

GEOMETRICAL AND NUMERICAL METHODS

UNFORTUNATE FACT OF LIFE : THE DIFFERENTIAL EQUATION

$$y' = f(x, y)$$

IS GENERALLY IMPOSSIBLE TO SOLVE EXACTLY.

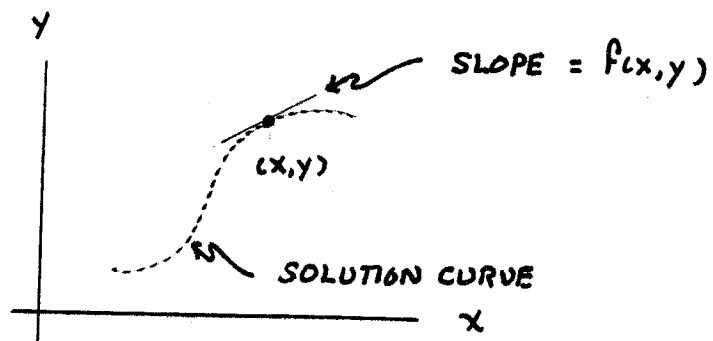
EVEN WHEN THIS IS THE CASE IT IS STILL POSSIBLE TO OBTAIN INFORMATION ABOUT THE SOLUTIONS, E. G.,

- GENERAL, QUALITATIVE BEHAVIOR (SHAPES OF THE INTEGRAL CURVES)
- APPROXIMATE VALUES OF SOLUTIONS

WE WILL CONSIDER ONE SIMPLE METHOD FOR EACH OF THESE.

- DIRECTION FIELDS
- EULER'S METHOD

IDEA BEHIND DIRECTION FIELDS : $y' = f(x, y)$



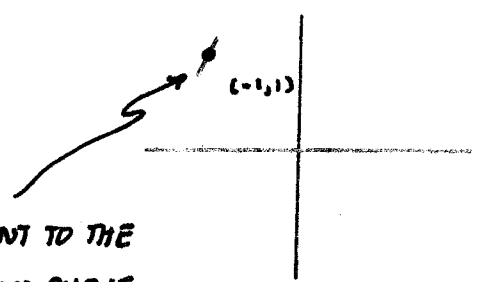
E.G., ANY SOLUTION TO

$$y' = x^2 + y^2$$

THROUGH THE POINT $(-1, 1)$ MUST HAVE SLOPE

$$(-1)^2 + (1)^2 = 2$$

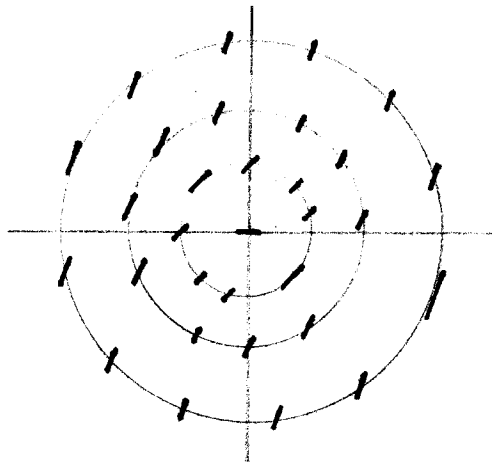
THERE.



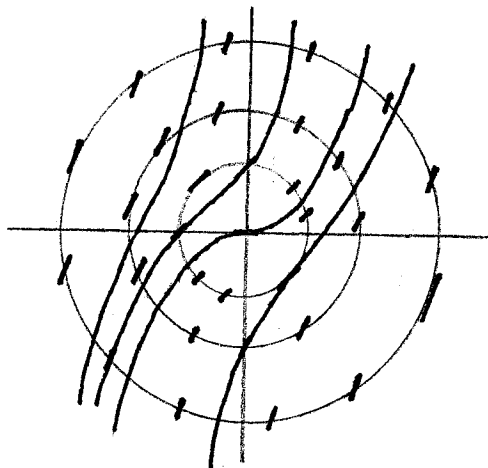
TANGENT TO THE
 SOLUTION CURVE
 TO $y' = x^2 + y^2$
 THROUGH $(-1, 1)$
 MUST HAVE SLOPE 2.

SUPPOSE WE DREW (OR HAD A COMPUTER DRAW) LOTS AND LOTS OF THESE
 LITTLE TANGENT LINES TO POINTS IN THE PLANE INDICATING THE SLOPE
 OF ANY SOLUTION CURVE AT THESE POINTS.

E.G., NOTING THAT $x^2 + y^2$ IS CONSTANT ON CIRCLES ABOUT THE ORIGIN,



THE PATTERNS OF THESE SLOPES (THE DIRECTION FIELD), SUGGESTS THE SHAPE OF THE CURVES THAT HAVE THESE TANGENTS, I. E., THE INTEGRAL CURVES OF THE EQUATION.



ANOTHER EXAMPLE :

$$y' = 4y(1-y)$$

NOTE : THIS IS AN EXAMPLE OF A LOGISTIC EQUATION AND IS USED TO DESCRIBE A CERTAIN POPULATION MODEL .

$$y = y(t) = \text{POPULATION AT TIME } t$$

WE ARE INTERESTED IN SOLUTIONS THAT SATISFY AN INITIAL CONDITION

$$y(0) = y_0$$

WHERE $y_0 > 0$.

NOTE THAT THE RIGHT-HAND SIDE (SLOPE) IS CONSTANT ON HORIZONTAL LINES $y = k$. WE'LL CHOOSE A NUMBER OF THESE FOR DRAWING THE DIRECTION FIELD.

$$y = k$$

0

 $\frac{1}{4}$ $\frac{1}{2}$ $\frac{3}{4}$

1

 $\frac{5}{4}$ $\frac{3}{2}$

$$y' = 4k(1-k)$$

0

 $\frac{3}{4}$

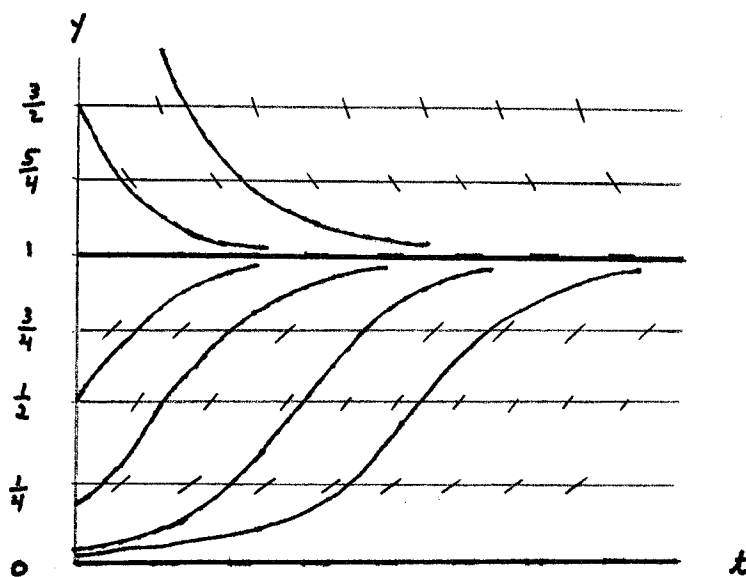
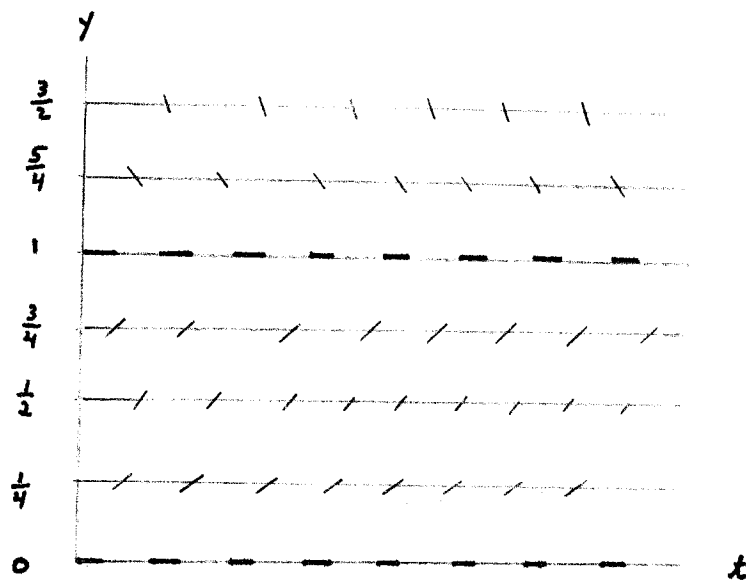
1

 $\frac{3}{4}$

0

 $-\frac{5}{4}$

-3

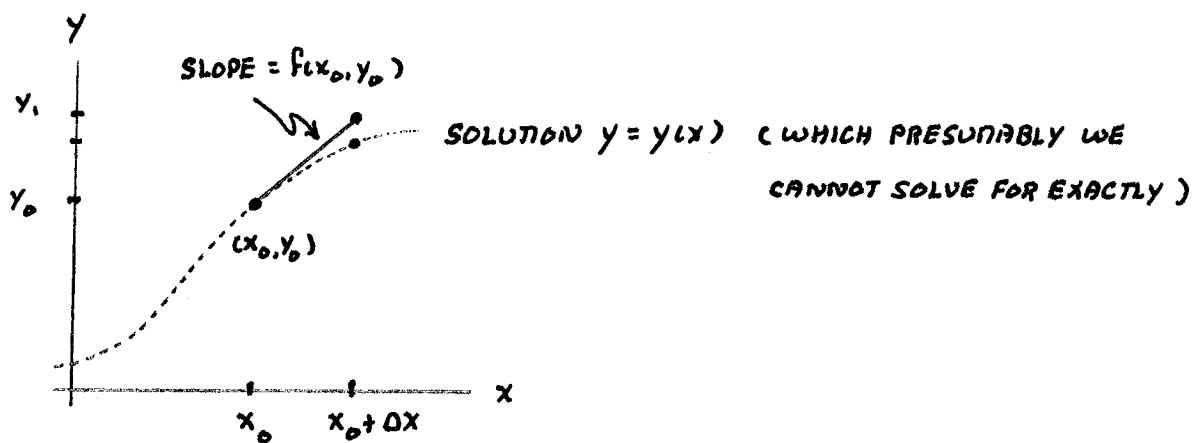


STABLE
POPULATION



IDEA BEHIND EULER'S METHOD :

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$



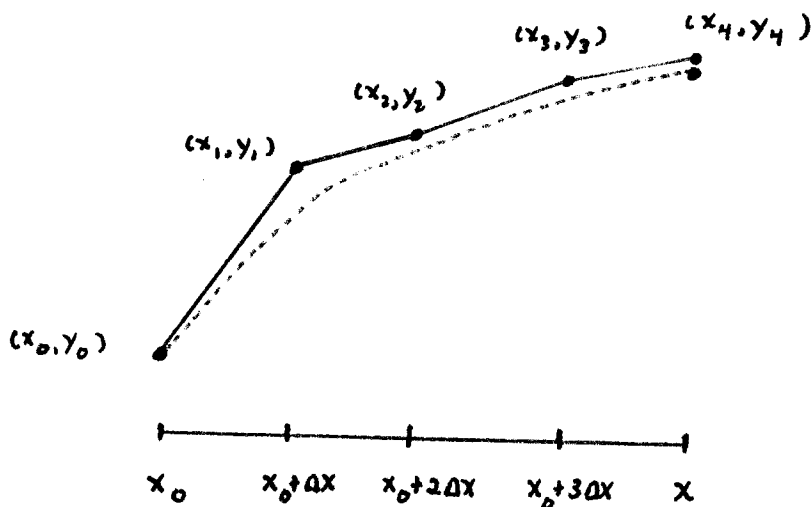
SUPPOSE YOU NEED $y(x_0 + \Delta x)$. IF Δx IS SMALL ENOUGH APPROXIMATE $y(x_0 + \Delta x)$ WITH THE HEIGHT OF THE TANGENT LINE ABOVE $x_0 + \Delta x$:

$$y - y_0 = f(x_0, y_0)(x - x_0)$$

$$x = x_0 + \Delta x$$

$$y_1 = y_0 + f(x_0, y_0)\Delta x \approx y(x_0 + \Delta x)$$

OR, IF YOU NEED $y(x)$ FOR SOME x THAT IS NOT CLOSE TO x_0 , APPROACH IT IN "STEPS", E.G.,



SUBDIVIDE $[x_0, x]$ INTO n EQUAL SUBINTERVALS OF LENGTH

$$\Delta x = \frac{x - x_0}{n} \quad (\text{STEP SIZE})$$

($n = 4$ IS SHOWN ABOVE)

$$y_1 = y_0 + f(x_0, y_0) \Delta x$$

$$y_2 = y_1 + f(x_1, y_1) \Delta x$$

$$y_3 = y_2 + f(x_2, y_2) \Delta x$$

$$y_4 = y_3 + f(x_3, y_3) \Delta x$$

⋮

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1}) \Delta x \approx y(x)$$

EXAMPLE :

$$\begin{cases} y' = 4y(1-y) \\ y(0) = \frac{1}{4} \end{cases}$$

$$f(x, y) = 4y(1-y) \\ (x_0, y_0) = (0, \frac{1}{4})$$

APPROXIMATE $y(0.3)$ WITH A STEP SIZE OF $\Delta x = 0.1$ (SO $n = 3$)

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0) \Delta x \\ &= \frac{1}{4} + f(0, \frac{1}{4})(0.1) \\ &= \frac{1}{4} + (4(\frac{1}{4})(1 - \frac{1}{4}))(0.1) \end{aligned}$$

$$= \frac{1}{4} + \left(\frac{3}{4}\right)(0.1)$$

$$= 0.25 + 0.075$$

$$= 0.325$$

$$(x_1, y_1) = (0.1, 0.325)$$

$$y_2 = y_1 + f(x_1, y_1) \Delta x$$

$$= 0.325 + \left(\frac{1}{4}(0.325)(1-0.325)\right)(0.1)$$

$$= 0.325 + (0.8775)(0.1)$$

$$= 0.4128$$

$$(x_2, y_2) = (0.2, 0.4128)$$

$$y_3 = y_2 + f(x_2, y_2) \Delta x$$

$$= 0.4128 + \left(\frac{1}{4}(0.4128)(1-0.4128)\right)(0.1)$$

$$= 0.5198$$

So

$$y(0.3) \approx y_3 = 0.5198$$

NOTE: IN THE NEXT SECTION WE WILL SOLVE THIS EQUATION EXACTLY AND FIND THAT

$$y(0.3) = \frac{0.25 e^{4(0.3)}}{1 + 0.25(e^{4(0.3)} - 1)} \approx 0.5253$$

(BUT ARRIVING AT 0.5253 ALSO INVOLVES APPROXIMATIONS)