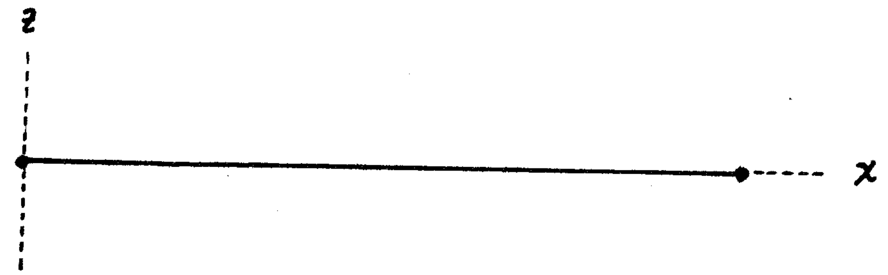
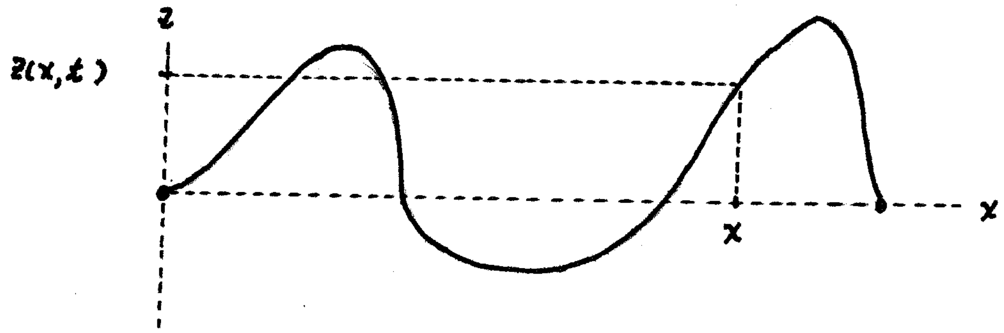
FUNCTIONS OF TWO OR MORE VARIABLES

THINK ABOUT A GUITAR STRING TIGHTLY STRETCHED BETWEEN TWO FIXED POINTS.



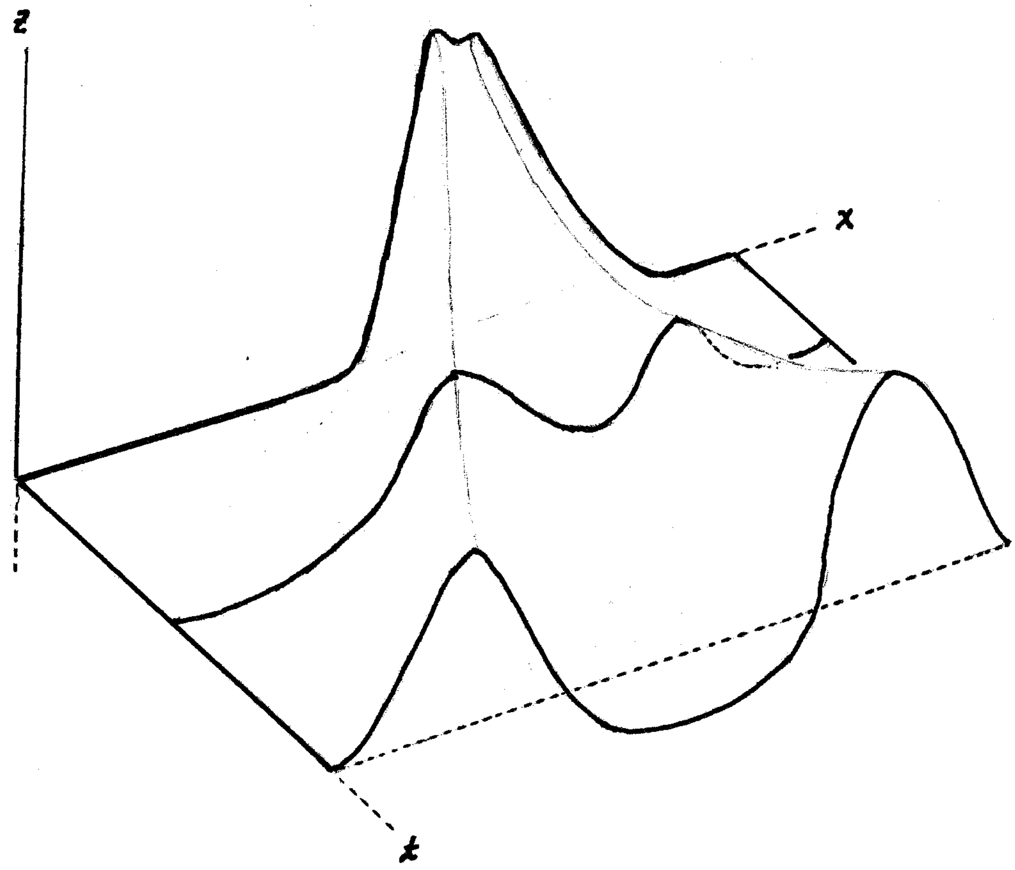
PLUCK THE STRING TO SET IT IN MOTION. A "SNAPSHOT" OF THE STRING AT SOME LATER TIME t MIGHT LOOK LIKE



$z(x,t)$ = HEIGHT OF THE STRING AT LOCATION x AND TIME t

TO DESCRIBE THE MOTION OF THE STRING WE NEED TO KNOW THIS SNAPSHOT AT EACH TIME t.

ALL OF THESE SNAPSHOTS CAN BE COMBINED INTO A SINGLE PICTURE BY "STACKING" THEM ALONG A t-AXIS :



$z(x,t)$ IS A TYPICAL EXAMPLE OF A "FUNCTION OF TWO VARIABLES" AND THE SURFACE ABOVE IS ITS "GRAPH". HERE'S THE DEFINITION:

A <u>FUNCTION</u> <u>OF</u> <u>TWO</u> <u>VARIABLES</u> (SAY, x AND y) IS A RULE f WHICH ASSIGNS TO EACH ORDERED PAIR (x,y) OF REAL NUMBERS IN SOME SUBSET OF THE xy-PLANE (CALLED THE <u>DOMAIN</u> OF THE FUNCTION) EXACTLY ONE REAL NUMBER

$$z = f(x,y)$$

CALLED THE <u>VALUE</u> <u>OF</u> f <u>AT</u> (x,y).

EXAMPLES :

1. $z = f(x,y) = x^2+y^2$

DOMAIN : All (x,y)

GRAPH : THE CIRCULAR PARABOLOID
$z = x^2+y^2$

2. $z = f(x,y) = \sqrt{1-x^2-y^2}$

DOMAIN : ALL (x,y) SATISFYING

$$1-x^2-y^2 \geq 0$$
$$1 \geq x^2+y^2$$
$$x^2+y^2 \leq 1$$

(THE UNIT DISC)

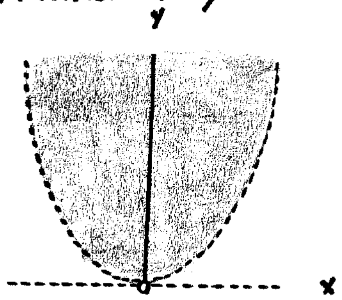
GRAPH : THE UPPER UNIT HEMISPHERE
$x^2+y^2+z^2=1, \quad z \geq 0$

3. $z = f(x,y) = \dfrac{1}{\sqrt{y-x^2}}$

DOMAIN : ALL (x,y) SATISFYING

$$y-x^2 > 0$$
$$y > x^2$$

ALL POINTS INSIDE THE PARABOLA $y = x^2$



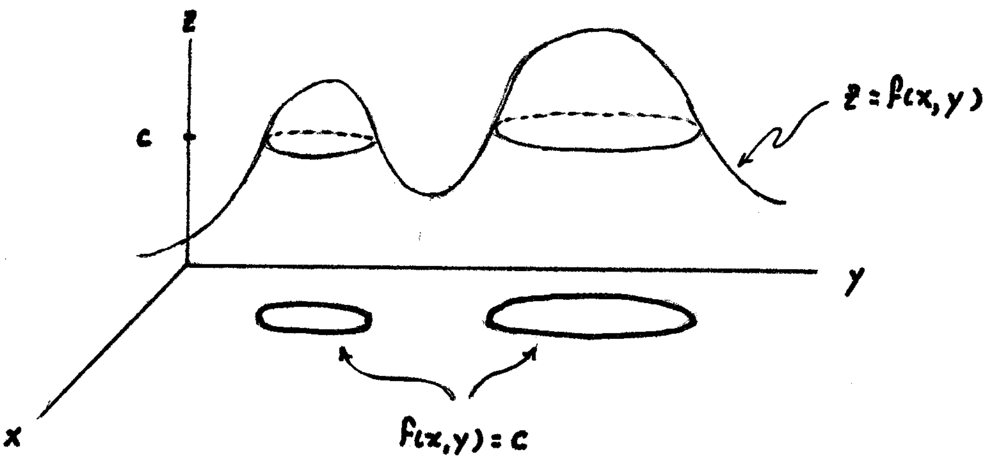
WE DO NOT YET KNOW THE GRAPH OF THIS ONE, BUT WE WILL SOON.

IT IS POSSIBLE TO OBTAIN SOMETHING LIKE A "PICTURE" OF A FUNCTION $z = f(x,y)$ WITHOUT DRAWING ITS GRAPH IN SPACE.

A <u>LEVEL CURVE</u> FOR $z = f(x,y)$ IS A CURVE IN THE xy-PLANE ON WHICH THE FUNCTION TAKES ONLY ONE VALUE, I.E., WITH AN EQUATION OF THE FORM

$$f(x,y) = c$$

FOR SOME CONSTANT c.

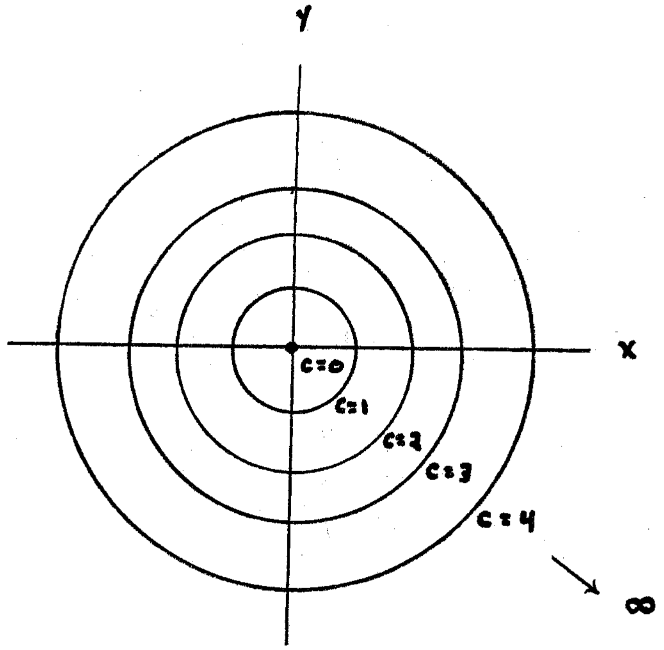


DRAW ENOUGH OF THESE, LABEL EACH WITH THE "c" IT CAME FROM (SO THAT YOU KNOW HOW HIGH IT SHOULD BE LIFTED TO GET TO THE GRAPH) AND YOU HAVE SOME IDEA WHAT THE SURFACE LOOKS LIKE.

<u>EXAMPLES</u>:

1. $z = f(x,y) = x^2 + y^2$

LEVEL CURVES : $x^2 + y^2 = c$ $\begin{cases} \text{EMPTY IF } c < 0 \\ (0,0) \text{ IF } c = 0 \\ \text{CIRCLE IF } c > 0 \end{cases}$



NOTE : $z = f(x,y) = \sqrt{x^2+y^2}$ HAS THE SAME TYPE OF LEVEL CURVES.

2. $z = f(x,y) = \sqrt{1-x^2-y^2}$

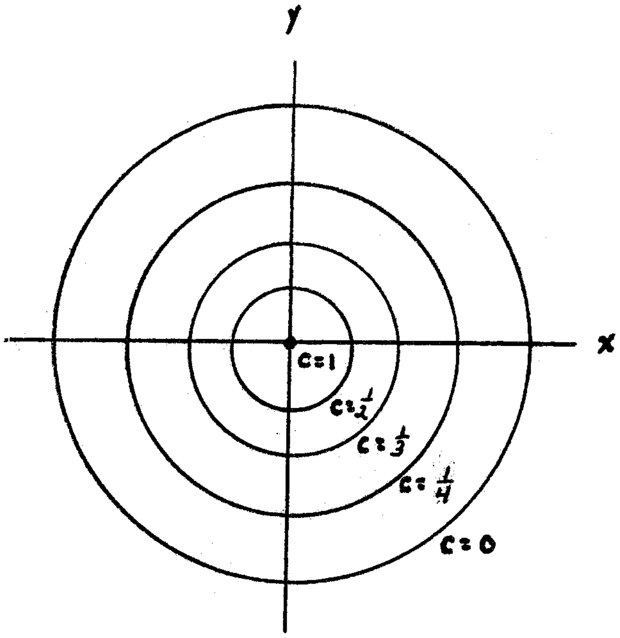
LEVEL CURVES : $\sqrt{1-x^2-y^2} = c$ NOTE : $c \geq 0$

$$1 - x^2 - y^2 = c^2$$

$$-x^2 - y^2 = -1 + c^2$$

$$x^2 + y^2 = 1 - c^2$$

$\begin{cases} \text{EMPTY IF } c > 1 \\ (0,0) \text{ IF } c = 1 \\ \text{CIRCLE IF } 0 \leq c < 1 \end{cases}$



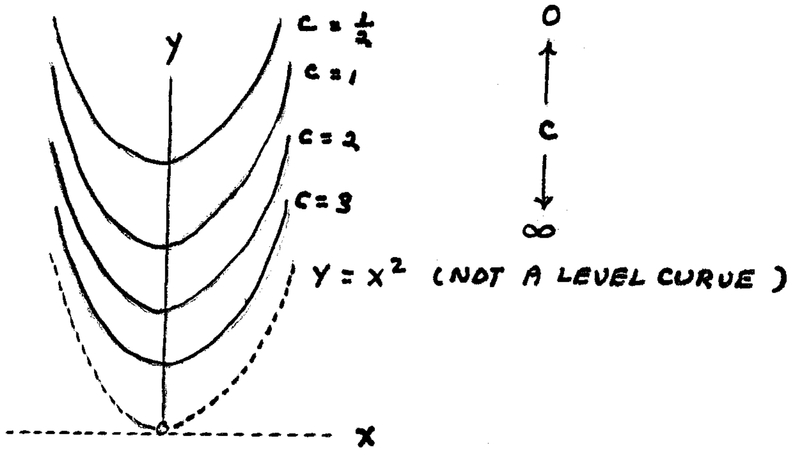
3. $z = f(x,y) = \dfrac{1}{\sqrt{y-x^2}}$

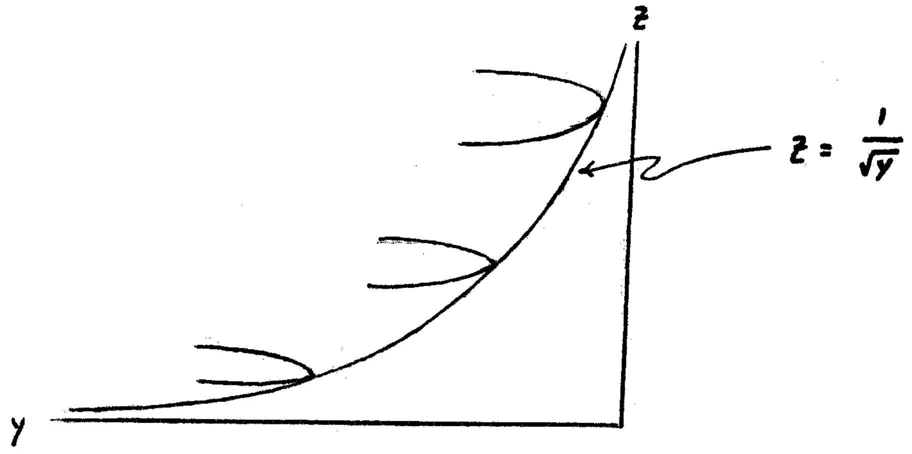
LEVEL CURVES : $\dfrac{1}{\sqrt{y-x^2}} = c$ $\quad$ NOTE : $c > 0$

$$\sqrt{y-x^2} = \frac{1}{c}$$

$$y - x^2 = \frac{1}{c^2}$$

$$y = x^2 + \frac{1}{c^2}$$ COPIES OF THE PARABOLA $y = x^2$





LIFT THE PARABOLA $y = x^2 + \frac{1}{c^2}$ UP FROM THE XY-PLANE TO HEIGHT C TO RETRIEVE THE GRAPH OF $z = \frac{1}{\sqrt{y - x^2}}$ (THE VERTEX WILL LAND ON THE CURVE $z = \frac{1}{\sqrt{y}}$, I.E., $x = 0$). THE SURFACE LOOKS LIKE THE HULL OF A (REALLY BIG) SHIP.

A FUNCTION OF THREE VARIABLES $w = f(x, y, z)$ IS DEFINED IN EXACTLY THE SAME WAY, BUT NOW ITS DOMAIN IS A SUBSET OF 3-SPACE, E.G.,

$$w = f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$$

IS DEFINED WHEN

$$1 - x^2 - y^2 - z^2 \geq 0$$

I.E.,

$$x^2 + y^2 + z^2 \leq 1$$

AND THIS IS THE SOLID <u>UNIT</u> <u>BALL</u> IN SPACE.

SINCE THE DOMAIN IS ALREADY 3-DIMENSIONAL, A "GRAPH" FOR SUCH A FUNCTION WOULD HAVE TO BE IN "4-SPACE". SINCE I DON'T KNOW HOW TO DRAW IN 4-SPACE WE WILL HAVE TO SETTLE FOR OTHER MEANS OF "PICTURING" THE FUNCTION. HERE'S A REASONABLE PROCEDURE:

IDENTIFY THE FUNCTION WITH SOMETHING PHYSICAL THAT YOU CAN RELATE TO, E.G.,

$f(x,y,z)$ = TEMPERATURE AT THE POINT (x,y,z) IN SPACE

AND THEN PICTURE ITS <u>LEVEL</u> <u>SURFACES</u>

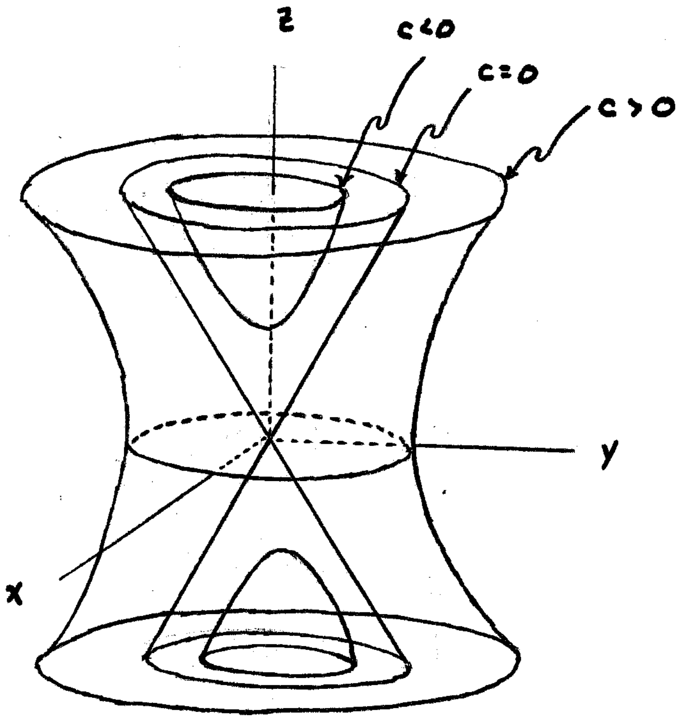
$$f(x,y,z) = c$$

(SURFACES OF CONSTANT TEMPERATURE, I.E., <u>ISOTHERMAL SURFACES</u>).

<u>EXAMPLE</u>: $f(x,y,z) = x^2+y^2-z^2$

LEVEL SURFACES:

$$x^2+y^2-z^2 = c \quad \begin{cases} \text{HYPERBOLOID OF 1 SHEET IF } c>0 \\ \text{CONE IF } c=0 \\ \text{HYPERBOLOID OF 2 SHEETS IF } c<0 \end{cases}$$



NOTE THAT IT'S EASY TO FIND THE LEVEL SURFACE THAT GOES THROUGH ANY GIVEN POINT, E.G., FOR $f(x,y,z) = x^2+y^2-z^2$ AND $(2,1,1)$ WE FIND THAT $f(2,1,1) = 2^2+1^2-1^2 = 4$ SO $(2,1,1)$ IS ON THE LEVEL SURFACE

$$x^2+y^2-z^2 = 4$$