GENERALIZED DUISTERMAAT - HECKMAN THEOREM I

CONSIDER AGAIN THE EXAMPLE OF THE HEIGHT FUNCTION H ON S^2. THE HAMILTONIAN VECTOR FIELD V_H HAS A PERIODIC FLOW AND WE OBSERVE NOW THAT THIS DETERMINES AN ACTION OF S^1 ON S^2 (PUSH POINTS OF S^2 AROUND THE INTEGRAL CURVES (CIRCLES) CONTAINING THEM).

$$g = e^{iT} \in S^1 \qquad\qquad p \in S^2$$

$$g \cdot p = e^{iT} \cdot p = \text{POINT ON } S^2 \text{ THAT IS } T \text{ UNITS ALONG THE UNIQUE INTEGRAL CURVE STARTING AT } p$$

OBSERVATIONS ON THIS ACTION :

$\mathrm{Lie}(S^1) = \mathrm{Im}\,\mathbb{C} = i\mathbb{R}$ AND EACH $\xi = ia \in \mathrm{Lie}(S^1)$ GIVES RISE TO A VECTOR FIELD $\xi^\#$ ON S^2 (THE INFINITESIMAL ACTION OF ξ ON S^2)

$$\xi^\#(p) = \frac{d}{dt}(\exp(-t\xi)\cdot p)\Big|_{t=0}$$

A SIMPLE CALCULATION SHOWS

$$\xi = ia \in \mathrm{Lie}(S^1) \quad \Rightarrow \quad \xi^\# = -a\frac{\partial}{\partial\theta} = -aV_H = V_{-aH}$$

EVERY $\xi^{\#}$, $\xi \in Lie(S^1)$, IS THE HAMILTONIAN VECTOR FIELD OF SOMETHING !

DEFINE

$$\mu : Lie(S^1) \longrightarrow C^{\infty}(S^2)$$

BY

$$\mu(\xi) = \mu(ia) = -aH$$

THEN

1. μ IS LINEAR

2. $\xi^{\#}$ IS THE HAMILTONIAN VECTOR FIELD OF $\mu(\xi)$

$$\xi^{\#} = V_{\mu(\xi)}$$

$$d\mu(\xi) = \iota_{\xi^{\#}}\sigma$$

3. μ IS EQUIVARIANT

$$\mu(g \cdot \xi) = g \cdot \mu(\xi)$$

$g \xi g^{-1}$

S^1-ACTION ON $C^{\infty}(S^2)$

$(g \cdot f)(p) = f(g^{-1} \cdot p)$

THUS MOTIVATED WE FORMULATE A GENERAL DEFINITION.

(M, σ) : COMPACT SYMPLECTIC MANIFOLD

G : COMPACT LIE GROUP ACTING SMOOTHLY ON M ON THE LEFT

THE ACTION IS SAID TO BE HAMILTONIAN IF THERE EXISTS A MAP

$$\mu : \mathfrak{g} \longrightarrow C^{\infty}(M)$$

SUCH THAT

1. μ IS LINEAR
2. FOR EACH $\xi \in \mathfrak{g}$ THE VECTOR FIELD $\xi^{\#}$ ON M DEFINED BY $\xi^{\#}(p) = \frac{d}{dt}(\exp(-t\xi)\cdot p)\Big|_{t=0}$ $\forall\, p \in M$ SATISFIES

$$\xi^{\#} = V_{\mu(\xi)}$$

I.E.,

$$d\mu(\xi) = \iota_{\xi^{\#}}\sigma$$

3. μ IS EQUIVARIANT, I.E., $\forall\, \xi \in \mathfrak{g}$ $\forall\, g \in G$

$$\mu(g\cdot\xi) = g\cdot\mu(\xi)$$

($\mu(\xi)$ IS CALLED THE SYMPLECTIC MOMENT OF ξ)

THE "GENERALIZED" DUISTERMAAT - HECKMAN THEOREM (WHICH WE WILL PROVE IMPLIES THE OLD - FASHIONED DUISTERMAAT - HECKMAN THEOREM) LOCALIZES INTEGRALS OF THE FORM

$$\int_M e^{i\mu(\xi)}\, dV_{\sigma}$$

TO STATE THE RESULT WE WILL NEED SOME GENERAL INFORMATION ABOUT THE ZERO SETS

$$Z(\xi^{\#}) = \{ p \in M : \xi^{\#}(p) = 0 \}$$

<u>NOTE</u> : FOR HAMILTONIAN ACTIONS, $d\mu(\xi) = \iota_{\xi^{\#}}\sigma$ AND NONDEGENERACY OF σ IMPLY

$$\xi^{\#}(p) = 0 \iff d\mu(\xi)(p) = 0$$

SO

$$Z(\xi^{\#}) = \mathrm{CRIT}\,(\mu(\xi))$$

UNTIL FURTHER NOTICE ($\Lsh$) WE WILL CONSIDER AN ARBITRARY ACTION OF A COMPACT LIE GROUP G ON A SMOOTH, ORIENTED n-MANIFOLD M.

$$(g, p) \in G \times M \longrightarrow g \cdot p \in M$$

$$L_g : M \longrightarrow M$$

$$L_g(p) = g \cdot p$$

$$M^G = \text{FIXED POINT SET OF THE } G\text{-ACTION} = \{ p \in M : g \cdot p = p \ \forall g \in G \}$$

$$\xi \in \mathfrak{g} \longrightarrow \xi^{\#} \in \Gamma(TM)$$

$$\xi^{\#}(p) = \frac{d}{dt}(\exp(-t\xi) \cdot p)\Big|_{t=0}$$

$$Z(\xi^{\#}) = \{ p \in M : \xi^{\#}(p) = 0 \}$$

CLEARLY,

$$M^G \subseteq Z(\xi^*)$$

$\forall \xi \in \mathfrak{g}$ BUT $Z(\xi^*)$ CAN BE LARGER THAN M^G.

HOWEVER,

$$p \in Z(\xi^*) \Rightarrow \exp(-t\xi)\cdot p = p$$

$\forall t$ SINCE $\xi^*(p) = 0 \Rightarrow$ THE INTEGRAL CURVE OF ξ^* STARTING AT p IS A POINT. THUS,

$Z(\xi^*)$ = THE FIXED POINT SET OF THE TORUS ACTION ON M OBTAINED BY RESTRICTING THE G-ACTION TO

$$T = \text{CLOSURE}_G \{\exp(-t\xi) : t \in \mathbb{R}\}$$

IN PARTICULAR, FOR S^1-ACTIONS

$$Z(\xi^*) = M^{S^1} \quad \forall \xi \in \mathrm{Lie}(S^1).$$

NOW SUPPOSE $p \in Z(\xi^*)$. DEFINE

$$L_p(\xi) : T_p(M) \longrightarrow T_p(M)$$

BY

$$L_p(\xi)(v_p) = (\mathcal{L}_{\xi^*} V)_p = [\xi^*, V]_p = -\frac{d}{dt}\left((L_{\exp(-t\xi)})_{*p}(v_p)\right)_{t=0}$$

WHERE V IS ANY VECTOR FIELD ON M WITH $V(p) = v_p$.

THE LINEAR TRANSFORMATION $L_p(\xi)$ (THOUGHT OF AS THE INFINITESIMAL ACTION INDUCED BY $\xi^{\#}$ ON $T_p(M)$) PLAYS A CRUCIAL ROLE IN THE LOCALIZATION FORMULA AND WE NEED SEVERAL OF ITS PROPERTIES.

CHOOSE A RIEMANNIAN METRIC $\langle\ ,\ \rangle_G$ ON M THAT IS G-INVARIANT (I.E., THE DIFFEOMORPHISMS $L_g : M \to M$ ARE ISOMETRIES).

EACH $\xi^{\#}$ IS THEN A KILLING VECTOR FIELD RELATIVE TO $\langle\ ,\ \rangle_G$

$$\mathcal{L}_{\xi^{\#}}(\langle\ ,\ \rangle_G) = 0$$

I.E., $\forall\ V, W \in \Gamma(TM)$

$$\xi^{\#}(\langle V, W\rangle_G) = \langle [\xi^{\#}, V], W\rangle_G + \langle V, [\xi^{\#}, W]\rangle_G$$

IF $p \in Z(\xi^{\#})$, THEN EVALUATING BOTH SIDES OF THIS EQUALITY AT p GIVES

$$0 = \langle L_p(\xi)(V(p)), W(p)\rangle_G + \langle V(p), L_p(\xi)(W(p))\rangle_G .$$

SO

$$L_p(\xi) : T_p(M) \to T_p(M)$$

IS <u>SKEW - SYMMETRIC</u> WITH RESPECT TO $\langle\ ,\ \rangle_G$.

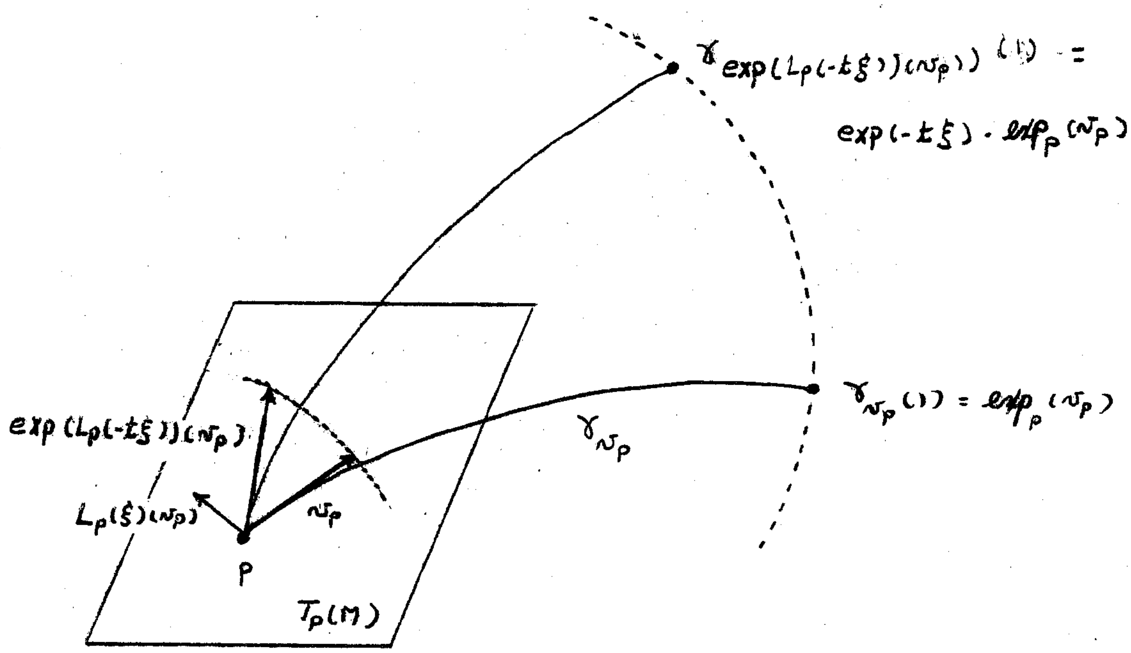
LET $\mathcal{E}xp_p$ BE THE (METRIC) EXPONENTIAL MAP ON $T_p(M)$ CORRESPONDING TO THE G-INVARIANT RIEMANNIAN METRIC $\langle\ ,\ \rangle_G$, I.E.,

$$\mathcal{E}xp_p(v_p) = \gamma_{v_p}(1)$$

WHERE γ_{v_p} IS THE GEODESIC OF $\langle\ ,\ \rangle_G$ WITH $\gamma'_{v_p}(0) = v_p$.

LOCAL DIFFEOMORPHISM. G-INVARIANCE IMPLIES THAT, LOCALLY,

$$\boxed{\mathcal{E}xp_p\left(\exp(L_p(-t\xi))(v_p)\right) = \exp(-t\xi)\cdot \mathcal{E}xp_p(v_p)}$$



FROM THIS IT FOLLOWS THAT

$$\boxed{(\mathcal{E}xp_p)_{*v_p}\left(\frac{d}{dt}\left(\exp(L_p(-t\xi))(v_p)\right)\Big|_{t=0}\right) = \xi^{\#}(\mathcal{E}xp_p(v_p))}$$

CONSEQUENTLY,

$$v_p \in \mathrm{KER}\,(L_p(\xi)) \iff \exp_p(v_p) \in Z(\xi^\#)$$

SO

NEAR EACH OF ITS POINTS p, $Z(\xi^\#)$ HAS A LOCAL MANIFOLD STRUCTURE OF DIMENSION

$$\dim\,(\mathrm{KER}\,(L_p(\xi)))$$

AND, IN PARTICULAR,

$$\boxed{p \text{ IS AN ISOLATED ZERO OF } \xi^\# \iff L_p(\xi) \text{ IS INVERTIBLE}}$$

p AN ISOLATED ZERO OF $\xi^\#$ $\Rightarrow$ $L_p(\xi)$ INVERTIBLE AND SKEW-SYMMETRIC WITH RESPECT TO $\langle\ ,\ \rangle_G$

$\Rightarrow$ $\dim M = 2k$

AND THERE IS AN ORIENTED, $\langle\ ,\ \rangle_G$ - ORTHONORMAL BASIS $\{e_1, \ldots, e_{2k}\}$ FOR $T_p(M)$ RELATIVE TO WHICH THE MATRIX OF $L_p(\xi)$ IS OF THE FORM

$$\begin{pmatrix} 0 & \lambda_1 & & & & \\ -\lambda_1 & 0 & & & \bigcirc & \\ & & \ddots & & & \\ & & & & 0 & \lambda_k \\ & \bigcirc & & & -\lambda_k & 0 \end{pmatrix}$$

FOR SOME NONZERO REAL NUMBERS $\lambda_1, \dots, \lambda_k$.

$$Pf\,(L_p(\xi)) = \lambda_1 \cdots \lambda_k$$

$\llcorner$

GENERALIZED DUISTERMAAT-HECKMAN THEOREM: LET M BE A COMPACT MANIFOLD OF DIMENSION $n = 2k$ WITH SYMPLECTIC FORM σ AND ORIENTED BY THE LIOUVILLE FORM $dV_\sigma = \frac{1}{k!}\, \sigma \wedge \overset{k}{\cdots} \wedge \sigma$. LET G BE A COMPACT LIE GROUP AND SUPPOSE THERE IS A HAMILTONIAN ACTION OF G ON M WITH SYMPLECTIC MOMENTS GIVEN BY $\mu : \mathfrak{g} \longrightarrow C^\infty(M)$. IF $\xi \in \mathfrak{g}$ IS SUCH THAT THE ZERO SET $Z(\xi^\#)$ OF $\xi^\#$ CONSISTS OF (A FINITE NUMBER OF) ISOLATED POINTS, THEN

$$\int_M e^{i\mu(\xi)}\, dV_\sigma = \sum_{p \in Z(\xi^\#)} (2\pi i)^k \frac{e^{i\mu(\xi)(p)}}{Pf(L_p(\xi))}$$

EVENTUALLY, WE WILL SHOW THAT THIS IS A CONSEQUENCE OF A STILL MORE GENERAL LOCALIZATION THEOREM FROM EQUIVARIANT COHOMOLOGY.