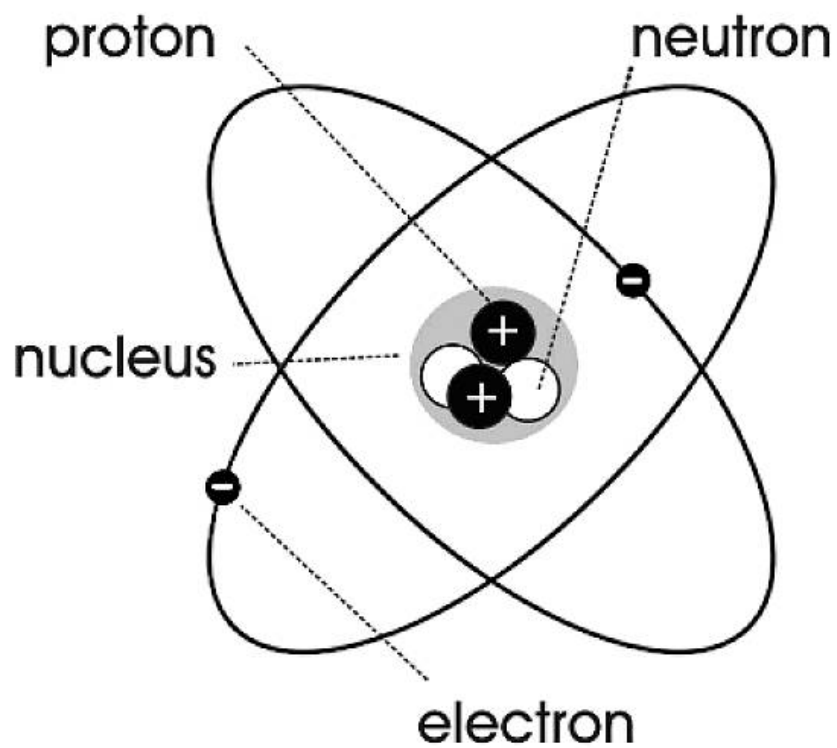
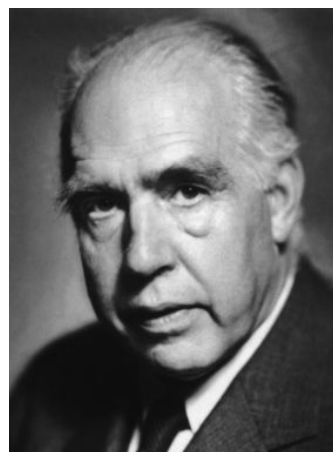


ATOMS:

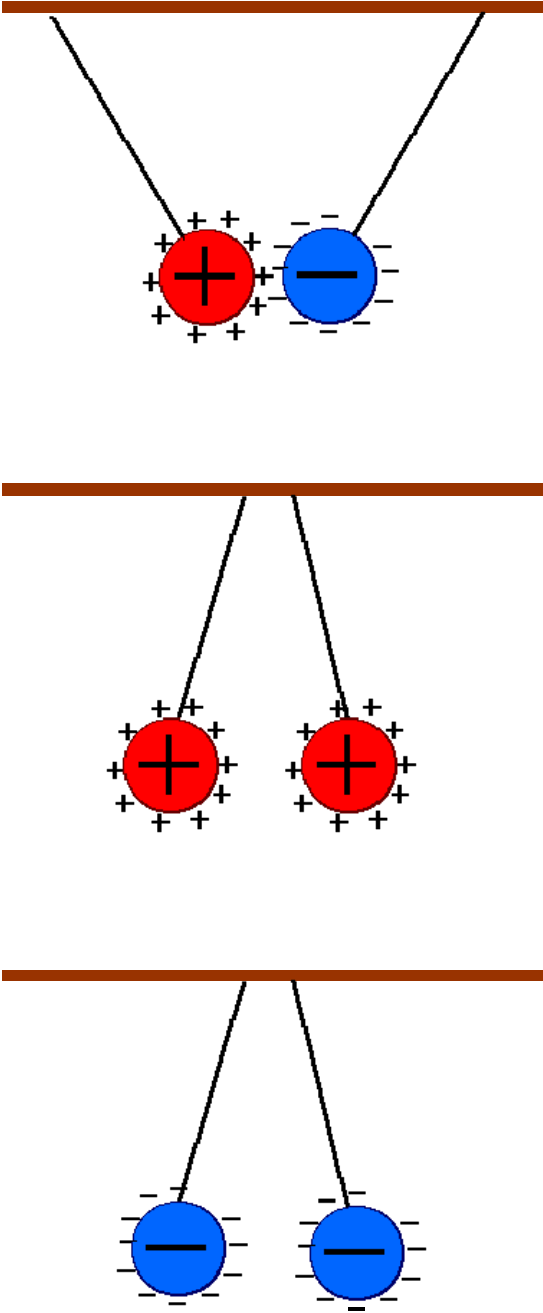


Ernest Rutherford

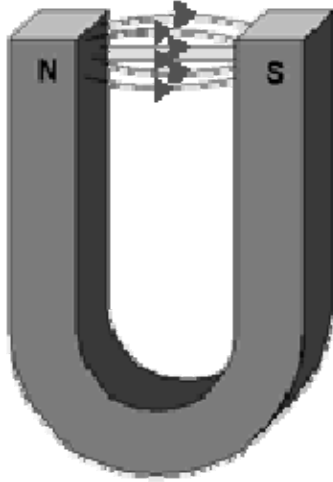


Neils Bohr

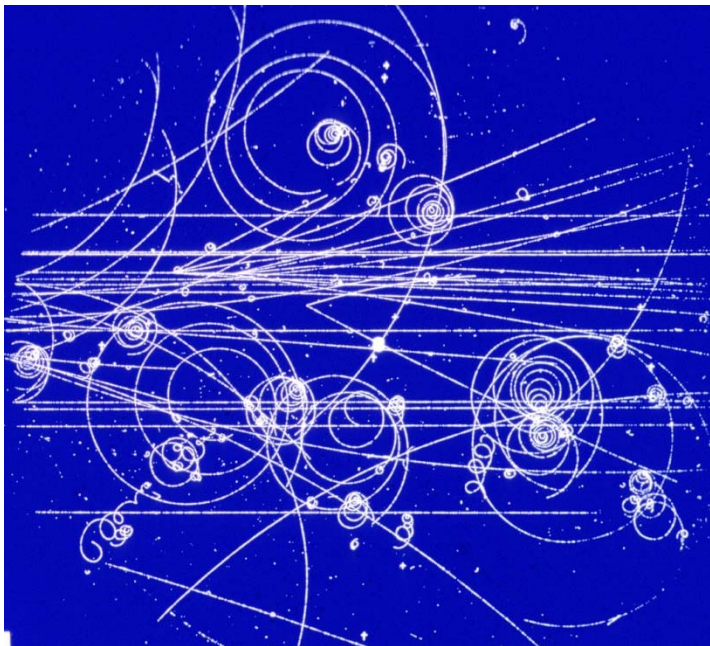
ELECTROSTATIC FIELDS: E



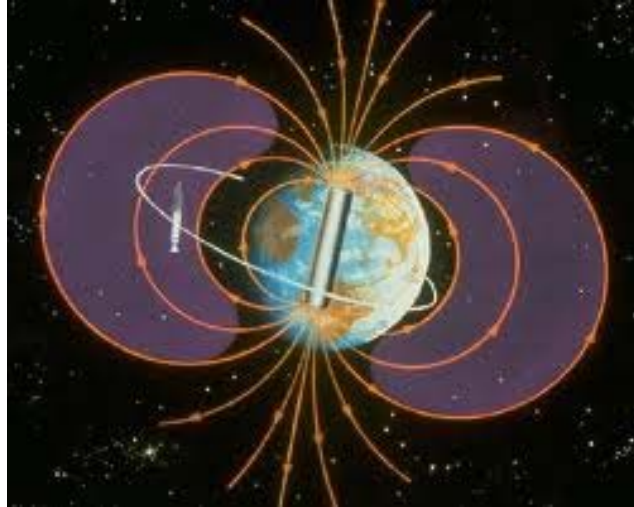
MAGNETIC FIELDS: B



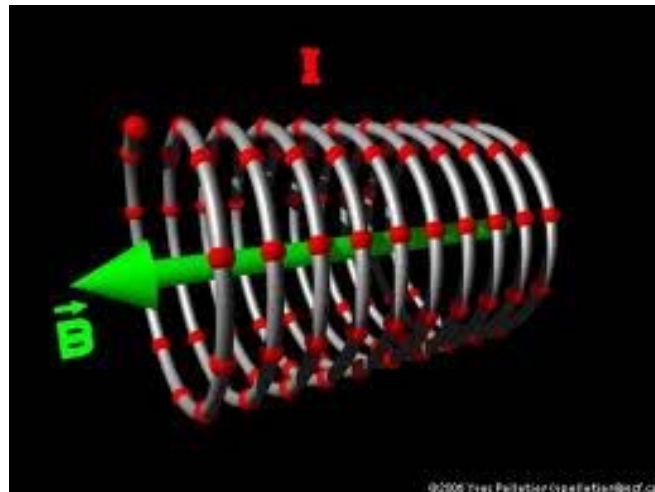
An electron at rest in the field will feel nothing, but ***throw*** it into the field and the electron is likely to go crazy:



$$\mathbf{F} = q\mathbf{V}\times\mathbf{B}$$



The Magnetic Field of the Earth



The Magnetic Field of a Solenoid

MAGNETIC POTENTIALS: A

Another “field” from which B can be computed.

$$\mathbf{B} = \mathit{curl} \mathbf{A}$$

In the old days (prior to 1960), everyone thought that A was a useful tool, but that it could have no real, physical effects (like pushing around little pieces of metal). The reason is that A is highly non-unique; for a given B there are many possible choices for A (each is called a choice of *gauge*).

$$\mathbf{B} = \mathit{curl} \mathbf{A} = \mathit{curl} (\mathbf{A} - \quad)$$

An analogy:



Label “height zero” wherever you like.

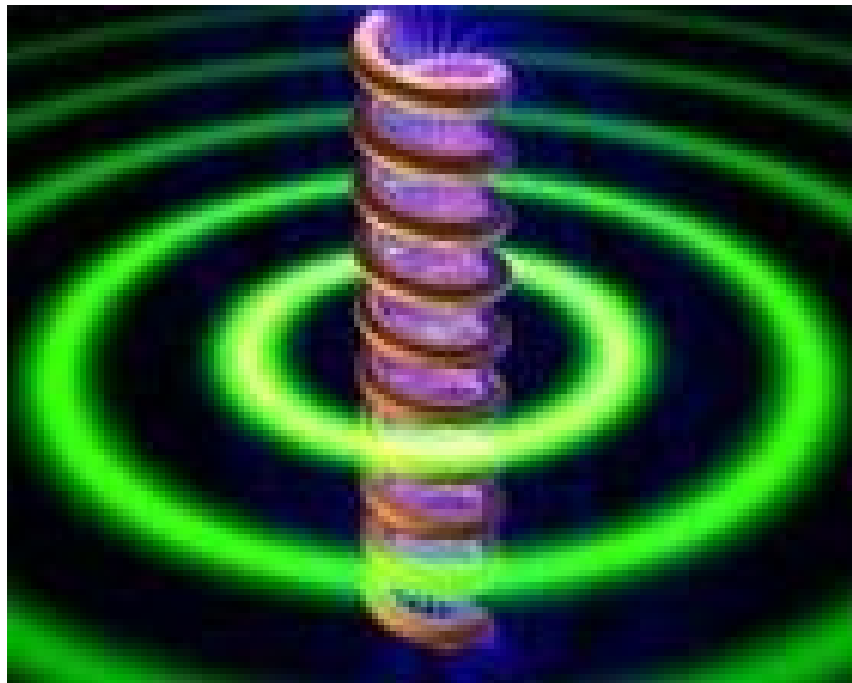
Only “height differences” matter to him.

A plays the role of a “zero level” benchmark.

This gets interesting for the solenoid.

If there is no current in the solenoid, then $\mathbf{B} = \mathbf{0}$ everywhere and we can take $\mathbf{A} = \mathbf{0}$ everywhere.

For a solenoid with current flowing through it, however,

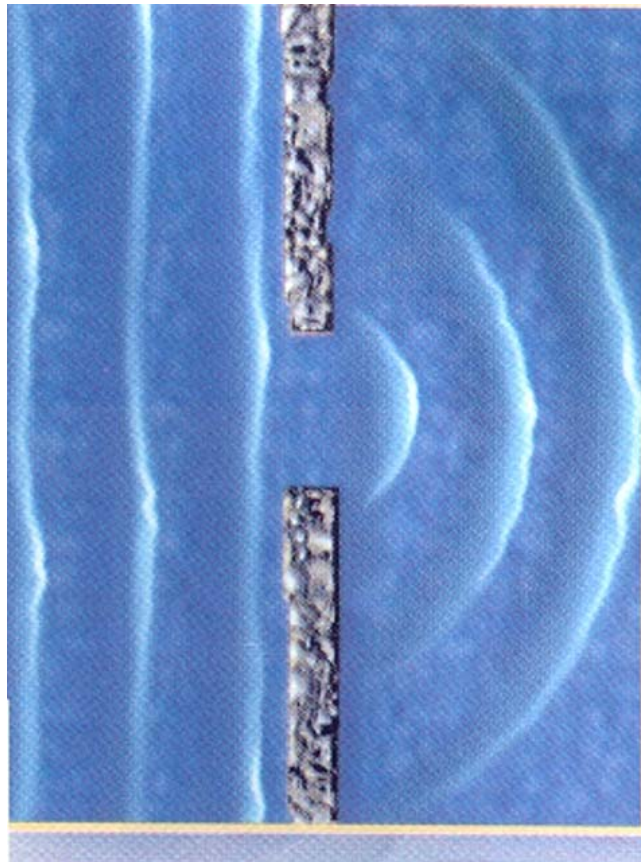


Potential for a Solenoid

even though \mathbf{B} is zero outside the solenoid, \mathbf{A} *cannot* be taken to be zero there (because the potential inside and the potential outside must “match up” on the outer surface of the solenoid).

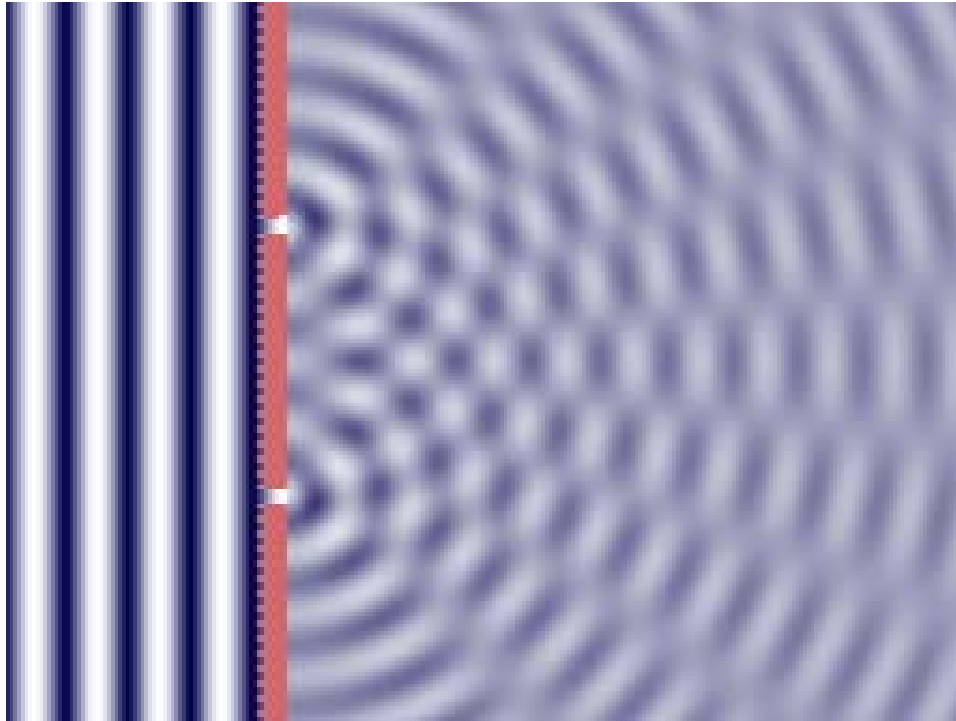
Now for something (apparently) quite different.

WATER WAVES:



1-Slit Experiment

A cork placed anywhere in the water will bob up and down periodically, but will not move to the left or right.



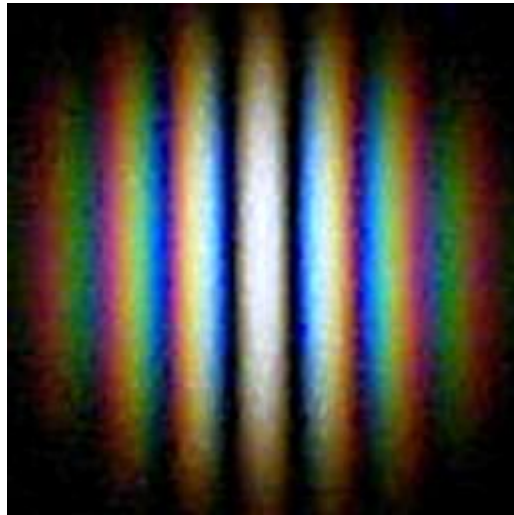
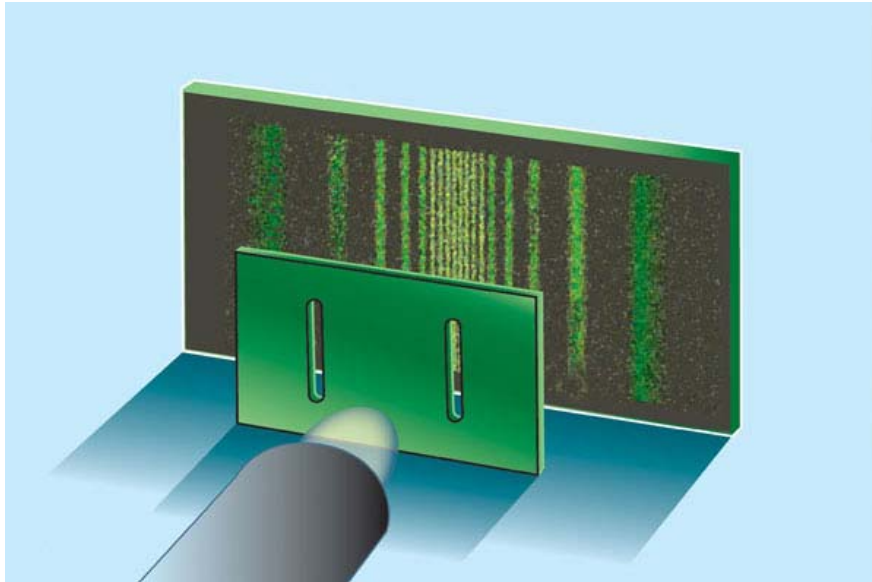
2-Slit Experiment: Interference

Notice the radial lines in the 2-slit experiment where the waves seem to have cancelled completely.

This is what waves do; they “interfere” with each other, “constructively” if they meet in phase and “destructively” if they meet out of phase.



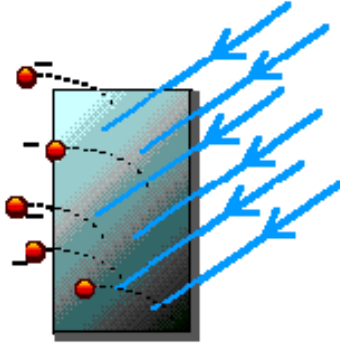
LIGHT BEHAVES LIKE A WAVE (SOMETIMES):



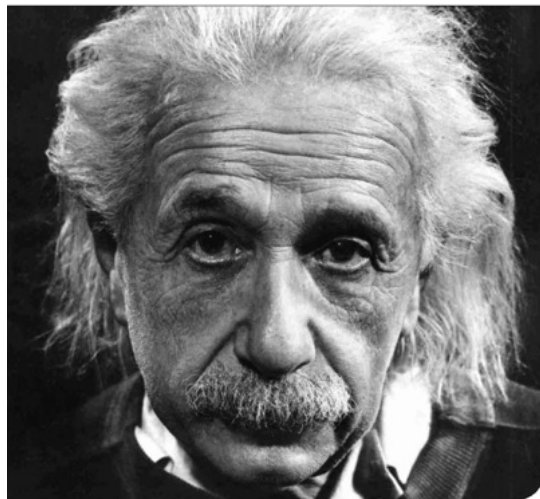
2-Slit Experiment for Light

A simple home experiment: Diffraction by a double hole. Two tiny holes were pierced with a sewing needle into an aluminum wrapping foil. Looking through these holes at a "point-like" light source, one can see a diffraction image similar to that shown here. For this image, the pierced foil has been fastened before the front lens of a digital camera, the light source was the reflection of a halogen lamp in a small silvery glass ball (as used for Christmas-tree decoration). The diameter of the holes was 0.07 mm, the centre-to-centre distance 0.3 mm.

LIGHT BEHAVES LIKE PARTICLES (SOMETIMES):



The Photoelectric Effect



Guess Who

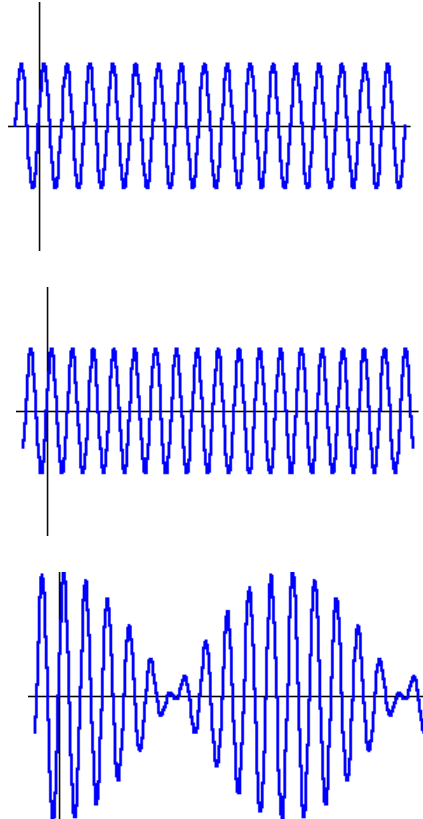
A GRADUATE STUDENT'S CRAZY IDEA:

Maybe *everything* behaves like a wave sometimes and like a particle sometimes.



Louis de Broglie

Waves behaving like particles? Well, maybe:

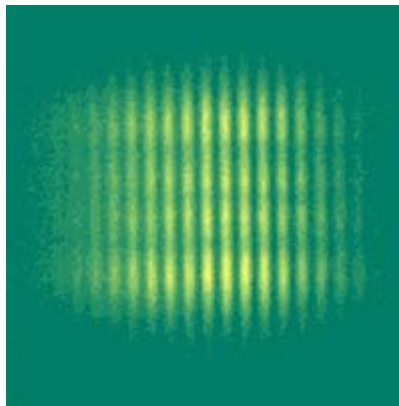


But, particles behaving like waves?

Sounds crazy, right?

Try sending a beam of *electrons* through two slits in a wall.
What would you expect to see at a screen past the wall?

Probably not what you actually do see:



2-Slit Experiment with Electrons

Bright spots are points where more electrons hit the screen.

Better yet, bright spots are points where the *probability* of detecting an electron (with, say, a Geiger counter) is greater.

*The electrons travel like waves and “interfere”,
but hit the screen like particles. Go figure!*

“Because atomic behavior is so unlike ordinary experience, it is very difficult to get used to, and it appears peculiar and mysterious to everyone - both to the novice and to the experienced physicist.”



Richard Feynman

The thing that is actually “waving” this time is what physicists call the *state* of the electron. They describe it with what they call the electron’s *wave function*, invariably denoted

$\psi(\mathbf{x}, t)$

and believed to contain all of the information one could ever obtain about the behavior of the electron. This information is encoded, at each point \mathbf{x} in space and each instant t of time, in the *complex number* $\psi(\mathbf{x}, t)$.

In this simple observation is the germ of an idea that evolved into one of the most fundamental principles of modern physics:

The Principle of Local Gauge Invariance

This principle is vastly more general than the context in which we find ourselves at the moment.

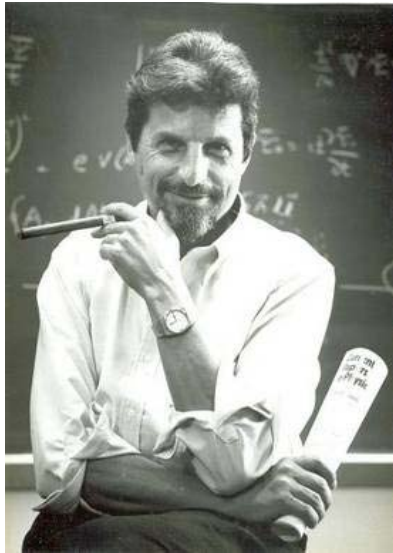
It is a guide to the construction of all the fundamental theories of modern physics (such as the Standard Model of Elementary Particles).

For us, here and now, we interpret the principle in the following way.

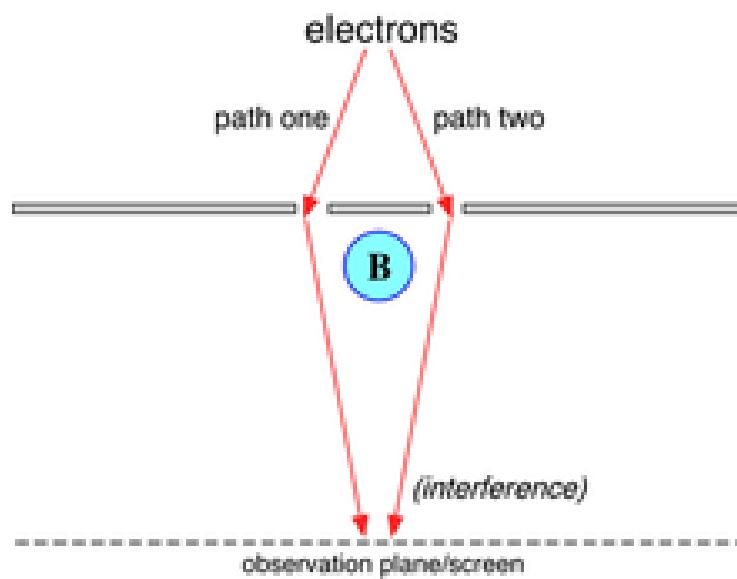
For a single electron, alone in the world, the phase doesn't count; change anyway you like in \mathbf{x}, t and the physical predictions are the same. However, phase differences determine how two waves (water waves, light waves, or wave functions) interfere when they get together.

Now for another experiment!

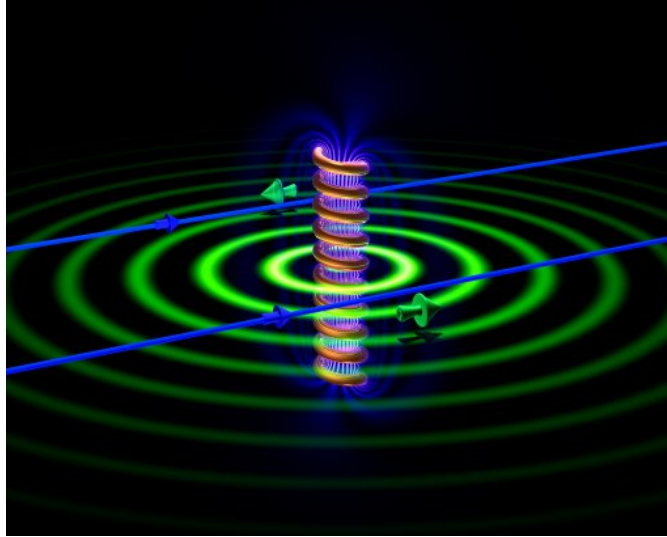
AHARONOV



BOHM

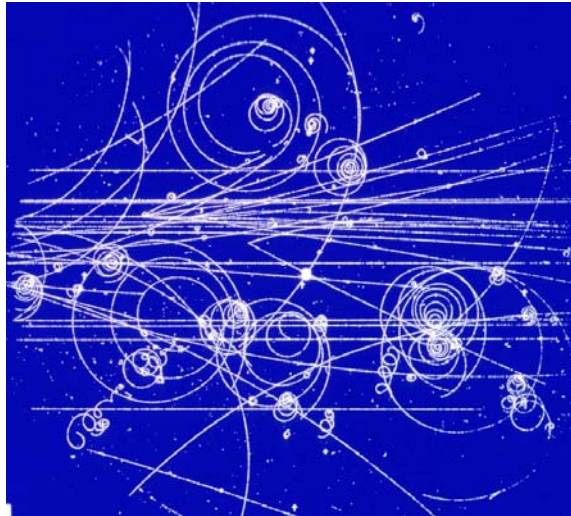


2-Slit Experiment with a Solenoid



The magnetic field \mathbf{B} is contained entirely inside the solenoid and the solenoid is shielded to keep the electrons out.

The electrons cannot “feel” \mathbf{B} at all so they won’t do anything crazy like this:



The motion of the electrons is unaffected by \mathbf{B} .

Remarkably, however, the electrons “know” that \mathbf{B} is there because, when they meet again at the screen, the interference pattern is *different* if there is a current flowing through the solenoid than it is when there is no current (i.e., when $\mathbf{B} = \mathbf{0}$).

What the electrons do encounter on their journey is the *magnetic potential* \mathbf{A} , which is *not* zero outside the solenoid, and they respond to it *not* by being pushed and pulled around in space, but by experiencing *phase changes* which alter the interference pattern at the screen, i.e. changes in

$$e^{-i\int \mathbf{A} \cdot d\mathbf{l}}$$

\mathbf{A} is a funny kind of “force” affecting not the motion, but rather the “internal structure” of the electrons. It is, in other words, the simplest example of what would now be called a

GAUGE FIELD.

Such fields are at the heart of much of contemporary theoretical physics. The phase of an electron determines how it interacts with other electrons, but the “internal structure” of an elementary particle is generally much more complex, determining its very identity as a particle.

ASIDE:

There is a very deep connection between the Principle of Local Gauge Invariance and the existence of Gauge Fields. This connection resides in a mathematical property of the equations that wave functions are required to satisfy (Schrödinger equation, Dirac equation, etc.). For the Schrödinger equation for a charged particle in a magnetic field, this is what it says:

, is a solution to

$$\nabla^2 \psi + \frac{1}{2} \psi = 0$$

if and only if

$$\nabla \psi + \frac{1}{2} \psi = 0$$

is a solution to

$$\nabla^2 \psi + \frac{1}{2} \psi = 0$$

This is proved in Appendix B.

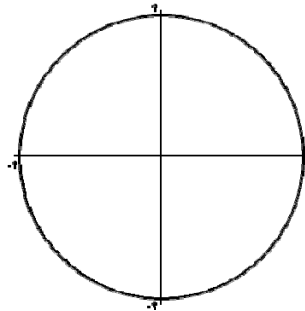
Gauge fields have reshaped the way physicists view the world at its most fundamental level.

Mathematicians have found that their world also has been reshaped by gauge fields, but in this world they are viewed quite differently. Let's see how.

Notice that the phase factors

$$e^{-i\phi}$$

are just complex numbers with amplitude 1 so we can picture them as points on the circle of radius 1 in the plane.



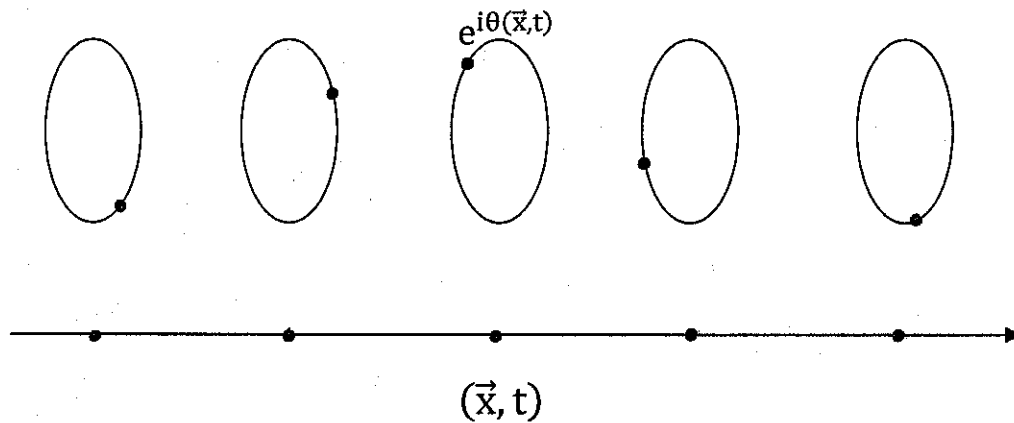
This collection of complex numbers is sometimes denoted $U(1)$ and is an example of what is called a “Lie group”.

The phase factor

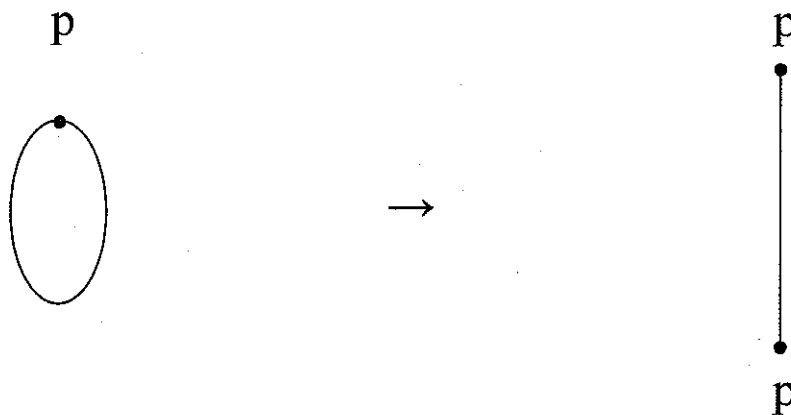
$$e^{-i\phi}$$

for any electron at any point \mathbf{x} and at any time t is just a dot on the circle.

Let's imagine a possible path for an electron and let's place above every point on it one of these circles as a sort of notebook in which to record the electron's phase factor at that point.

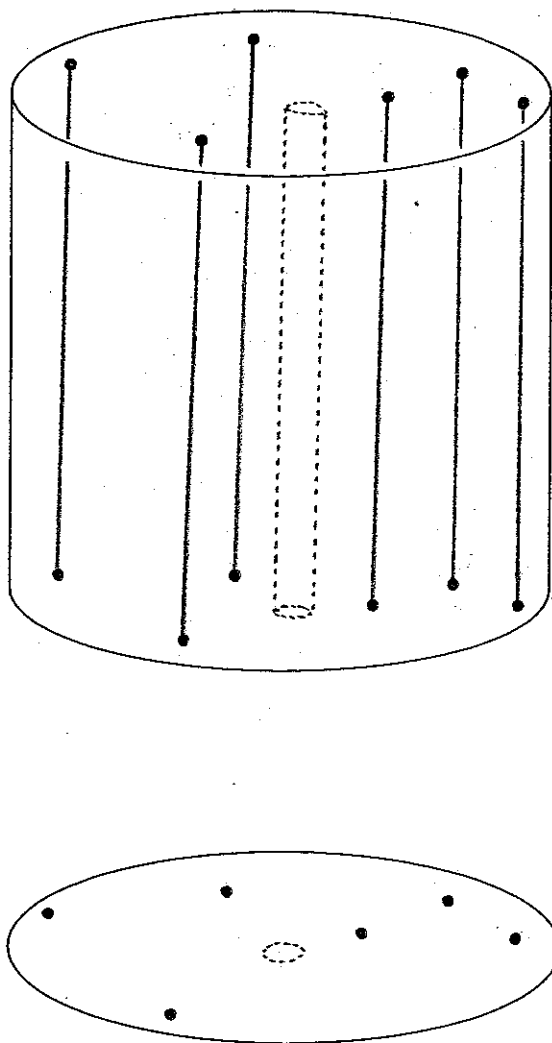


In fact, there are probably lots of electrons around so we should really put a circle above *every* point. This might be a little tough to draw unless we agree to picture a circle this way:



(a line segment with the endpoints glued together)

Now let's put such a "circle" above every point where the Aharonov-Bohm electrons can move.

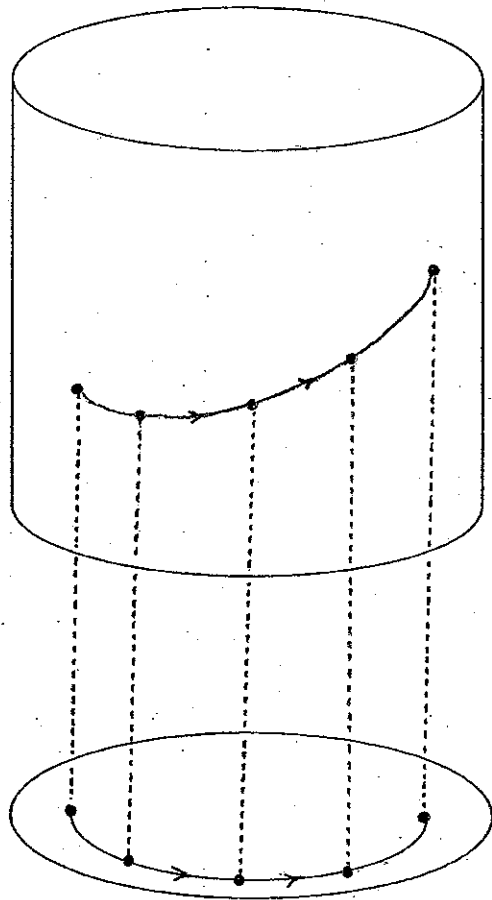


Think of it as a "bundle" of circles above the space in which the electrons can move.

Now we can record the phase factor for any electron that passes through one of these points as a dot on the circle above that point.

Imagine a path that might be followed by one of our electrons. As the path is traversed, the magnetic potential A dictates how the phase of the electron varies and we record these variations in the notebooks up above.

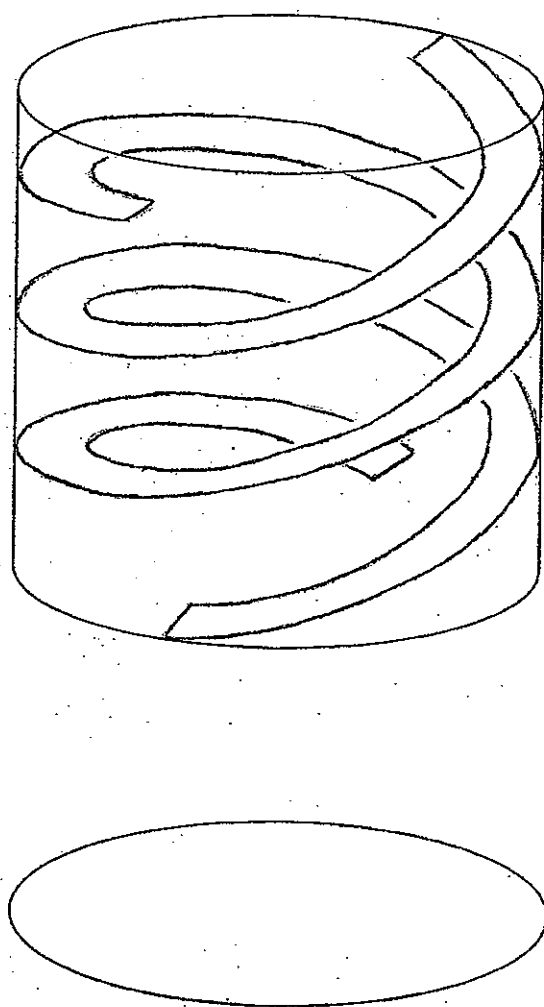
The result is a path through the bundle of circles that sets above the path of the electron.



Think of A as providing a “path lifting procedure” from the space in which the electrons move to the bundle of circles; the path below records the position and the path above records the phase.

A provides the bundles of circles with a structure that guides the evolution of the phase along any path an electron chooses to follow.

Think of it as a family of ramps in a parking structure.



All of these pictures look very familiar to mathematicians, although they would use different words to describe them.

Bundle of Circles = **Principal U(1)-Bundle**

Path Lifting Procedure = **Connection**

Magnetic Field = **Curvature**

For a mathematician, a *gauge field* is just a *connection on a principal bundle* and these were intensively studied by geometers and topologists long before anyone noticed that physicists were studying the same things for entirely different reasons.

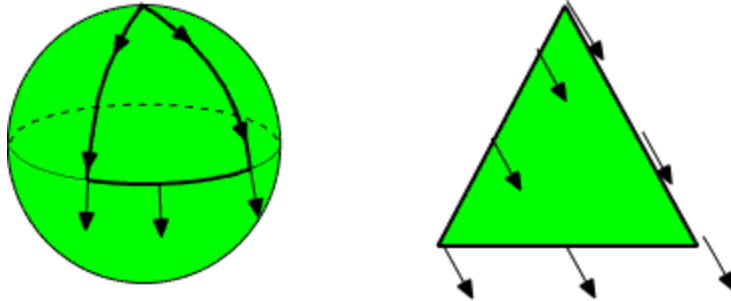


Elie Cartan



Charles Ehresmann

The reason for the terminology can be found in the geometrical phenomenon that Cartan and Ehresmann were generalizing.



Unlike the plane, the sphere is curved and this “curvature” can be detected by “parallel translating” a tangent vector around a closed curve on the sphere and noting an “angular shift” in its position.

Picturing the tangent planes to the sphere as sitting “above” the sphere (to form a “bundle of planes”) this parallel translation amounts to a path lifting from the sphere to the bundle which “connects” tangent planes at different points and for which angular shifts measure “curvature”.

Eventually, of course, someone did notice that the physicists and mathematicians were doing the same thing and then



Gauge Theory as a branch of mathematics exploded upon the scene, serving up some of the deepest results in all of topology and geometry and sparking an intense exchange between mathematics and physics that remains unabated to this day.

A few of the things I would like to tell you about now are

- Yang-Mills Theory
- Donaldson Theory
- Witten's Topological Quantum Field Theory
- Seiberg-Witten Theory
- The Witten Conjecture

but that, as they say, is another story.

Thanks for listening

RECOMMENDED READING

1. Schumm, Bruce A., **Deep Down Things**, The Johns Hopkins University Press, Baltimore and London, 2004.
2. Bernstein, Herbert J and Anthony V. Phillips, “Fiber Bundles and Quantum Theory,” *Scientific American*, July, 1981.
3. ‘t Hooft, G., “Gauge Theories of the Forces between Elementary Particles,” *Scientific American*, Volume 242, Issue 6, 1980, 90-116.
4. Feynman, Richard P., Robert B Leighton and Matthew Sands, **The Feynman Lectures in Physics: Quantum Mechanics**, Addison-Wesley Publishing Company, Reading, Massachusetts, 1965.