GENERALIZED DUISTERMAAT-HECKMAN $\Rightarrow$ DUISTERMAAT-HECKMAN I

COMMON ASSUMPTIONS : M IS A COMPACT MANIFOLD OF DIMENSION $n = 2k$ WITH SYMPLECTIC FORM σ AND ORIENTED BY THE LIOUVILLE FORM

$$dV_\sigma = \frac{1}{k!}\,\sigma \wedge \overset{k}{\cdots} \wedge \sigma$$

DUISTERMAAT-HECKMAN :

$H \in C^\infty(M)$ IS A MORSE FUNCTION

V_H IS THE HAMILTONIAN VECTOR FIELD OF H $(dH = \iota_{V_H}\sigma)$ AND IS ASSUMED TO HAVE A PERIODIC FLOW

THEN $\forall\, T > 0$

$$\int_M e^{iTH} = \sum_{dH(p)=0} \left(\frac{2\pi}{T}\right)^k \frac{e^{\pi i\,(\mathrm{SGN}(\mathcal{H}_H(p)))/4}\, e^{iTH(p)}}{\sqrt{|\det(\mathcal{H}_H(p)(e_i,e_j))|}}$$

WHERE $\mathcal{H}_H(p)$ IS THE HESSIAN OF H AT p AND $\{e_1,\dots,e_{2k}\}$ IS A BASIS FOR $T_p(M)$ WITH $(dV_\sigma)_p(e_1,\dots,e_{2k}) = 1$.

GENERALIZED DUISTERMAAT-HECKMAN :

G IS A COMPACT LIE GROUP

THERE IS A HAMILTON ACTION OF G ON M $(\mu : \mathfrak{g} \to C^\infty(M))$

$\xi \in \mathfrak{g}$ IS SUCH THAT $Z(\xi^\#)$ IS DISCRETE

THEN

$$\int_M e^{i\mu(\xi)} dV_\sigma = \sum_{\xi^\#(p)=0} (2\pi i)^k \frac{e^{i\mu(\xi)(p)}}{\mathrm{Pf}(L_p(\xi))}$$

ASSUME THAT THE GENERALIZED DUISTERMAAT-HECKMAN THEOREM HAS BEEN PROVED, H IS A MORSE FUNCTION ON M AND V_H HAS PERIODIC FLOW.

JUST AS FOR THE HEIGHT FUNCTION ON S^2,

$$\text{PERIODIC FLOW} \longrightarrow S^1\text{-ACTION ON } M$$

AND THE ACTION IS HAMILTONIAN

$$\mu : \mathrm{Lie}(S^1) \to C^\infty(M)$$

$$\mu(\xi) = \mu(ia) = -aH$$

$\xi = ia \Rightarrow \xi^\# = V_{-aH} = -aV_H$ VANISHES ONLY AT THE CRITICAL POINTS OF H WHICH ARE ISOLATED AND SO, BY COMPACTNESS OF M, DISCRETE.

APPLY GENERALIZED DUISTERMAAT-HECKMAN WITH $\xi = i(-T)$, $T > 0$, SO THAT $\mu(\xi) = TH$.

$$\int_M e^{iTH} dV_\sigma = \sum_{\xi^\#(p)=0} (2\pi i)^k \frac{e^{iTH(p)}}{\mathrm{Pf}(L_p(-iT))}$$

$$= \sum_{dH(p)=0} \left(\frac{2\pi}{T}\right)^k \frac{(iT)^k e^{iTH(p)}}{\mathrm{Pf}(L_p(-iT))}$$

ALL THAT REMAINS IS TO SHOW THAT

$$\frac{(iT)^k}{Pf(L_p(-iT))} = \frac{e^{\pi i (\mathrm{SGN}(\mathcal{H}_H(p)))/4}}{\sqrt{|\det(\mathcal{H}_H(p)(e_i, e_j))|}}$$

THIS BEGINS WITH A

<u>LEMMA</u> : H A MORSE FUNCTION, V_H ITS HAMILTONIAN VECTOR FIELD, $p \in Z(V_H) = \mathrm{CRIT}(H)$ AND $\mathcal{H}_H(p) : T_p(M) \times T_p(M) \to \mathbb{R}$ THE HESSIAN OF H AT p. THEN

$$\mathcal{H}_H(p)(v_p, w_p) = -\sigma_p\left((\mathcal{L}_{V_H} V)_p , w_p \right)$$

WHERE V IS ANY VECTOR FIELD ON M WITH $V(p) = v_p$.

NOTHING DEEP HERE. JUST STRING TOGETHER HALF-A-DOZEN "WELL-KNOWN" IDENTITIES FROM GEOMETRY ($\mathcal{L}_V \circ \iota_W = \iota_{[V,W]} + \iota_W \circ \mathcal{L}_V$, ETC.)

NOW,

$$L_p(-iT)(v_p) = (\mathcal{L}_{(-iT)^*} V)_p = (\mathcal{L}_{TV_H} V)_p$$
$$= T(\mathcal{L}_{V_H} V)_p$$

SO

$$\boxed{T\mathcal{H}_H(p)(v_p, w_p) = -\sigma_p\left(L_p(-iT)(v_p), w_p\right)}$$

IN PARTICULAR, IF $\{e_1, \dots, e_{2k}\}$ IS ANY BASIS FOR $T_p(M)$,

$$T \mathcal{H}_H(p)(e_i, e_j) = -\sigma_p\,(L_p(-iT)(e_i), e_j)$$

WRITING THESE OUT IN MATRIX FORM AND TAKING DETERMINANTS GIVES

$$T^{2k} \det(\mathcal{H}_H(p)(e_i, e_j)) = \det(-L_p(-iT)) \det(\sigma_p(e_i, e_j))$$

NOW CHOOSE AN ORIENTED, $\langle\ ,\ \rangle_{S'}$ - ORTHONORMAL BASIS $\{e_1, \dots, e_{2k}\}$ RELATIVE TO WHICH

$$(\sigma_p(e_i, e_j)) = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

AND THE MATRIX OF $L_p(-iT)$ HAS THE FORM

$$\begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & \lambda_k \\ & & & -\lambda_k & 0 \end{pmatrix}$$

FOR NONZERO REAL NUMBERS $\lambda_1, \dots, \lambda_k$. THEN

$$T^{2k} \det(\mathcal{H}_H(p)(e_i, e_j)) = \lambda_1^2 \cdots \lambda_k^2$$

$$T^k \sqrt{|\det(\mathcal{H}_H(p)(e_i, e_j))|} = \text{SIGN}(\lambda_1 \cdots \lambda_k)\, \mathrm{Pf}(L_p(-iT))$$

NOW ALL THAT REMAINS IS TO SHOW THAT

$$\text{SIGN}(\lambda_1 \cdots \lambda_k) = (-i)^k e^{\pi i (\text{SGN}(\mathcal{H}_H(p)))/4}$$

THIS IS EASY TO CHECK WHEN $k = 1$ BECAUSE

$$\begin{pmatrix} T\mathcal{H}_H(p)(e_1, e_1) & T\mathcal{H}_H(p)(e_1, e_2) \\ T\mathcal{H}_H(p)(e_2, e_1) & T\mathcal{H}_H(p)(e_2, e_2) \end{pmatrix} = \begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

AND THE GENERAL CASE FOLLOWS BY INDUCTION.

FINALLY, NOTING THAT $\text{SGN}(\mathcal{H}_H(p))$ IS INDEPENDENT OF THE CHOICE OF BASIS AND $\det(\mathcal{H}_H(p)(e_i, e_j))$ IS THE SAME FOR ALL BASES SATISFYING $(dV_\sigma)_p(e_1, \ldots, e_{2k}) = 1$ COMPLETES THE PROOF.