GENERALIZED DUISTERMAAT-HECKMAN $\Rightarrow$ DUISTERMAAT-HECKMAN II

DUISTERMAAT-HECKMAN THEOREM : LET M BE A COMPACT MANIFOLD OF DIMENSION $n = 2k$ WITH SYMPLECTIC FORM σ AND ORIENTED BY THE LIOUVILLE FORM $dV_\sigma = \frac{1}{k!}\sigma \wedge \overset{k}{\cdots} \wedge \sigma$. LET $H \in C^\infty(M)$ BE A MORSE FUNCTION AND V_H ITS HAMILTONIAN VECTOR FIELD ($dH = \iota_{V_H}\sigma$). ASSUME THE FLOW OF V_H IS PERIODIC. THEN

$$(1) \quad \int_M e^{iTH}\, dV_\sigma = \sum_{\substack{p \in M \\ dH(p)=0}} \left(\frac{2\pi}{T}\right)^k \frac{e^{\pi i(\mathrm{sgn}(\mathcal{H}_H(p)))/4}\, e^{iTH(p)}}{\sqrt{|\det(\mathcal{H}_H(p)(e_i,e_j))|}}$$

FOR ANY $T > 0$, WHERE $\mathcal{H}_H(p) : T_p(M) \times T_p(M) \longrightarrow \mathbb{R}$ IS THE HESSIAN OF H AT p AND $\{e_1, \dots, e_{2k}\}$ IS A BASIS FOR $T_p(M)$ WITH $(dV_\sigma)_p(e_1, \dots, e_{2k}) = 1$.

GENERALIZED DUISTERMAAT-HECKMAN THEOREM : LET M BE A COMPACT MANIFOLD OF DIMENSION $n = 2k$ WITH SYMPLECTIC FORM σ AND ORIENTED BY THE LIOUVILLE FORM $dV_\sigma = \frac{1}{k!}\sigma \wedge \overset{k}{\cdots} \wedge \sigma$. LET G BE A COMPACT LIE GROUP AND SUPPOSE THERE IS A HAMILTONIAN ACTION OF G ON M WITH CORRESPONDING SYMPLECTIC MOMENTS GIVEN BY $\mu : \mathfrak{g} \to C^\infty(M)$. IF $\xi \in \mathfrak{g}$ IS SUCH THAT THE ZERO SET OF $\xi^\#$ ($\xi^\#(p) = \frac{d}{dt}(\exp(-t\xi)\cdot p)|_{t=0}$) CONSISTS OF A FINITE NUMBER OF POINTS, THEN

$$(2) \quad \int_M e^{i\mu(\xi)}\, dV_\sigma = \sum_{\substack{p \in M \\ \xi^\#(p)=0}} \frac{(2\pi i)^k\, e^{i\mu(\xi)(p)}}{\mathrm{Pf}(L_p(\xi))}$$

WHERE $L_p(\xi) : T_p(M) \longrightarrow T_p(M)$ IS GIVEN BY $L_p(\xi)(v_p) = (\mathcal{L}_{\xi^\#}V)_p = [\xi^\#, V]_p$ FOR ANY VECTOR FIELD V ON M WITH $V(p) = v_p$.

WE WILL PROVE THAT THE DUISTERMAAT - HECKMAN THEOREM FOLLOWS FROM THE GENERALIZED DUISTERMAAT - HECKMAN THEOREM. FIRST WE GENERALIZE THE DISCUSSION OF THE HEIGHT FUNCTION ON S^2 TO FIND THE APPROPRIATE GROUP ACTION.

THE MORSE FUNCTION H ON M IS ASSUMED TO HAVE A HAMILTONIAN VECTOR FIELD V_H WITH PERIODIC FLOW AND THIS GIVES RISE TO AN S^1-ACTION ON M ($e^{iT} \cdot p$ PUSHES p T UNITS ALONG THE UNIQUE INTEGRAL CURVE OF V_H STARTING AT p). THUS, WE WILL APPLY THE GENERALIZED DUISTERMAAT - HECKMAN THEOREM WITH $G = S^1$.

WE CLAIM THAT THIS S^1-ACTION ON M IS HAMILTONIAN. WE IDENTIFY THE LIE ALGEBRA OF S^1 WITH $\mathrm{Im}\,\mathbb{C} = \{ ia : a \in \mathbb{R} \}$ AND WILL SHOW THAT

$$\xi = ia \quad \Rightarrow \quad \xi^{\#} = V_{-aH}$$

(THE HAMILTONIAN VECTOR FIELD OF THE FUNCTION $-aH$). TO SEE THIS, RECALL THAT THE S^1-ACTION OF e^{iT} MOVES p A PARAMETER DISTANCE T ALONG THE INTEGRAL CURVE OF V_H THAT STARTS AT p. THUS,

$$\begin{aligned} (i(-1))^{\#}(p) &= \frac{d}{dt}(\exp(-t(-i)) \cdot p)\Big|_{t=0} \\ &= \frac{d}{dt}(e^{it} \cdot p)\Big|_{t=0} \\ &= V_H(p) \end{aligned}$$

MOREOVER, $V_{-aH} = -a V_H$ BECAUSE

$$d(-aH) = -a\,dH = -a\,\iota_{V_H}\sigma = \iota_{-aV_H}\sigma .$$

THUS,

$$\xi^{\#}(p) = (ia)^{\#}(p) = -a\,(i(-1))^{\#}(p) = -a\,V_H(p) = V_{-aH}(p) .$$

WE ARE THEREFORE LED TO DEFINE

$$\mu : \operatorname{Im}\mathbb{C} \longrightarrow C^{\infty}(M)$$

$$\mu(ia) = -aH$$

THIS IS SURELY LINEAR AND WE HAVE JUST VERIFIED THAT

$$d\mu(\xi) = \iota_{\xi^{\#}}\sigma .$$

IT REMAINS TO SHOW THAT μ IS EQUIVARIANT, I.E.,

$$\mu(g\cdot\xi) = g\cdot\mu(\xi)$$

$\forall g \in S^1\ \forall \xi \in \operatorname{Im}\mathbb{C}$. BUT $g\cdot\xi = g\xi g^{-1} = \xi$ BECAUSE S^1 IS ABELIAN SO

$$\mu(g\cdot\xi) = \mu(\xi).$$

MOREOVER, THE ACTION OF S^1 ON $C^{\infty}(M)$ IS GIVEN BY $(g\cdot f)(p) = f(g^{-1}\cdot p)$ SO

$$\begin{aligned}
(g\cdot\mu(\xi))(p) &= \mu(\xi)(g^{-1}\cdot p)\\
&= \mu(ia)(g^{-1}\cdot p)\\
&= (-aH)(g^{-1}\cdot p)\\
&= -a\,H(g^{-1}\cdot p)\\
&= -a\,H(p) \qquad \text{(SEE THE REMARK BELOW)}\\
&= \mu(\xi)(p)
\end{aligned}$$

BECAUSE H IS CONSTANT ON THE S^1-ORBITS (INTEGRAL CURVES OF V_H) DUE TO

$$V_H(H) = dH(V_H) = (\iota_{V_H}\sigma)(V_H) = \sigma(V_H, V_H) = 0.$$

THUS, WE HAVE A HAMILTONIAN ACTION OF S^1 ON M. MOREOVER, EVERY NONZERO ELEMENT $\xi = ia$ OF THE LIE ALGEBRA HAS $\xi^{\#}$ WITH THE PROPERTY THAT $Z(\xi^{\#})$ IS FINITE SINCE $\xi^{\#} = V_{-aH} = -a V_H$ WHICH VANISHES ONLY AT THE CRITICAL POINTS OF THE MORSE FUNCTION H (WHICH ARE ISOLATED AND SO, BY COMPACTNESS OF M, FINITE IN NUMBER).

NOW WE APPLY THE GENERALIZED DUISTERMAAT - HECKMAN THEOREM. WITH $T > 0$ AND $\xi = i(-T) \in \operatorname{Im}\mathbb{C}$ WE HAVE $\mu(\xi) = TH$ SO

$$\int_M e^{iTH}\, dV_\sigma = \sum_{\substack{p \in M \\ \xi^{\#}(p)=0}} \frac{(2\pi i)^k\, e^{iTH(p)}}{Pf(L_p(-iT))}$$

$$= \sum_{\substack{p \in M \\ dH(p)=0}} \left(\frac{2\pi}{T}\right)^k \frac{(iT)^k\, e^{iTH(p)}}{Pf(L_p(-iT))}$$

WHICH WILL REDUCE TO THE OLD-FASHIONED DUISTERMAAT - HECKMAN THEOREM IF WE SHOW THAT

(3) $$\frac{(iT)^k}{Pf(L_p(-iT))} = \frac{e^{\pi i(\operatorname{sgn}(\mathcal{H}_H(p)))/4}}{\sqrt{|\det \mathcal{H}_H(p)(e_i, e_j)|}}$$

THIS TAKES A BIT OF PROVING, HOWEVER, AND WE BEGIN THE PROCESS WITH A LEMMA.

LEMMA: LET H BE MORSE, V_H ITS HAMILTONIAN VECTOR FIELD, $p \in Z(V_H)$ AND $\mathcal{H}_H(p) : T_p(M) \times T_p(M) \longrightarrow \mathbb{R}$ THE HESSIAN OF H AT p. THEN

$$\mathcal{H}_H(p)(v_p, w_p) = -\sigma_p\left((\mathcal{L}_{V_H} V)_p, w_p\right),$$

WHERE V IS ANY VECTOR FIELD ON M WITH $V(p) = v_p$.

PROOF: BY DEFINITION,

$$\mathcal{H}_H(p)(v_p, w_p) = v_p(W(H))$$

FOR ANY VECTOR FIELD W ON M WITH $W(p) = w_p$. NOW,

$$W(H) = \mathcal{L}_W(H) = dH(W) = (\iota_{V_H}\sigma)(W) = (\iota_W \circ \iota_{V_H})(\sigma).$$

WE COMPUTE

$$V(W(H)) = \mathcal{L}_V(W(H)) = \underbrace{\mathcal{L}_V \circ \iota_W}_{\iota_{[V,W]} + \iota_W \circ \mathcal{L}_V} \circ\, \iota_{V_H}(\sigma)$$

$$= \iota_{[V,W]} \circ \iota_{V_H}(\sigma) + \iota_W \circ \underbrace{\mathcal{L}_V \circ \iota_{V_H}}_{\iota_{[V,V_H]} + \iota_{V_H} \circ \mathcal{L}_V}(\sigma)$$

$$= \sigma(V_H, [V,W]) - \sigma([V_H, V], W) + (\mathcal{L}_V\sigma)(V_H, W).$$

NOW EVALUATE AT p AND USE $V_H(p) = 0$ TO GET

$$\mathcal{H}_H(p)(v_p, w_p) = 0 - \sigma_p([V_H, V]_p, w_p) + 0$$

$$= -\sigma_p((\mathcal{L}_{V_H} V)_p, w_p). \qquad \square$$

NOW, FOR ANY $T > 0$,

$$L_p(-iT)(\omega_p) = (\mathcal{L}_{(-iT)^\#} V)_p = (\mathcal{L}_{T(i(-1))^\#} V)_p$$

$$= T(\mathcal{L}_{(i(-1))^\#} V)_p$$

$$= T(\mathcal{L}_{V_H} V)_p \qquad \text{(SEE PAGE 2)}$$

SO THE LEMMA GIVES

$$\text{(4)} \qquad \mathcal{H}_H(p)(\omega_p, \omega_p) = \frac{1}{T}(-\sigma_p(L_p(-iT)(\omega_p), \omega_p))$$

IN PARTICULAR, IF $\{e_1, \ldots, e_{2k}\}$ IS <u>ANY</u> BASIS FOR $T_p(M)$,

$$T\,\mathcal{H}_H(p)(e_i, e_j) = -\sigma_p(L_p(-iT)(e_i), e_j)$$

$\forall\, i, j = 1, \ldots, 2k$. WRITING

$$L_p(-iT)(e_i) = L_i^\ell e_\ell$$

THIS BECOMES

$$T\,\mathcal{H}_H(p)(e_i, e_j) = -L_i^\ell\, \sigma_p(e_\ell, e_j)$$

$\forall\, i, j = 1, \ldots, 2k$. WRITING THESE OUT AS A MATRIX EQUATION GIVES

$$\text{(5)} \quad \begin{pmatrix} T\,\mathcal{H}_H(p)(e_1,e_1) & \cdots & T\,\mathcal{H}_H(p)(e_1,e_{2k}) \\ \vdots & & \vdots \\ T\,\mathcal{H}_H(p)(e_{2k},e_1) & \cdots & T\,\mathcal{H}_H(p)(e_{2k},e_{2k}) \end{pmatrix} = \begin{pmatrix} -L_1^1 & \cdots & -L_1^{2k} \\ \vdots & & \\ -L_{2k}^1 & \cdots & -L_{2k}^{2k} \end{pmatrix} \begin{pmatrix} \sigma_p(e_1,e_1) & \cdots & \sigma_p(e_1,e_{2k}) \\ \vdots & & \vdots \\ \sigma_p(e_{2k},e_1) & \cdots & \sigma_p(e_{2k},e_{2k}) \end{pmatrix}$$

TAKING DETERMINANTS ON BOTH SIDES GIVES

$$(6) \qquad T^{2k} \det\left(\mathcal{H}_H(p)(e_i, e_j)\right) = \det(L^\ell{}_i)\det\left(\sigma_p(e_\ell, e_j)\right)$$

(NO SUM OVER ℓ HERE, OF COURSE).

NOW WE NEED TO MAKE A SPECIAL CHOICE OF THE BASIS. WE WANT A BASIS $\{e_1, \dots, e_{2k}\}$ SATISFYING

1. $$\left(\sigma_p(e_\ell, e_j)\right) = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

I.E., $\sigma_p = e^1 \wedge e^2 + e^3 \wedge e^4 + \dots + e^{2k-1} \wedge e^{2k}$

<u>NOTE</u>: THE DARBOUX THEOREM GIVES A BASIS SATISFYING THIS CONDITION. ALSO NOTE THAT IT FOLLOWS FROM THIS THAT

$$\frac{1}{k!}\sigma_p \wedge \overset{k}{\cdots} \wedge \sigma_p = e^1 \wedge \dots \wedge e^{2k}$$

I.E.,

$$(dV_\sigma)_p(e_1, \dots, e_{2k}) = 1$$

SO $\{e_1, \dots, e_{2k}\}$ IS ORIENTED FOR THE LIOUVILLE FORM.

2. THE MATRIX $(L^\ell{}_i)$ OF $L_p(-iT)$ HAS THE FORM

$$(L^\ell{}_i) = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_k \\ & & & -\lambda_k & 0 \end{pmatrix}$$

<u>NOTE</u>: WE HAVE ALREADY SEEN THAT $L_p(-iT)$ IS SKEW-SYMMETRIC WITH RESPECT TO ANY

S^1- INVARIANT METRIC $\langle\ ,\ \rangle_{S^1}$ ON M AND SO IT HAS A MATRIX RELATIVE TO SOME ORIENTED, $\langle\ ,\ \rangle_{S^1}$ - ORTHONORMAL BASIS OF THE REQUIRED FORM. WE KNOW ALSO THAT $L_p(-iT)$ IS INVERTIBLE SO, IN ANY SUCH REPRESENTATION, NONE OF THE λ_i CAN BE ZERO.

THE TRICK IS TO FIND A SINGLE BASIS RELATIVE TO WHICH <u>BOTH</u> $(\sigma_p(e_\ell, e_j))$ AND $(L^\ell{}_i)$ HAVE THE REQUIRED FORM. WE WILL TURN TO THE EXISTENCE OF SUCH A BASIS SHORTLY, BUT FIRST WE ASSUME IT AND FINISH THE PROOF.

ASSUMING THAT $(\sigma_p(e_\ell, e_j))$ AND $(L^\ell{}_i)$ ARE OF THE REQUIRED FORM, (6) BECOMES

$$T^{2k} \det(\mathcal{H}_H(p)(e_i, e_j)) = \lambda_1^2 \cdots \lambda_k^2$$

SO

$$T^{k} |\det(\mathcal{H}_H(p)(e_i, e_j))|^{\frac{1}{2}} = \mathrm{SIGN}(\lambda_1 \cdots \lambda_k)\, \lambda_1 \cdots \lambda_k$$

(NOTE THAT $\det(\mathcal{H}_H(p)(e_i, e_j))$ IS ACTUALLY POSITIVE HERE). THUS, SINCE OUR BASIS IS ORIENTED WITH RESPECT TO THE LIOUVILLE FORM,

$$T^{k} |\det(\mathcal{H}_H(p)(e_i, e_j))|^{\frac{1}{2}} = \mathrm{SIGN}(\lambda_1 \cdots \lambda_k)\, \mathrm{Pf}(L_p(-iT))$$

OR

$$\frac{(iT)^k}{\mathrm{Pf}(L_p(-iT))} = \frac{i^k\, \mathrm{SIGN}(\lambda_1 \cdots \lambda_k)}{\sqrt{|\det(\mathcal{H}_H(p)(e_i, e_j))|}}$$

COMPARING THIS WITH (3) WE FIND THAT IT ONLY REMAINS TO SHOW

$$(7) \qquad \operatorname{SIGN}(\lambda_1 \cdots \lambda_k) = (-i)^k e^{\pi i (\operatorname{SGN}(\mathcal{H}_H(p)))/4}$$

FIRST SUPPOSE $k = 1$. THEN (5), WITH THE CHOICE OF BASIS JUST DESCRIBED, GIVES

$$\begin{pmatrix} T\mathcal{H}_H(p)(e_1,e_1) & T\mathcal{H}_H(p)(e_1,e_2) \\ T\mathcal{H}_H(p)(e_2,e_1) & T\mathcal{H}_H(p)(e_2,e_2) \end{pmatrix} = \begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$(\mathcal{H}_H(p)(e_i,e_j)) = \begin{pmatrix} \lambda_1/T & 0 \\ 0 & \lambda_1/T \end{pmatrix}$$

NOW CONSIDER TWO CASES:

1. $\lambda_1 > 0$ $(\operatorname{SIGN}(\lambda_1) = 1)$ THEN $\operatorname{SGN}(\mathcal{H}_H(p)) = 2$ SO

$$(-i)^1 e^{\pi i (\operatorname{SGN}(\mathcal{H}_H(p)))/4} = -i e^{\pi i/2} = e^{-\frac{\pi}{2} i} e^{\frac{\pi}{2} i} = 1$$
$$= \operatorname{SIGN}(\lambda_1)$$

2. $\lambda_1 < 0$ $(\operatorname{SIGN}(\lambda_1) = -1)$ THEN $\operatorname{SGN}(\mathcal{H}_H(p)) = -2$ SO

$$(-i)^1 e^{\pi i (\operatorname{SGN}(\mathcal{H}_H(p)))/4} = -i e^{-\pi i/2} = e^{-\frac{\pi}{2} i} e^{-\frac{\pi}{2} i} = -1$$
$$= \operatorname{SIGN}(\lambda_1)$$

FROM THIS THE GENERAL RESULT FOLLOWS BY INDUCTION. TO SEE THIS WRITE (5) IN THE SELECTED BASIS AS

$$(\mathcal{H}_H(p)(e_i, e_j)) = \begin{pmatrix} \lambda_1/T & 0 & & & & 0 \\ 0 & \lambda_1/T & & & & \\ & & \ddots & & & \\ & 0 & & & \lambda_k/T & 0 \\ & & & & 0 & \lambda_k/T \end{pmatrix}$$

so $\mathrm{SGN}\,(\mathcal{H}_H(p))/4 \;=\; \mathrm{SGN}\begin{pmatrix} \lambda_1/T & 0 \\ 0 & \lambda_1/T \end{pmatrix} + \cdots + \mathrm{SGN}\begin{pmatrix} \lambda_k/T & 0 \\ 0 & \lambda_k/T \end{pmatrix}$

AND THEREFORE

$$(-i)^k e^{\pi i(\mathrm{SGN}(\mathcal{H}_H(p)))/4} =$$

$$\left((-i)e^{\frac{\pi i}{4}\mathrm{SGN}\begin{pmatrix} \lambda_1/T & 0 \\ 0 & \lambda_1/T \end{pmatrix}}\right)\cdots\left((-i)e^{\frac{\pi i}{4}\mathrm{SGN}\begin{pmatrix} \lambda_k/T & 0 \\ 0 & \lambda_k/T \end{pmatrix}}\right).$$

THUS, (7) CAN BE WRITTEN

$$(\mathrm{SIGN}\,\lambda_1)\cdots(\mathrm{SIGN}\,\lambda_k) = \left((-i)e^{\frac{\pi i}{4}\mathrm{SGN}\begin{pmatrix} \lambda_1/T & 0 \\ 0 & \lambda_1/T \end{pmatrix}}\right)\cdots\left((-i)e^{\frac{\pi i}{4}\mathrm{SGN}\begin{pmatrix} \lambda_k/T & 0 \\ 0 & \lambda_k/T \end{pmatrix}}\right)$$

WHICH CLEARLY FOLLOWS FROM THE $k=1$ CASE AND INDUCTION.

AT THIS POINT WE HAVE PROVED THE RESULT FOR SPECIAL BASES OF THE TYPE DESCRIBED ON PAGES 7-8, BUT STILL HAVE NOT ESTABLISHED THE EXISTENCE OF SUCH BASES. WE WILL TURN TO THIS SHORTLY, BUT FIRST WE OBSERVE THAT THE RIGHT-HAND SIDE OF (1) IS INDEPENDENT OF THE CHOICE OF BASIS $\{e_1, \ldots, e_{2k}\}$ WITH $(dV_\sigma)_p(e_1, \ldots, e_{2k}) = 1$ SO THAT PROVING THE EXISTENCE OF OUR SPECIAL BASIS WILL FINISH THE PROOF THAT GENERALIZED D-H $\Rightarrow$ D-H.

FIRST NOTICE THAT, SINCE $\mathcal{H}_H(p)$ IS A NONDEGENERATE, SYMMETRIC, BILINEAR FORM ON $T_p(M)$, THE NUMBER OF NEGATIVE (POSITIVE) EIGENVALUES OF ANY MATRIX REPRESENTATION OF IT CAN BE (INVARIANTLY) DESCRIBED AS THE DIMENSION OF A MAXIMAL SUBSPACE ON WHICH $\mathcal{H}_H(p)$ IS NEGATIVE (POSITIVE) DEFINITE. THUS,

$$\mathrm{SGN}\,(\mathcal{H}_H(p))$$

IS ACTUALLY INDEPENDENT OF <u>ANY</u> CHOICE OF BASIS.

NOW, IF $\{\hat{e}_1, \ldots, \hat{e}_{2k}\}$ IS ANOTHER BASIS AND

$$\hat{e}_a = \Lambda_a^{\ \alpha} e_\alpha$$

THEN

$$\mathcal{H}_H(p)(\hat{e}_a, \hat{e}_b) = \Lambda_a^{\ \alpha} \Lambda_b^{\ \beta} \mathcal{H}_H(p)(e_\alpha, e_\beta)$$

SO

$$\det(\mathcal{H}_H(p)(\hat{e}_a, \hat{e}_b)) = (\det \Lambda)^2 \det(\mathcal{H}_H(p)(e_\alpha, e_\beta))$$

BUT WE CLAIM THAT IF $\{\hat{e}_1, \ldots, \hat{e}_{2k}\}$ ALSO SATISFIES $(dV_\sigma)_p(\hat{e}_1, \ldots, \hat{e}_{2k}) = 1$, THEN $\det \Lambda = 1$ AND THIS WILL ESTABLISH OUR RESULT. TO SEE THIS, NOTE THAT, IN OUR SPECIAL BASIS $\{e_1, \ldots, e_{2k}\}$,

$$\sigma_p = e^1 \wedge e^2 + e^3 \wedge e^4 + \cdots + e^{2k-1} \wedge e^{2k}$$

SO

$$(dV_\sigma)_p = \frac{1}{k!} \overbrace{\sigma_p \wedge \cdots \wedge \sigma_p}^{k} = e^1 \wedge \cdots \wedge e^{2k}.$$

THUS,

$$\begin{aligned} 1 = (dV_\sigma)_p(\hat{e}_1, \ldots, \hat{e}_{2k}) &= (e^1 \wedge \cdots \wedge e^{2k})(\hat{e}_1, \ldots, \hat{e}_{2k}) \\ &= (\det \Lambda)(e^1 \wedge \cdots \wedge e^{2k})(e_1, \ldots, e_{2k}) \\ &= \det \Lambda. \end{aligned}$$

ALL THAT REMAINS IS TO PROVE THE EXISTENCE OF THE BASIS $\{e_1, \dots, e_{2\ell}\}$ FOR $T_p(M)$ SATISFYING THE TWO CONDITIONS ON PAGE 7. FOR THIS WE WILL APPEAL TO THE FOLLOWING RESULT (STATED, WITHOUT PROOF, ON PAGE 223 OF [3]).

> LET (V, σ) BE A SYMPLECTIC VECTOR SPACE, AND LET L BE AN INVERTIBLE, SEMISIMPLE ENDOMORPHISM OF V WITH PURE IMAGINARY EIGENVALUES. THE QUADRATIC FORM $H(v, w) = -\sigma(Lv, w)$ IS NONDEGENERATE, AND L IS A SKEW-ADJOINT ENDOMORPHISM OF V WITH RESPECT TO H. CHOOSE A BASIS FOR V OF VECTORS p_i, q_i, $1 \le i \le \ell$, SUCH THAT $\sigma(p_i, q_j) = \delta_{ij}$, $\sigma(p_i, p_j) = \sigma(q_i, q_j) = 0$, $Lp_i = -\lambda_i q_i$ AND $Lq_i = \lambda_i p_i$.

WITH THE ORDERING $\{p_1, q_1, \dots, p_\ell, q_\ell\} = \{e_1, \dots, e_{2\ell}\}$ THIS GIVES

$$(\sigma(e_i, e_j)) = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & 0 \\ & & \ddots & & \\ & 0 & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

AND THE MATRIX OF L IS

$$\begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & 0 \\ & & \ddots & & \\ & & & 0 & \lambda_\ell \\ & 0 & & -\lambda_\ell & 0 \end{pmatrix}$$

WITH $V = T_p(M)$ AND $L = L_p(-i)$ WE SHOW THAT L SATISFIES THE REQUIRED CONDITIONS. WE KNOW ALREADY THAT L IS INVERTIBLE SO WE MUST SHOW THAT L IS SEMISIMPLE (V DECOMPOSES INTO A DIRECT SUM OF L-INVARIANT SUBSPACES, EACH OF WHICH ITSELF HAS NO PROPER L INVARIANT SUBSPACE) AND HAS PURE IMAGINARY EIGENVALUES).

FIRST NOTE THAT

$$\langle Lv, w \rangle_{S'} = - \langle v, Lw \rangle_{S'}$$

IMPLIES

$$\langle L^2 v, w \rangle_{S'} = - \langle Lv, Lw \rangle_{S'} = \langle v, L^2 w \rangle_{S'}$$

SO L^2 IS SELF-ADJOINT WITH RESPECT TO $\langle \, , \rangle_{S'}$. MOREOVER,

$$\langle L^2 v, v \rangle_{S'} = - \langle Lv, Lv \rangle_{S'} \leq 0$$

(AND L^2 INVERTIBLE) IMPLIES THAT L^2 HAS NEGATIVE EIGENVALUES ($L^2 v = \lambda v \Rightarrow 0 \geq \lambda \langle v, v \rangle_{S'} \Rightarrow \lambda \leq 0$, BUT $\lambda \neq 0$). LET

$$-\lambda_1^2 \, , \ldots , \, -\lambda_\ell^2$$

WITH

$$0 < \lambda_1 < \cdots < \lambda_\ell$$

BE THE DISTINCT EIGENVALUES OF L^2 AND

$$V = V_1 \oplus \cdots \oplus V_\ell$$

THE ($\langle \, , \rangle_{S'}$-ORTHOGONAL) EIGENSPACE DECOMPOSITION OF V FOR L^2.

LET

: $L(V_j) \subseteq V_j$

$$L_j = L \,|\, V_j \qquad , \; j = 1, \ldots, \ell .$$

E $L^2 v = -\lambda_j^2 v$

IES $L^3 v = -\lambda_j^2 Lv$

$L^2(Lv) = -\lambda_j^2 Lv$

$Lv \in V_j$

THEN

$$L_j^2 = -\lambda_j^2 \, 1$$

SO, IF WE DEFINE

$$J_j = \frac{1}{\lambda_j} L_j \qquad , \; j = 1, \ldots, \ell ,$$

THEN

$$J_j^2 = -1 .$$

THUS, J_j IS A COMPLEX STRUCTURE ON V_j (SO $J = J_1 \oplus \cdots \oplus J_\ell$ IS A COMPLEX STRUCTURE ON V).

NOW USE J TO REGARD V AS A COMPLEX VECTOR SPACE V^J IN THE USUAL WAY ($(a+bi)v = av + bJv$). EACH V_j IS A COMPLEX LINEAR SUBSPACE V_j^J OF V^J BECAUSE $J(V_j) = J_j(V_j) \subseteq V_j$. L IS COMPLEX LINEAR ON V^J BECAUSE IT COMMUTES WITH J.

LET $\{f_{j1}, \dots, f_{jk_j}\}$ BE A BASIS FOR V_j^J. THEN $\{f_{j1}, \dots, f_{jk_j}, Jf_{j1}, \dots, Jf_{jk_j}\}$ IS A BASIS FOR V_j (OVER $\mathbb{R}$).

$$L_j(f_{j1}) = \lambda_j J_j(f_{j1}) = \lambda_j i f_{j1}$$
$$L_j(Jf_{j1}) = L_j(if_{j1}) = i L_j(f_{j1}) = \lambda_j i J_j f_{j1}$$

SO f_{j1} AND $J_j f_{j1}$ ARE BOTH EIGENVECTORS OF L_j WITH EIGENVALUE $\lambda_j i$. ALSO NOTE THAT

$$L_j(f_{j1}) = \lambda_j (J_j f_{j1})$$
$$L_j(J_j f_{j1}) = \lambda_j J_j^2 f_{j1} = -\lambda_j f_{j1}$$

SIMILARLY FOR THE REMAINING PAIRS $\{f, Jf\}$. THUS, WE HAVE A BASIS $\{f_1^j, f_2^j, \dots, f_{2k_j-1}^j, f_{2k_j}^j\} = \{J_j f_{j1}, f_{j1}, \dots, J_j f_{jk_j}, f_{jk_j}\}$ FOR V_j CONSISTING OF EIGENVECTORS FOR L_j WITH EIGENVALUE $\lambda_j i$ AND SATISFYING

$$L_j(f_1^j) = -\lambda_j f_2^j$$
$$L_j(f_2^j) = \lambda_j f_1^j$$

AND SIMILARLY FOR THE REMAINING PAIRS. THIS ALSO GIVES A MINIMAL L_j-INVARIANT DECOMPOSITION OF $V_j = \mathrm{Span}\{f_1^j, f_2^j\} \oplus \cdots \oplus \mathrm{Span}\{f_{2k_j-1}^j, f_{2k_j}^j\}$ AND SHOWS THAT THE ONLY EIGENVALUE OF L_j IS $\lambda_j i$. NOW REPEAT FOR EACH $j = 1, \dots, \ell$

TO OBTAIN A BASIS $\{e_1, e_2, \dots, e_{2k-1}, e_{2k}\}$ FOR V WITH

$$L(e_{2j-1}) = -\lambda_j e_{2j}$$
$$L(e_{2j}) = \lambda_j e_{2j-1}$$

AND, RELATIVE TO THE COMPLEX STRUCTURE J,

$$L(e_{2j-1}) = \lambda_j i\, e_{2j-1}$$
$$L(e_{2j}) = \lambda_j i\, e_{2j}$$

AND,

$$V = \operatorname{Span}\{e_1, e_2\} \oplus \dots \oplus \operatorname{Span}\{e_{2k-1}, e_{2k}\}$$

THE DECOMPOSITION INTO MINIMAL L-INVARIANT SUBSPACES.