R}^n)$.

NOW, $H^0_c(\mathbb{R}^n)$ IS JUST THE SPACE OF CLOSED 0-FORMS WITH COMPACT SUPPORT ON $\mathbb{R}^n$. BY CONNECTIVITY, A CLOSED 0-FORM ON $\mathbb{R}^n$ IS A CONSTANT FUNCTION AND THE ONLY ONE OF THESE WITH COMPACT SUPPORT ON $\mathbb{R}^n$ IS THE FUNCTION THAT IS IDENTICALLY ZERO SO $H^0_c(\mathbb{R}^n)$ CONSISTS OF JUST THIS ONE FUNCTION. THUS, THE INTEGRALS ABOVE ARE BOTH ZERO FOR ANY CHOICE OF $\tau_p \in \Omega^n(\mathbb{R}^n)$. <u>ANY</u> n-FORM ON $\mathbb{R}^n$ IS A THOM FORM FOR $\{p\}$ ($H^n(\mathbb{R}^n) = 0$ SO THEY ALL REPRESENT THE SAME (TRIVIAL) COHOMOLOGY CLASS). TURNING NOW TO THE CLASS $[\tau_p'] \in H^n_c(\mathbb{R}^n)$ FOR

WHICH

$$\int_{\{p\}} \iota^*\beta = \int_{\mathbb{R}^n} \beta \wedge \tau_p'$$

$\forall [\beta] \in H^0(\mathbb{R}^n)$ WE OBSERVE THE FOLLOWING : $H^0(\mathbb{R}^n) \cong \mathbb{R}$ IS THE SPACE OF CONSTANT FUNCTIONS ON $\mathbb{R}^n$ AND $H_c^n(\mathbb{R}^n) \cong \mathbb{R}$, WHERE THE ISOMORPHISM IS $\int_{\mathbb{R}^n} : H_c^n(\mathbb{R}^n) \to \mathbb{R}$. THUS, THE INTEGRAL ON THE LEFT-HAND SIDE IS JUST THE (CONSTANT) VALUE $\beta(p)$ OF β AT p AND THAT ON THE RIGHT-HAND SIDE IS $\beta(p) \int_{\mathbb{R}^n} \tau_p'$. CONSEQUENTLY, $[\tau_p']$ IS REPRESENTED BY ANY "BUMP" n-FORM ON $\mathbb{R}^n$ WITH TOTAL INTEGRAL 1. THE CORRESPONDING CLASS $[\tau_p'] \in H_c^n(\mathbb{R}^n)$ IS NONTRIVIAL (AND IS, IN FACT, THE NATURAL GENERATOR FOR $H_c^n(\mathbb{R}^n)$), EVEN THOUGH τ_p' IS ALSO A REPRESENTATIVE OF $[\tau_p] = 0 \in H^n(\mathbb{R}^n)$.

IN THE SAME WAY ONE SEES THAT THE POINCARÉ DUAL OF N IN N IS REPRESENTED BY THE CONSTANT FUNCTION 1.

OUR NEXT OBJECTIVE IS TO RELATE THE NOTIONS OF "POINCARÉ DUAL" AND "THOM CLASS". TO DO THIS WE MUST FIRST RECALL THAT ANY $M \hookrightarrow N$ CAN BE VIEWED AS LIVING (VIA THE 0-SECTION) IN THE NORMAL BUNDLE OF M IN N. MORE PRECISELY :

M IS AGAIN A TOPOLOGICALLY CLOSED, ORIENTED SUBMANIFOLD OF DIMENSION n IN AN ORIENTED MANIFOLD N OF DIMENSION $d = n + k$, $k \geq 0$, AND $\iota : M \hookrightarrow N$ IS THE INCLUSION MAP.

A <u>TUBULAR</u> <u>NEIGHBORHOOD</u> OF M IN N IS AN OPEN NEIGHBORHOOD W OF M IN N WHICH IS ALSO THE TOTAL SPACE OF A VECTOR BUNDLE

$$\pi : W \longrightarrow M$$

OF RANK k OVER M FOR WHICH THE 0-SECTION IS THE INCLUSION MAP.

THE <u>TUBULAR</u> <u>NEIGHBORHOOD</u> <u>THEOREM</u> ASSERTS THAT M HAS A TUBULAR NEIGHBORHOOD IN N WHICH IS, IN FACT, EQUIVALENT TO THE NORMAL BUNDLE OF M IN N.

<u>NOTE</u>: ROUGHLY STATED, WHAT WE WANT TO SHOW IS THAT

"THE POINCARÉ DUAL OF M IN N IS THE THOM CLASS OF ITS NORMAL BUNDLE"

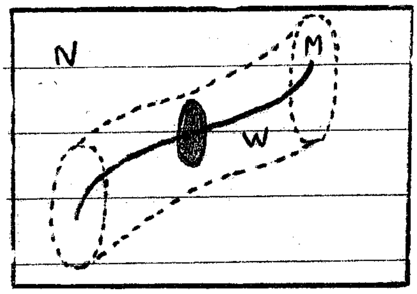
FROM WHICH ONE CONCLUDES THAT

"THE THOM CLASS OF A REAL, ORIENTABLE VECTOR BUNDLE IS THE POINCARÉ DUAL OF (THE IMAGE OF) ITS 0-SECTION."

MORE PRECISELY:

THEOREM : LET M^n BE A TOPOLOGICALLY CLOSED, ORIENTED SUBMANIFOLD OF THE ORIENTED MANIFOLD N^{n+k} AND LET $\pi : W \to M$ BE A TUBULAR NEIGHBORHOOD OF M IN N (EQUIVALENT TO THE NORMAL BUNDLE OF M IN N). THEN THE POINCARÉ DUAL OF M IN N AND THE THOM CLASS OF $\pi : W \to M$ CAN BE REPRESENTED BY THE SAME FORM.

SKETCH OF THE PROOF : LET $j : \Omega^*_{cv}(W) \to \Omega^*(N)$ BE THE MAP "EXTENSION BY ZERO" (DEFINED SINCE VERTICAL SUPPORTS ARE COMPACT).



LET υ BE A REPRESENTATIVE OF THE THOM CLASS OF $\pi : W \to M$ (WHICH WE IDENTIFY WITH THE NORMAL BUNDLE OF M IN N). THEN $j(\upsilon)$ HAS SUPPORT CONTAINED IN W, WHERE IT AGREES WITH υ. WE SHOW THAT $j(\upsilon)$ ALSO REPRESENTS THE POINCARÉ DUAL OF M IN N BY PROVING

$$\int_M \iota^*\beta = \int_N \beta \wedge j(\upsilon)$$

$\forall\ [\beta] \in H^n_c(N)$.

LET β BE ANY CLOSED n-FORM WITH COMPACT SUPPORT ON N.

- THE INTEGRAL OF β OVER M IS THE SAME AS THE INTEGRAL OVER M OF β'S RESTRICTION TO W.
- THE INTEGRAL OF $\beta \wedge j(U)$ OVER N IS THE SAME AS THE INTEGRAL OVER W OF THE RESTRICTION OF $\beta \wedge j(U)$ TO W, WHICH IS $(\beta|W) \wedge U$.

THUS, LETTING β NOW DENOTE THE RESTRICTION OF β TO W, IT WILL SUFFICE TO PROVE

$$\int_M \iota^*\beta = \int_W \beta \wedge U$$

WHERE $\iota : M \hookrightarrow W$ IS NOW IDENTIFIED WITH THE 0-SECTION. NOW, $\pi : W \to M$ AND $\iota : M \hookrightarrow W$ SATISFY $\pi \circ \iota = id_M$ AND $\iota \circ \pi \simeq id_W$ SO, IN COHOMOLOGY, π^* AND ι^* ARE INVERSE ISOMORPHISMS. THUS, β AND $\pi^*(\iota^*\beta)$ DIFFER BY AN EXACT FORM:

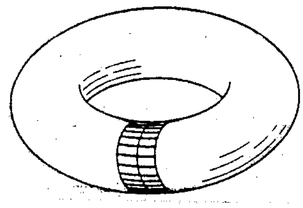
$$\beta = \pi^*(\iota^*\beta) + d\alpha$$

THUS,

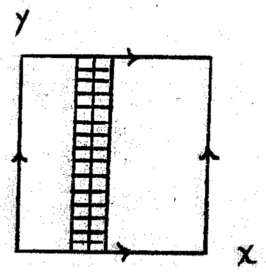
$$\begin{aligned} \int_W \beta \wedge U &= \int_W (\pi^*(\iota^*\beta) + d\alpha) \wedge U \\ &= \int_W \pi^*(\iota^*\beta) \wedge U \qquad \text{BY STOKES' THEOREM SINCE } d\alpha \wedge U = d(\alpha \wedge U) \\ &= \int_M \iota^*\beta \wedge \pi_* U = \int_M \iota^*\beta \wedge 1 = \int_M \iota^*\beta \end{aligned}$$

$\square$

EXAMPLE : POINCARÉ DUAL OF A CIRCLE IN A TORUS



THINK OF THE TORUS AS THE QUOTIENT OF A SQUARE AND THE CIRCLE AS THE IMAGE OF SOME COORDINATE LINE.



THE NORMAL BUNDLE TO THE CIRCLE IS IDENTIFIED WITH THE IMAGE OF THE STRIP SHOWN. THE POINCARÉ DUAL / THOM FORM IS REPRESENTED BY A BUMP 1-FORM WITH SUPPORT IN THE TUBULAR NEIGHBORHOOD AND INTEGRAL 1 OVER EACH FIBER. IN THE COORDINATES SHOWN IT CAN BE WRITTEN AS

$$\rho(x)\,dx$$

WHERE ρ IS A BUMP FUNCTION OF TOTAL INTEGRAL 1.

THIS RELATIONSHIP BETWEEN THOM CLASSES AND POINCARÉ DUALS YIELDS A USEFUL PROPERTY OF POINCARÉ DUALS WHICH WE SIMPLY STATE (PROOF ON PAGE 69 OF [6]).

"UNDER POINCARÉ DUALITY, THE TRANSVERSAL INTERSECTION OF CLOSED, ORIENTED SUBMANIFOLDS CORRESPONDS TO THE WEDGE PRODUCT OF FORMS."

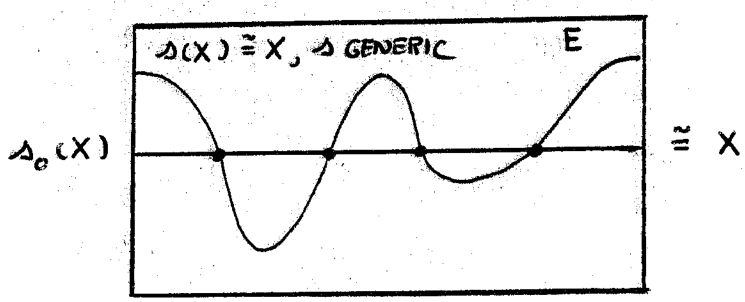
MORE PRECISELY :

LET M_1 AND M_2 BE CLOSED, ORIENTED SUBMANIFOLDS OF N THAT INTERSECT TRANSVERSALLY (SO $\operatorname{codim}(M_1 \cap M_2) = \operatorname{codim} M_1 + \operatorname{codim} M_2$). THEN

$$[\tau_{M_1 \cap M_2}] = [\tau_{M_1} \wedge \tau_{M_2}]$$

THESE RESULTS, TOGETHER WITH $e(E) = s^* U(E)$, GIVE THE FOLLOWING GEOMETRICAL INTERPRETATION OF THE EULER CLASS.

FOR ANY ORIENTED REAL VECTOR BUNDLE $\pi : E \to X$ OVER AN ORIENTED MANIFOLD X, THE EULER CLASS $e(E)$ IS THE POINCARÉ DUAL IN X OF THE ZERO SET $Z(s)$ OF ANY GENERIC SECTION s.



$$Z(s) = s(X) \cap s_0(X)$$

THOM CLASS OF E = POINCARÉ DUAL OF $s_0(X)$ IN E

EULER CLASS OF E = POINCARÉ DUAL OF $Z(s)$ IN X

COMBINING THESE RESULTS ONE OBTAINS THE FOLLOWING ESSENTIAL PROPERTIES OF THOM AND EULER CLASSES.

$\pi : E \to X$ AN ORIENTED, REAL VECTOR BUNDLE OF FIBER DIMENSION k OVER THE ORIENTED n-MANIFOLD X WITH $k \le n$. U A REPRESENTATIVE OF THE THOM CLASS $U(E)$ AND $s : X \to E$ A GENERIC SECTION. THEN $s^* U(E) = e(E)$ AND

1. FOR ANY $\theta \in \Omega_c^{n-k}(X)$,

$$\int_X \theta \wedge s^* U = \int_{Z(s)} \iota^* \theta$$

WHERE $\iota : Z(s) \hookrightarrow X$.

NOTE: IN TQFT THE LEFT-HAND SIDE IS A FORMAL FEYNMAN PATH INTEGRAL REPRESENTING THE EXPECTATION VALUE OF THE OBSERVABLE θ, BUT $Z(s)$ MAY BE FINITE-DIMENSIONAL.

2. IF $n = k$, THEN

$$\chi(E) = \int_X e(E) = \int_X s^* U = \int_X 1 \wedge s^* U$$

$$= \int_{Z(s)} \iota^* 1 = \int_{Z(s)} 1 = \text{INTERSECTION NUMBER OF } s(X) \text{ AND } s_0(X).$$