GENERALIZED DUISTERMAAT-HECKMAN THEOREM II

WE CONSIDER AGAIN THE EXAMPLE OF THE HEIGHT FUNCTION ON S^2.

THE HEIGHT FUNCTION H TOGETHER WITH THE SYMPLECTIC FORM σ GIVES A HAMILTONIAN VECTOR FIELD V_H WHICH HAS A FLOW THAT IS PERIODIC AND THIS GIVES RISE TO AN OBVIOUS ACTION OF S^1 ON S^2 ("ROTATE POINTS OF S^2 AROUND THE INTEGRAL CURVES (CIRCLES) CONTAINING THEM ").

IN SOMEWHAT MORE DETAIL :

$$g = e^{iT} \in S^1$$

$$p = \varphi(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) \in S^2$$

$$g \cdot p = e^{iT} \cdot (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

$$= (\sin\phi\cos(\theta+T), \sin\phi\sin(\theta+T), \cos\phi)$$

(OF COURSE, N AND S ARE FIXED POINTS OF THE ACTION)

SOME OBSERVATIONS ABOUT THIS ACTION :

THE LIE ALGEBRA OF S^1 IS IDENTIFIED WITH $\operatorname{Im}\mathbb{C} = i\mathbb{R}$ AND EACH $\xi = ia \in \operatorname{Im}\mathbb{C}$ GIVES RISE TO A VECTOR FIELD $\xi^\#$ ON S^2 (THE INFINITESIMAL ACTION OF ξ ON S^2) DEFINED BY

$$\xi^{\#}(p) = \frac{d}{dt}(\exp(-t\xi)\cdot p)\Big|_{t=0} = \frac{d}{dt}(e^{-iat}\cdot p)\Big|_{t=0}$$

$$= \frac{d}{dt}(\sin\phi\cos(\theta-at), \sin\phi\sin(\theta-at), \cos\phi)\Big|_{t=0}$$

$$= -a(-\sin\phi\sin\theta, \sin\phi\cos\theta, 0)$$

$$= -a\,\partial_\theta(p)$$

SO

$$\xi = ia \Rightarrow \xi^{\#} = -a\partial_\theta = -aV_H \qquad (H = \text{HEIGHT FUNCTION})$$

$$= V_{-aH}$$

<u>EVERY</u> $\xi^{\#}$, $\xi \in \mathrm{Im}\,\mathbb{C}$ <u>IS</u> <u>A</u> <u>HAMILTONIAN</u> <u>VECTOR</u> <u>FIELD</u>! PURSUE THIS A BIT FURTHER:

DEFINE A MAP $\mu : \mathrm{Im}\,\mathbb{C} \to C^\infty(S^2)$ BY

$$\mu(\xi) = \mu(ia) = -aH.$$

THEN

1. μ IS LINEAR.
2. $\xi^{\#}$ IS THE HAMILTONIAN VECTOR FIELD ON S^2 DETERMINED BY $\mu(\xi)$.
3. μ IS EQUIVARIANT, I.E., FOR ANY $g \in S^1$,

$$\mu(g\cdot\xi) = g\cdot\mu(\xi)$$

<u>PROOF</u> <u>OF</u> #3: $g\cdot\xi = g\xi g^{-1} = \xi$ BECAUSE $g = e^{it}$ AND $\xi = ia$ COMMUTE. THUS, $\mu(g\cdot\xi) = \mu(\xi)$. THE ACTION OF S^1 ON $C^\infty(S^2)$ IS DEFINED BY $(g\cdot f)(p) = f(g^{-1}\cdot p)$ SO $(g\cdot\mu(\xi))(p) = \mu(\xi)(g^{-1}\cdot p) = (-aH)(g^{-1}\cdot p) = (-aH)(p) = \mu(\xi)(p)$ BECAUSE $-H$ IS CONSTANT ON THE ORBITS. THUS, $g\cdot\mu(\xi) = \mu(\xi) = \mu(g\cdot\xi)$. $\square$

SOME GENERAL DEFINITIONS:

LET (M, σ) BE A COMPACT SYMPLECTIC MANIFOLD OF DIMENSION $n = 2k$ AND G A COMPACT LIE GROUP THAT ACTS SMOOTHLY ON M ON THE LEFT. THE ACTION IS HAMILTONIAN IF THERE IS A MAP

$$\mu : \mathfrak{g} \longrightarrow C^{\infty}(M)$$

SUCH THAT

1. μ IS LINEAR.
2. FOR EACH $\xi \in \mathfrak{g}$ THE VECTOR FIELD $\xi^{\#}$ ON M DEFINED BY $\xi^{\#}(p) = \frac{d}{dt}(\exp(-t\xi)\cdot p)|_{t=0}$ $\forall p \in M$ SATISFIES
$$\xi^{\#} = V_{\mu(\xi)} \qquad (d\mu(\xi) = \iota_{\xi^{\#}}\sigma)$$
3. μ IS EQUIVARIANT:
$$\mu(g\cdot\xi) = g\cdot\mu(\xi)$$

$\mu(\xi)$ IS CALLED THE SYMPLECTIC MOMENT OF ξ.

NOTE: IT FOLLOWS FROM #2 AND THE NONDEGENERACY OF σ THAT THE CRITICAL POINTS OF $\mu(\xi)$ COINCIDE WITH THE ZEROS OF $\xi^{\#}$. EVERY FIXED POINT OF THE G-ACTION IS A ZERO OF EVERY $\xi^{\#}$.

IN ORDER TO STATE THE GENERALIZED DUISTERMAAT-HECKMAN THEOREM WE WILL NEED MORE DETAILED INFORMATION ABOUT THE ZERO SETS OF THE VECTOR FIELDS $\xi^{\#}$. WE PAUSE NOW TO DERIVE SOME BASIC RESULTS ALONG THESE LINES.

SOME GENERALITIES ON ZERO SETS AND FIXED POINTS :

LET G BE A COMPACT LIE GROUP WITH LIE ALGEBRA $\mathfrak{g}$. SUPPOSE G ACTS ON THE SMOOTH, ORIENTED n-MANIFOLD M ON THE LEFT :

$$(g, p) \in G \times M \longrightarrow g \cdot p \in M$$

<u>NOTE</u> : IT IS ALWAYS POSSIBLE TO SELECT A RIEMANNIAN METRIC $\langle\ ,\ \rangle_G$ ON M THAT IS G-INVARIANT, I.E., FOR WHICH THE DIFFEOMORPHISMS

$$L_g : M \to M$$
$$L_g(p) = g \cdot p$$

ARE ISOMETRIES. WE ASSUME SUCH A METRIC HAS BEEN SELECTED AND FIXED.

INFINITESIMAL ACTION OF $\mathfrak{g}$ ON M :

$$\xi \in \mathfrak{g} \longrightarrow \xi^{\#} \in \Gamma(TM)$$

$$\xi^{\#}(p) = \frac{d}{dt}(\exp(-t\xi)\cdot p)\Big|_{t=0}$$

RELATIVE TO $\langle\ ,\ \rangle_G$, EACH $\xi^{\#}$ IS A KILLING VECTOR FIELD, I.E.,

$$\mathcal{L}_{\xi^{\#}}(\langle\ ,\ \rangle_G) = 0$$

SO $\forall\ V, W \in \Gamma(TM)$

(1) $$\xi^{\#}(\langle V, W\rangle_G) = \langle [\xi^{\#}, V], W\rangle_G + \langle V, [\xi^{\#}, W]\rangle_G$$

NOW FIX A $\xi \in \mathfrak{g}$ AND LET $p \in M$ BE A ZERO OF $\xi^{\#}$:

$$\xi^{\#}(p) = 0$$

DEFINE

$$L_p(\xi) : T_p(M) \longrightarrow T_p(M)$$

BY

$$L_p(\xi)(v_p) = (\mathcal{L}_{\xi^{\#}} V)_p = [\xi^{\#}, V]_p = -\frac{d}{dt}\left((L_{\exp(-t\xi)})_{*p}(v_p)\right)\Big|_{t=0}$$

WHERE V IS ANY VECTOR FIELD ON M WITH

$$V(p) = v_p$$

(THINK OF $L_p(\xi)$ AS THE INFINITESIMAL ACTION INDUCED BY $\xi^{\#}$ ON $T_p(M)$).

<u>LEMMA</u> : $L_p(\xi) : T_p(M) \to T_p(M)$ IS SKEW-SYMMETRIC WITH RESPECT TO THE INNER PRODUCT ON $T_p(M)$ INDUCED BY THE G-INVARIANT METRIC $\langle\ ,\ \rangle_G$, I.E.,

$$\langle L_p(\xi)(v_p), w_p \rangle_G = - \langle v_p, L_p(\xi)(w_p) \rangle_G .$$

<u>PROOF</u> : CHOOSE VECTOR FIELDS V AND W ON M WITH $V(p) = v_p$ AND $W(p) = w_p$. EVALUATE (1)

$$\langle [\xi^{\#}, V], W \rangle_G + \langle V, [\xi^{\#}, W] \rangle_G = \xi^{\#}(\langle V, W \rangle_G)$$

AT p TO OBTAIN

$$\langle [\xi^{\#}, V]_p, W(p) \rangle_G + \langle V(p), [\xi^{\#}, W]_p \rangle_G = \xi^{\#}(p)(\langle V, W \rangle_G)$$

I.E.,

$$\langle L_p(\xi)(v_p), v_p \rangle_G + \langle v_p, L_p(\xi)(v_p) \rangle_G = 0$$

BECAUSE $\xi^{\#}(p) = 0$. $\square$

<u>ALTERNATIVE</u> <u>DESCRIPTION</u> <u>OF</u> $L_p(\xi)$: LET ∇ = LEVI-CIVITA CONNECTION OF $\langle\ ,\ \rangle_G$. TORSION FREE GIVES

$\mathcal{L}_{\xi^{\#}} V = \nabla_{\xi^{\#}} V - \nabla_V \xi^{\#}$ SO

$$L_p(\xi)(v_p) = (\nabla_{\xi^{\#}} V)_p - (\nabla_V \xi^{\#})_p$$

FOR EACH $\xi \in \mathfrak{z}$ WE LET

$$Z(\xi^{\#}) = \{ p \in M : \xi^{\#}(p) = 0 \}.$$

ALSO LET

$$M^G = \text{FIXED POINT SET OF THE } G\text{-ACTION}.$$

THEN CLEARLY

$$M^G \subseteq Z(\xi^{\#}) \qquad \forall\, \xi \in \mathfrak{z}$$

<u>LEMMA</u> : IF $G = S^1$, THEN $M^{S^1} = Z(\xi^{\#})$ FOR EVERY $\xi \neq 0$ IN $\mathrm{Lie}(S^1) = i\mathbb{R}$.

<u>PROOF</u> : $\xi \neq 0 \Rightarrow \xi$ SPANS $\mathrm{Lie}(S^1)$, I.E.,

$$\mathrm{Lie}(S^1) = \{ -t\xi : t \in \mathbb{R} \}$$

THE EXPONENTIAL MAP FOR S^1 IS SURJECTIVE SO THE ORBIT OF ANY $p \in M$ COINCIDES WITH $\{ \exp(-t\xi) \cdot p : t \in \mathbb{R} \}$, I.E., WITH THE INTEGRAL CURVE OF $\xi^{\#}$ THROUGH p. IF $\xi^{\#}(p) = 0$, THEN THIS INTEGRAL CURVE IS A POINT, I.E.,

P IS A FIXED POINT OF THE ACTION. □

IN GENERAL, $Z(\xi^\#)$ CAN BE LARGER THAN M^G. HOWEVER, $Z(\xi^\#)$ ALWAYS COINCIDES WITH THE FIXED POINT SET OF THE ACTION ON M OF THE SUBGROUP

$$T = \text{CLOSURE}_G \ \{ \exp(-t\xi) : t \in \mathbb{R} \}$$

OF G. BEING COMPACT, CONNECTED, AND ABELIAN, T IS A TORUS. THUS,

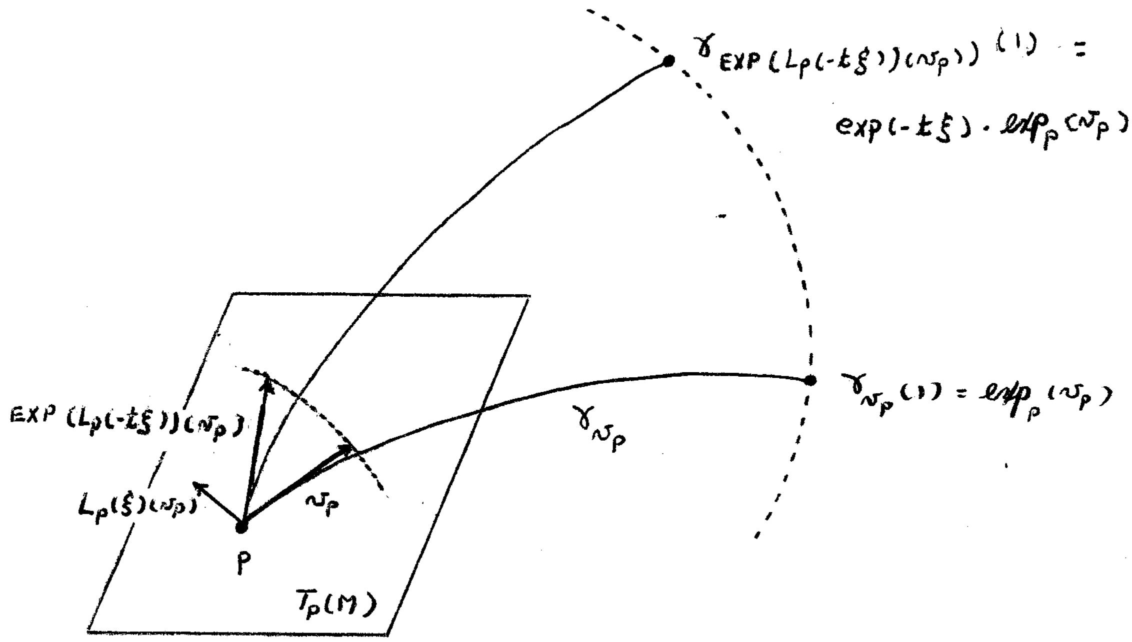
THE ZERO SET OF $\xi^\#$ IS ALWAYS THE FIXED POINT SET OF A TORUS ACTION ON M.

WE WILL NEED SOMEWHAT MORE DETAILED INFORMATION ABOUT $Z(\xi^\#)$. FOR THIS WE LET $\exp_p$ BE THE (METRIC) EXPONENTIAL MAP ON $T_p(M)$ CORRESPONDING TO THE G-INVARIANT RIEMANNIAN METRIC $\langle\ ,\ \rangle_G$ ON M. THUS,

$$\exp_p(v_p) = \gamma_{v_p}(1),$$

WHERE γ_{v_p} IS THE GEODESIC OF $\langle\ ,\ \rangle_G$ WITH $\gamma_{v_p}'(0) = v_p$. $\exp_p$ IS A LOCAL DIFFEOMORPHISM. WE CLAIM THAT THE G-INVARIANCE OF $\langle\ ,\ \rangle_G$ IMPLIES THAT, LOCALLY,

(2) $$\exp_p\left(\mathrm{EXP}\,(L_p(-t\xi))(v_p)\right) = \exp(-t\xi)\cdot \exp_p(v_p)$$



FIX A t. THE MAP $m \longrightarrow \exp(-t\xi)\cdot m$ IS AN ISOMETRY OF M BY G-INVARIANCE OF THE METRIC $\langle\ ,\ \rangle_G$. THUS,

$$\alpha_{-t\xi}(s) := \exp(-t\xi)\cdot\gamma_{v_p}(s) = (L_{\exp(-t\xi)}\circ\gamma_{v_p})(s)$$

IS A GEODESIC WITH

$$\alpha_{-t\xi}(0) = \exp(-t\xi)\cdot p = p$$

BECAUSE $\xi^{\#}(p) = 0$. MOREOVER,

(3) $$\alpha_{-t\xi}'(0) = \mathrm{EXP}\,(L_p(-t\xi))(v_p)$$

SO THAT $\alpha_{-t\xi}(s) = \gamma_{\mathrm{EXP}(L_p(-t\xi))(v_p)}(s)$ AND (2) FOLLOWS BY TAKING $s=1$.

FROM (2) WE CONCLUDE THE FOLLOWING : DENOTE BY $V_p(\xi)$ THE VECTOR FIELD ON $T_p(M)$ GIVEN BY

$$V_p(\xi)(v_p) = \frac{d}{dt}\left(\mathrm{EXP}\,(L_p(-t\xi))(v_p)\right)\Big|_{t=0}$$

THEN, LOCALLY,

$$(\exp_p)_{*v_p}(V_p(\xi)(v_p)) = \frac{d}{dt}\left[\exp_p\,(\mathrm{EXP}\,(L_p(-t\xi))(v_p))\right]_{t=0}$$

$$= \frac{d}{dt}\left[\exp(-t\xi)\cdot \exp_p(v_p)\right]_{t=0}$$

(4) $$(\exp_p)_{*v_p}(V_p(\xi)(v_p)) = \xi^{\#}(\exp_p(v_p))$$

SO, ON SOME NEIGHBORHOOD OF 0 IN $T_p(M)$, $V_p(\xi)(v_p) = 0 \Leftrightarrow \xi^{\#}(\exp_p(v_p)) = 0$

AND SO

$$\boxed{v_p \in \mathrm{KER}\,(L_p(\xi)) \iff \exp_p(v_p) \in Z(\xi^{\#})}$$

IN PARTICULAR, $\exp_p$ MAPS A NEIGHBORHOOD OF 0 IN $\mathrm{KER}\,(L_p(\xi))$ DIFFEOMORPHICALLY ONTO A NEIGHBORHOOD OF $p = \exp_p(0)$ IN $Z(\xi^{\#})$ SO

NEAR EACH OF ITS POINTS p, $Z(\xi^{\#})$ HAS
A LOCAL MANIFOLD STRUCTURE OF DIMENSION
$\dim\,(\mathrm{KER}\,L_p(\xi))$.

THIS DIMENSION NEED NOT BE THE SAME FOR EACH $p \in Z(\xi^{\#})$, BUT IS CONSTANT ON CONNECTED COMPONENTS OF $Z(\xi^{\#})$.

IN PARTICULAR, $Z(\xi^{\#})$ IS 0-DIMENSIONAL AT p IF AND ONLY IF $\mathrm{KER}\,(L_p(\xi))$ IS TRIVIAL SO

$$\boxed{p \text{ IS AN ISOLATED ZERO OF } \xi^{\#} \iff L_p(\xi) : T_p(M) \to T_p(M) \text{ IS INVERTIBLE}}$$

UNTIL FURTHER NOTICE WE WILL ASSUME THAT p IS AN ISOLATED ZERO OF $\xi^{\#}$.

THUS, $L_p(\xi)$ IS INVERTIBLE AND SKEW-SYMMETRIC WITH RESPECT TO $\langle\ ,\ \rangle_G$. IT THEREFORE FOLLOWS THAT THE DIMENSION OF M MUST BE EVEN

$$n = 2k$$

AND THAT THERE EXISTS AN ORIENTED, $\langle\ ,\ \rangle_G$ - ORTHONORMAL BASIS $\{e_1, \ldots, e_{2k}\}$ FOR $T_p(M)$ RELATIVE TO WHICH

$$L_p(\xi)(e_{2j-1}) = -\lambda_j e_{2j}$$

$$L_p(\xi)(e_{2j}) = \lambda_j e_{2j-1}$$

$$j = 1, \ldots, k$$

WHERE EACH λ_j IS A NONZERO REAL NUMBER. THUS, THE MATRIX OF $L_p(\xi)$ RELATIVE TO $\{e_1, \ldots, e_{2k}\}$ IS

$$\begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & 0 \\ & & \ddots & & \\ 0 & & & 0 & \lambda_k \\ & & & -\lambda_k & 0 \end{pmatrix}$$

ANY $2k \times 2k$ SKEW-SYMMETRIC MATRIX A HAS A PFAFFIAN (SEE THE SUPPLEMENT ON "PFAFFIANS"). ACCORDING TO OUR CONVENTIONS, THE PFAFFIAN OF THE MATRIX ABOVE IS $\lambda_1 \cdots \lambda_k$. SINCE $B \in SO(2k)$ IMPLIES $Pf(BAB^{-1}) = Pf(A)$, THE LINEAR TRANSFORMATION $L_p(\xi)$ ITSELF HAS A WELL-DEFINED PFAFFIAN

$$Pf(L_p(\xi))$$

(THE PFAFFIAN OF ITS MATRIX RELATIVE TO ANY ORIENTED, $\langle\ ,\ \rangle_G$-ORTHONORMAL BASIS FOR $T_p(M)$). CHANGING THE ORIENTATION CHANGES THE SIGN OF THE PFAFFIAN.

IF $v \in T_p(M)$ AND IF WE WRITE $v = v^i e_i$, THEN

$$L_p(\xi)(v) = \lambda_1(v^2 e_1 - v^1 e_2) + \cdots + \lambda_k(v^{2k} e_{2k-1} - v^{2k-1} e_{2k}).$$

<u>NOTE FOR FUTURE REFERENCE</u>: THE BASIS $\{e_1, \ldots, e_{2k}\}$ AND THE EXPONENTIAL MAP $\exp_p$ GIVE NORMAL COORDINATES $x^1, \ldots, x^{2k}$ ON A NEIGHBORHOOD U_p OF P AND THEN (4) TOGETHER

WITH THE LAST EQUALITY GIVES

$$\xi^{\#} | U_p = \lambda_1 \left(x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} \right) + \cdots + \lambda_k \left(x^{2k} \frac{\partial}{\partial x^{2k-1}} - x^{2k-1} \frac{\partial}{\partial x^{2k}} \right).$$

IF p HAPPENS TO BE A FIXED POINT OF THE G-ACTION, THEN U_p CAN BE CHOSEN G-INVARIANT (RESTRICT TO SOME ε-BALL RELATIVE TO $\langle\, ,\, \rangle_G$).

WITH THIS WE CAN STATE THE

> <u>GENERALIZED</u> <u>DUISTERMAAT</u>-<u>HECKMAN</u> <u>THEOREM</u> : LET M BE A COMPACT MANIFOLD OF DIMENSION $n = 2k$ WITH SYMPLECTIC FORM σ AND ORIENTED BY THE LIOUVILLE FORM $dV_\sigma = \frac{1}{k!} \overbrace{\sigma \wedge \cdots \wedge \sigma}^{k}$. LET G BE A COMPACT LIE GROUP AND SUPPOSE THERE IS A HAMILTONIAN ACTION OF G ON M WITH SYMPLECTIC MOMENTS GIVEN BY $\mu : \mathfrak{g} \to C^\infty(M)$. IF $\xi \in \mathfrak{g}$ IS SUCH THAT THE ZERO SET OF $\xi^{\#}$ CONSISTS OF A FINITE NUMBER OF ISOLATED POINTS, THEN
>
> $$(5) \qquad \int_M e^{i\mu(\xi)} dV_\sigma = \sum_{\substack{p \in M \\ \xi^{\#}(p) = 0}} (2\pi i)^k \left(Pf(L_p(\xi)) \right)^{-1} e^{i\mu(\xi)(p)}$$

WE WILL EVENTUALLY PROVE THIS AS A CONSEQUENCE OF A STILL MORE GENERAL LOCALIZATION THEOREM FROM EQUIVARIANT COHOMOLOGY. FOR THE TIME BEING WE WILL BE CONTENT TO SHOW THAT THE ORIGINAL DUISTERMAAT-HECKMAN THEOREM FOLLOWS FROM THIS GENERALIZED VERSION.