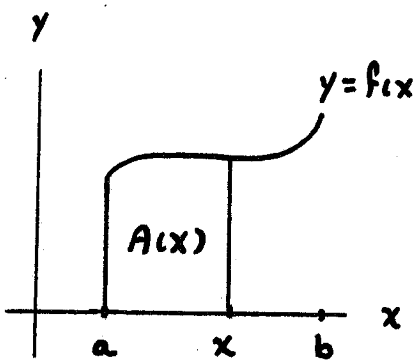
THE FUNDAMENTAL THEOREM OF CALCULUS :

THERE ARE TWO PARTS TO THIS. AS MOTIVATION FOR THE FIRST, RECALL THAT



$$A'(x) = f(x)$$

$$\frac{d}{dx} A(x) = f(x)$$

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

THE FUNDAMENTAL THEOREM OF CALCULUS SAYS THAT THIS IS ALWAYS TRUE :

> IF $f(x)$ IS CONTINUOUS ON THE INTERVAL I AND a IS IN I, THEN
>
> $$\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$
>
> I.E., $F(x) = \int_a^x f(t)\,dt$ IS AN ANTIDERIVATIVE FOR $f(x)$.

EXAMPLES :

1. $\frac{d}{dx}\int_1^x \cos t\,dt = \cos x$

2. $\frac{d}{dx}\int_5^x \cos t\,dt = \cos x$

3. DEFINE A FUNCTION $y = f(x)$ BY

$$f(x) = \int_{\sqrt{3}}^x \arctan t\,dt .$$

FIND $f(\sqrt{3})$, $f'(\sqrt{3})$ AND $f''(\sqrt{3})$.

$$f(\sqrt{3}) = \int_{\sqrt{3}}^{\sqrt{3}} \arctan t\,dt = 0$$

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx}\int_{\sqrt{3}}^x \arctan t\,dt$$

$$= \arctan x$$

$$f'(\sqrt{3}) = \arctan\sqrt{3} = \frac{\pi}{3}$$

$$f''(x) = (\arctan x)' = \frac{1}{1+x^2}$$

$$f''(\sqrt{3}) = \frac{1}{1+(\sqrt{3})^2} = \frac{1}{4}$$

4. $$\frac{d}{dx}\int_x^1 \frac{\sin t}{t}\,dt = \frac{d}{dx}\left(-\int_1^x \frac{\sin t}{t}\,dt\right)$$

$$= -\frac{d}{dx}\int_1^x \frac{\sin t}{t}\,dt = -\frac{\sin x}{x}$$

THE OTHER HALF OF THE FUNDAMENTAL THEOREM OF CALCULUS IS VERY CLOSELY RELATED TO THIS.

SUPPOSE $f(x)$ IS CONTINUOUS ON $[a, b]$. THEN $\int_a^x f(t)\,dt$ IS AN ANTIDERIVATIVE FOR $f(x)$ ON $[a, b]$.

ANY OTHER ANTIDERIVATIVE $F(x)$ FOR $f(x)$ ON $[a, b]$ DIFFERS FROM THIS ONE BY SOME CONSTANT :

$$F(x) = \int_a^x f(t)\,dt + C$$

NOW NOTICE THAT

$$\begin{aligned} F(b) - F(a) &= \left[\int_a^b f(t)\,dt + C \right] - \left[\int_a^a f(t)\,dt + C \right] \\ &= \int_a^b f(t)\,dt + C - 0 - C \\ &= \int_a^b f(x)\,dx \end{aligned}$$

IF $f(x)$ IS CONTINUOUS ON $[a, b]$ AND $F(x)$ IS ANY ANTIDERIVATIVE FOR $f(x)$ ON $[a, b]$, THEN

$$\begin{aligned} \int_a^b f(x)\,dx &= F(b) - F(a) \\ &= F(x) \Big|_a^b \end{aligned}$$

EXAMPLES :

1. $\int_1^2 x^3 dx = \frac{1}{4}x^4 \Big|_1^2 = \frac{1}{4}(2^4 - 2^1) = \frac{1}{4}(14) = \frac{7}{2}$

2. $\int_0^{2\pi} \cos x \, dx = \sin x \Big|_0^{2\pi} = \sin 2\pi - \sin 0 = 0 - 0 = 0$

3. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx = \text{ARCSIN } x \Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \text{ARCSIN } \frac{1}{2} - \text{ARCSIN } (-\frac{1}{2})$

$= \frac{\pi}{6} - (-\frac{\pi}{6}) = \frac{2\pi}{6} = \frac{\pi}{3}$

4. $\int_0^{\frac{\pi}{4}} \tan x \, dx = \ln|\sec x| \Big|_0^{\frac{\pi}{4}} = \ln|\sec \frac{\pi}{4}| - \ln|\sec 0|$

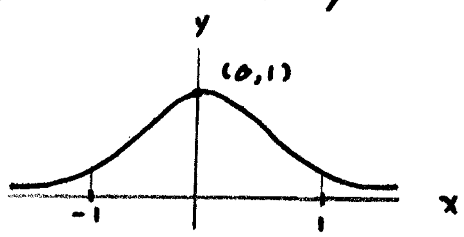
$= \ln|\sqrt{2}| - \ln|1| = \ln\sqrt{2} = \frac{1}{2}\ln 2.$

5. $\int_{-e}^{-1} \frac{1}{x} dx = \ln|x| \Big|_{-e}^{-1} = \ln|-1| - \ln|-e| = \ln 1 - \ln e$

$= 0 - 1$

$= -1$

6. FIND THE AREA UNDER THE GRAPH OF $y = f(x) = \frac{1}{1+x^2}$ FROM $x = -1$ TO $x = 1$.



$A = \int_{-1}^{1} \frac{1}{1+x^2} dx = \text{ARCTAN } x \Big|_{-1}^{1} = \text{ARCTAN }(1) - \text{ARCTAN }(-1)$

$= \frac{\pi}{4} - (-\frac{\pi}{4}) = \frac{\pi}{2}$

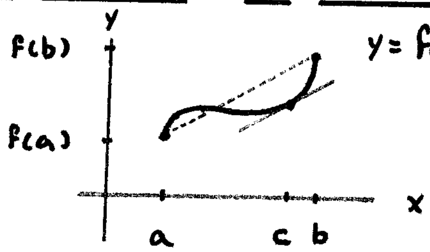
NOTICE THAT THE FOLLOWING IS OBVIOUSLY STUPID :

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^{1} = -\frac{1}{1} - \left(-\frac{1}{-1}\right)$$

$$= -1 - 1 = -2$$

WHY IS IT OBVIOUSLY STUPID AND WHAT WENT WRONG ?

RECALL THE MEAN VALUE THEOREM (FOR DERIVATIVES) FROM LAST TERM :

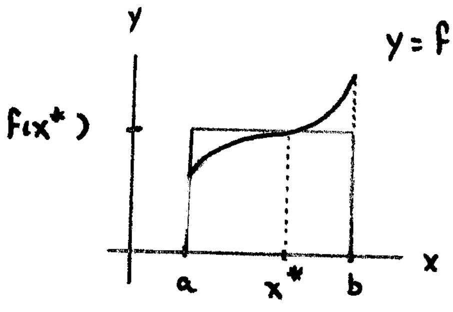


$y = f(x)$, CONTINUOUS ON $[a,b]$ AND DIFFERENTIABLE ON (a,b)

FOR SOME C IN (a,b),

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

THERE IS A SIMILAR MEAN VALUE THEOREM (FOR INTEGRALS) :



$y = f(x)$, CONTINUOUS ON $[a,b]$

FOR SOME x^* IN $[a,b]$,

$$\int_a^b f(x)\,dx = f(x^*)(b-a)$$

<u>EXAMPLE</u> : $f(x) = x^2$ IS CONTINUOUS ON $[1,4]$

$$\int_1^4 x^2\,dx = \tfrac{1}{3}x^3 \Big|_1^4 = \tfrac{1}{3}(4^3 - 1^3) = \frac{63}{3} = 21$$

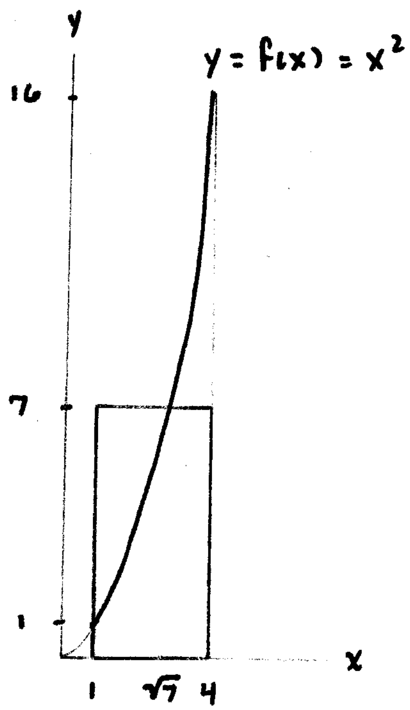
ACCORDING TO THE M.V.T. THERE SHOULD BE SOME x^* IN $[1,4]$ AT WHICH

$$\int_1^4 x^2\,dx = (x^*)^2(4-1)$$

$$21 = 3(x^*)^2$$

$$7 = (x^*)^2$$

THE SOLUTION IN $[1,4]$ IS $x^* = \sqrt{7} \approx 2.65$



NOW WE RETURN TO

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

ODDLY ENOUGH, FUNCTIONS OF THE TYPE $\int_a^x f(t)\,dt$ DO ARISE OFTEN IN APPLICATIONS, E.G., THE <u>FRESNEL FUNCTION</u>

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$$

IS INPORTANT IN PHYSICS (OPTICS).

SINCE $\sin\left(\frac{\pi t^2}{2}\right)$ HAS NO ELEMENTARY ANTIDERIVATIVE, $S(x)$ REALLY CANNOT BE SIMPLIFIED ANY FURTHER.

HOWEVER, ITS DERIVATIVES ARE EASY:

$$S'(x) = \frac{d}{dx} S(x) = \frac{d}{dx} \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$$

$$= \sin\left(\frac{\pi x^2}{2}\right)$$

$$S''(x) = \left(\sin\left(\frac{\pi x^2}{2}\right)\right)' = \pi x \qquad \left(\frac{\pi x^2}{2}\right)$$

ETC.

HOWEVER, WHAT ABOUT THE DERIVATIVES OF

$$S(x^2) = \int_0^{x^2} \sin\left(\frac{\pi t^2}{2}\right) dt$$

OR

$$S(e^x) = \int_0^{e^x} \sin\left(\frac{\pi t^2}{2}\right) dt$$

ETC. ?

FOR THIS SORT OF THING WE NEED TO COMBINE OUR NEW DIFFERENTIATION FORMULA

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

WITH THE CHAIN RULE

$$\frac{d}{dx}F(u(x)) = F'(u(x))\,u'(x)$$

TO GET

$$\boxed{\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))\,u'(x)}$$

EXAMPLES :

1. $$\frac{d}{dx}\int_3^{x^3} t^2\sin t\,dt = (x^3)^2\sin(x^3)\,(x^3)'$$
$$= x^6\sin(x^3)\,(3x^2) = 3x^8\sin(x^3)$$

2. FIND y' IF $y = \int_{e^x}^{0}\sin^3 t\,dt$

$$y' = \frac{d}{dx}\int_{e^x}^{0}\sin^3 t\,dt = -\frac{d}{dx}\int_0^{e^x}\sin^3 t\,dt = -\sin^3(e^x)\,(e^x)'$$
$$= -e^x\sin^3(e^x)$$

3. $$\frac{d}{dx}\int_{\cos x}^{5x}\cos(t^2)\,dt = \frac{d}{dx}\left[\int_{\cos x}^{0}\cos(t^2)\,dt + \int_0^{5x}\cos(t^2)\,dt\right] =$$

$$\frac{d}{dx}\left[-\int_0^{\cos x}\cos(t^2)\,dt + \int_0^{5x}\cos(t^2)\,dt\right] =$$

$$-\cos(\cos^2 x)(-\sin x) + \cos((5x)^2)(5) =$$

$$\cos(\cos^2 x)\sin x + 5\cos(25x^2)$$