MATH 680 LECTURE 23

THE RIEMANNIAN MANIFOLD FOR WHICH THE POINCARÉ UPPER HALF-PLANE, POINCARE DISC AND MINKOWSKI HYPERBOLA ARE ALL ISOMETRIC MODELS IS CALLED THE <u>HYPERBOLIC PLANE</u> AND DENOTED H^2. ONE OF ITS MOST IMPORTANT PROPERTIES IS THAT IT HAS LOTS OF ISOMETRIES. I'LL ASK YOU TO PROVE THIS IN WHICHEVER POINCARÉ MODEL IS YOUR FAVORITE.

<u>EXERCISE</u> : PROVE <u>ONE</u> OF THE FOLLOWING AND THEN DEDUCE THE OTHER ONE FROM IT.

(a) LET $F : \mathbb{C} \to \mathbb{C}$ BE A FRACTIONAL LINEAR TRANSFORMATION OF THE FORM

$$F(z) = \frac{az+b}{cz+d},$$

WHERE a, b, c, d ARE <u>REAL</u> NUMBERS AND $ad-bc = 1$. SHOW THAT F CARRIES THE UPPER HALF-PLANE $\operatorname{Im} z$ ONTO ITSELF AND IS, IN FACT, AN ISOMETRY OF THE POINCARÉ UPPER HALF-PLANE ONTO ITSELF.

(b) LET $F : \mathbb{C} \to \mathbb{C}$ BE A FRACTIONAL LINEAR TRANSFORMATION OF THE FORM

$$F(z) = \frac{az+\bar{c}}{cz+\bar{a}}$$

WHERE $a, c \in \mathbb{C}$ AND $|a|^2 - |c|^2 = 1$.
SHOW THAT F CARRIES THE UNIT DISC
$|z| < 1$ ONTO ITSELF AND IS, IN FACT,
AN ISOMETRY OF THE POINCARÉ DISC.

<u>EXERCISE</u>: SHOW THAT, GIVEN TWO POINTS $p_1, p_2 \in H^2$, THERE IS AN ISOMETRY OF H^2 ONTO ITSELF THAT CARRIES p_1 ONTO p_2.

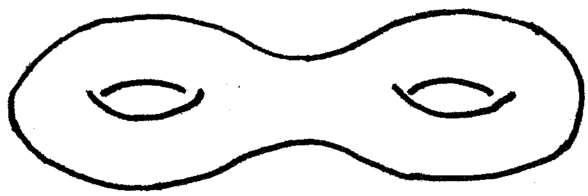
IN THIS SENSE H^2, DESPITE OUR EUCLIDEAN VIEW OF IT FROM $\mathbb{R}^3$, IS "HOMOGENEOUS" (LOOKS LOCALLY THE SAME AT EACH POINT).

DIFFERENTIAL GEOMETRY IS ALL ABOUT BUILDING GLOBAL OBJECTS FROM LOCAL MODELS SO OUR NEXT STEP SHOULD BE CLEAR:

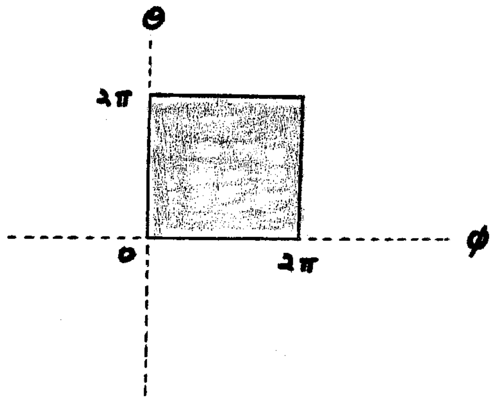
A <u>HYPERBOLIC</u> <u>2</u>-<u>MANIFOLD</u> IS A 2-DIMENSIONAL RIEMANNIAN (M, g) THAT IS LOCALLY ISOMETRIC TO H^2 (I.E., EACH POINT OF WHICH HAS A NEIGHBORHOOD ISOMETRIC TO AN OPEN SET IN H^2).

WE DON'T HAVE ANY EXAMPLES (OTHER THAN H^2 ITSELF) YET, BUT OUR EVENTUAL GOAL ("UNIFORMIZATION THEOREM") IS TO DESCRIBE ALL OF THE (COMPACT, CONNECTED, ORIENTABLE) HYPERBOLIC 2-MANIFOLDS.

NOTE: ONE EXAMPLE WILL BE THE 2-HOLED TORUS



WITH A CERTAIN RIEMANNIAN METRIC DEFINED ON IT (NOT THE USUAL ONE). I WILL EVENTUALLY FOLLOW A DIFFERENT APPROACH, BUT WILL JUST BRIEFLY SKETCH WHERE SUCH A RIEMANNIAN METRIC MIGHT COME FROM. RECALL HOW WE PRODUCED THE "FLAT TORUS":



$$P : \mathbb{R}^2 \to \mathbb{R}^4$$

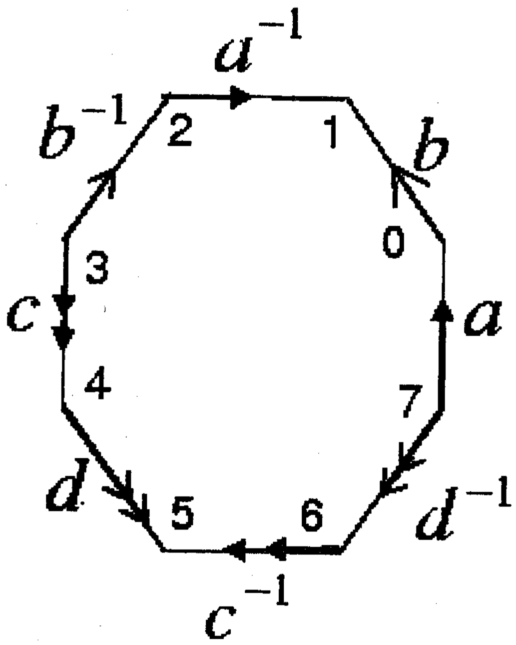
$$P(\phi, \theta) = (\cos(\phi+\theta), \sin(\phi+\theta), \cos(\phi-\theta), \sin(\phi-\theta))$$

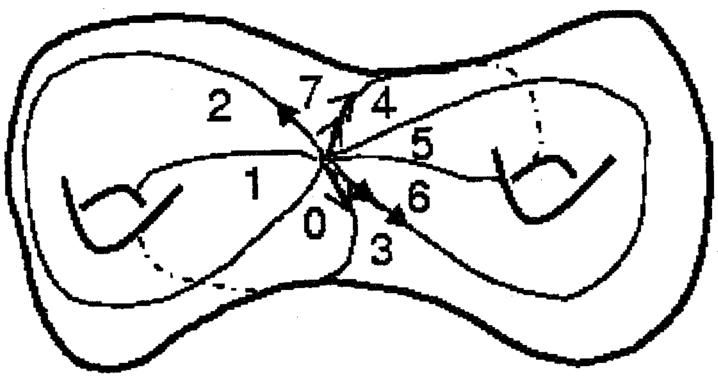
LOCAL EMBEDDING

$$P([0,2\pi] \times [0,2\pi]) \cong S^1 \times S^1$$

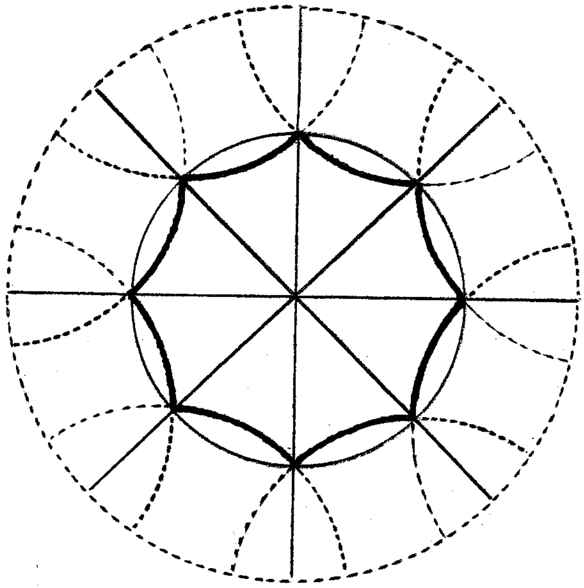
TRANSFER THE RIEMANNIAN METRIC OF $\mathbb{R}^2$ LOCALLY TO $S^1 \times S^1$ VIA P.

IT TURNS OUT THAT THE 2-HOLED TORUS CAN BE OBTAINED BY A CERTAIN IDENTIFICATION OF THE EDGES OF AN OCTAGON





AND THAT IF ONE BEGINS WITH A CERTAIN SPECIFIC GEODESIC OCTAGON IN THE POINCARÉ DISC



(THE HYPERBOLIC RADIUS OF THE INNER CIRCLE NEEDS TO BE CHOSEN CAREFULLY), THEN ONE CAN MIMIC THE PROCEDURE FOR TRANSFERING THE EUCLIDEAN METRIC TO THE FLAT TORUS AND TRANSFER THE HYPERBOLIC METRIC TO THE 2-HOLED TORUS (LOCALLY).

I WILL NOT PURSUE THIS SORT OF "COMBINATORIAL" APPROACH ANY FURTHER SINCE, IN A SENSE, IT HIDES UNDER THE RUG WHAT IS (I THINK, ANYWAY) THE MOST SIGNIFICANT ASPECT OF WHAT IS REALLY GOING ON HERE : A VERY DEEP AND BEAUTIFUL CONNECTION BETWEEN TOPOLOGY, GEOMETRY, AND ANALYSIS (SPECIFICALLY, PARTIAL DIFFERENTIAL EQUATIONS).

WE ARE NOW ENTERING INTO A HUGE SUBJECT ("GEOMETRIC ANALYSIS") AND WILL NOT EVEN BE ABLE TO SCRATCH THE SURFACE.

BEFORE WE CAN EVEN START WE NEED A NEW WAY OF LOOKING AT HYPERBOLIC 2-MANIFOLDS AND FOR THIS WE MUST FINALLY COME TO TERMS WITH VARIOUS NOTIONS OF "CURVATURE".

ALTHOUGH IT COULD BE DONE MUCH MORE GENERALLY I WILL INTRODUCE THESE NOTIONS ONLY FOR THE LEVI-CIVITA CONNECTION ON A RIEMANNIAN MANIFOLD, WHERE WE HAVE SEEN SOME MODEST FORM OF MOTIVATION:

<u>RECALL</u>: (M, g) = RIEMANNIAN MANIFOLD, (U, φ) = CHART WITH COORDINATE FUNCTIONS $x^1, \ldots, x^n$ AND METRIC COMPONENTS g_{ij}. THEN THERE IS ANOTHER COORDINATE SYSTEM ON SOME OPEN SUBSET OF U FOR WHICH THE METRIC COMPONENTS ARE δ_{ij} IF AND ONLY IF $R_{ijk}^{\;\;\;\;\ell} = 0$, $i, j, k, \ell = 1, \ldots, n$, WHERE

$$R_{ijk}^{\;\;\;\;\ell} = \frac{\partial \Gamma_{ik}^{\ell}}{\partial x^j} + \Gamma_{j\alpha}^{\ell}\,\Gamma_{ik}^{\alpha} - \frac{\partial \Gamma_{jk}^{\ell}}{\partial x^i} - \Gamma_{i\alpha}^{\ell}\,\Gamma_{jk}^{\alpha}.$$

AND

$$\Gamma^{i}_{jk} = \frac{1}{2} g^{i\alpha} \left(\frac{\partial g_{k\alpha}}{\partial x^{j}} + \frac{\partial g_{j\alpha}}{\partial x^{k}} - \frac{\partial g_{jk}}{\partial x^{\alpha}} \right)$$

THESE R^{i}_{jkl} WILL EVENTUALLY APPEAR AS THE COMPONENTS OF THE "RIEMANN CURVATURE TENSOR" IN THE CHART (U, φ), BUT OUR DEFINITION WILL BE INDEPENDENT OF COORDINATES AND MAY LOOK A BIT STRANGE FIRST TIME THROUGH. FOR ONE LAST BIT OF MOTIVATION, DO THE FOLLOWING :

<u>EXERCISE</u> : LET ∇ BE THE STANDARD CONNECTION ON $\mathbb{R}^n$ ($\nabla_V W = \nabla_V (w^i \frac{\partial}{\partial x^i}) = V(w^i) \frac{\partial}{\partial x^i}$). NOW LET X, Y AND Z BE THREE SMOOTH VECTOR FIELDS ON $\mathbb{R}^n$. SHOW THAT

$$\nabla_X (\nabla_Y Z) - \nabla_Y (\nabla_X Z) = \nabla_{[X,Y]} Z ,$$

I.E.,

$$- \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z = 0$$

THE FAILURE OF THIS CONDITION TO BE SATISFIED IN A GENERAL MANIFOLD WITH CONNECTION IS MEASURED BY "CURVATURE".

JUST AS A CONNECTION CAN BE VIEWED AS A MAP FROM ORDERED PAIRS OF VECTOR FIELDS TO VECTOR FIELDS, SO THE <u>RIEMANN CURVATURE TENSOR</u> R OF A CONNECTION ∇ CAN BE THOUGHT OF AS A MAP

$$R : \mathcal{T}(TM) \times \mathcal{T}(TM) \times \mathcal{T}(TM) \longrightarrow \mathcal{T}(TM)$$

CARRYING AN ORDERED TRIPLE (X, Y, Z) OF VECTOR FIELDS ON M TO A VECTOR FIELD

$$R(X,Y)Z \in \mathcal{T}(TM)$$

ON M DEFINED BY

$$\boxed{R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z}$$

EQUIVALENTLY, R CAN BE THOUGHT OF AS ASSIGNING TO ANY ORDERED PAIR (X, Y) OF VECTOR FIELDS ON M AN OPERATOR

$$R(X,Y) : \mathcal{T}(M) \longrightarrow \mathcal{T}(M)$$

$$Z \longrightarrow R(X,Y)Z$$

<u>EXERCISE</u> : PROVE EACH OF THE FOLLOWING :

(a) $R(X,Y)Z = -R(Y,X)Z \qquad \forall\ X, Y, Z \in \mathcal{T}(TM)$

(b) $R(X,Y)Z$ IS LINEAR IN EACH OF X, Y AND Z SEPARATELY, E.G.,

$$R(a_1 X_1 + a_2 X_2, Y)Z = a_1 R(X_1, Y)Z + a_2 R(X_2, Y)Z.$$

(c) $R(X,Y)Z$ IS A $C^\infty(M)$-MODULE HOMOMORPHISM IN EACH OF X, Y AND Z SEPARATELY, E.G., $\forall f \in C^\infty(M)$,

$$R(X, fY)Z = f R(X,Y)Z.$$

AT THIS POINT WE WILL ONCE AGAIN SPECIALIZE TO THE CASE OF THE LEVI-CIVITA CONNECTION ∇ ON A RIEMANNIAN MANIFOLD (M, g).

<u>EXERCISE</u>: LET $X, Y, Z, W \in \mathcal{T}(TM)$. PROVE EACH OF THE FOLLOWING:

(a) $g(R(X,Y)Z, W) = -g(Z, R(X,Y)W)$

(b) $g(R(X,Y)Z, W) + g(R(Z,X)Y, W) + g(R(Y,Z)X, W) = 0$

NOW, WITH THESE FEW BASIC PROPERTIES HANDLED, LET'S SEE WHAT THE CONNECTION IS BETWEEN $R(X,Y)Z$ AND $R^{l}_{\ ;ijk}$.

SO SUPPOSE (M, g) IS A RIEMANNIAN MANIFOLD, R IS ITS RIEMANN CURVATURE TENSOR AND (U, φ) IS A CHART WITH COORDINATE FUNCTIONS $x^1, \dots, x^n$. FOR CONVENIENCE WE WILL WRITE

$$\partial_i := \frac{\partial}{\partial x^i} \quad , \quad i = 1, \dots, n$$

NOW COMPUTE

$$\begin{aligned}
R(\partial_i, \partial_j)\partial_k &= -\nabla_{\partial_i} \nabla_{\partial_j} \partial_k + \nabla_{\partial_j} \nabla_{\partial_i} \partial_k + \nabla_{[\partial_i, \partial_j]} \partial_k \\
&= -\nabla_{\partial_i}(\Gamma^\alpha_{jk} \partial_\alpha) + \nabla_{\partial_j}(\Gamma^\alpha_{ik} \partial_\alpha) + 0 \\
&= -\Gamma^\alpha_{jk}(\nabla_{\partial_i} \partial_\alpha) - (\partial_i \Gamma^\alpha_{jk})\partial_\alpha + \Gamma^\alpha_{ik}(\nabla_{\partial_j} \partial_\alpha) \\
&\qquad + (\partial_j \Gamma^\alpha_{ik})\partial_\alpha \\
&= -\Gamma^\alpha_{jk} \Gamma^\ell_{i\alpha} \partial_\ell - (\partial_i \Gamma^\alpha_{jk})\partial_\alpha + \Gamma^\alpha_{ik} \Gamma^\ell_{j\alpha} \partial_\ell \\
&\qquad + (\partial_j \Gamma^\alpha_{ik})\partial_\alpha \\
&= -\Gamma^\ell_{i\alpha} \Gamma^\alpha_{jk} \partial_\ell - (\partial_i \Gamma^\ell_{jk})\partial_\ell + \Gamma^\ell_{j\alpha} \Gamma^\alpha_{ik} \partial_\ell \\
&\qquad + (\partial_j \Gamma^\ell_{ik})\partial_\ell \\
&= \left(\frac{\partial \Gamma^\ell_{ik}}{\partial x^j} - \frac{\partial \Gamma^\ell_{jk}}{\partial x^i} + \Gamma^\ell_{j\alpha} \Gamma^\alpha_{ik} - \Gamma^\ell_{i\alpha} \Gamma^\alpha_{jk} \right) \partial_\ell \\
&= R^\ell_{\ ijk} \partial_\ell
\end{aligned}$$

THE SYMMETRY $R(X,Y)Z = -R(Y,X)Z$ NOW BECOMES

$$R^{\ell}_{\ ijk} = -R^{\ell}_{\ jik}$$

THOSE SYMMETRIES INVOLVING THE METRIC ARE MORE CONVENIENTLY EXPRESSED IN TERMS OF

$$\begin{aligned} R_{ijk\ell} &= g(R(\partial_i,\partial_j)\partial_k, \partial_\ell) \\ &= g(R^{\beta}_{\ ijk}\partial_\beta, \partial_\ell) \\ &= R^{\beta}_{\ ijk}\, g(\partial_\beta,\partial_\ell) \\ &= R^{\beta}_{\ ijk}\, g_{\beta\ell} \qquad (\text{"LOWER THE INDEX"}) \end{aligned}$$

THUS, FOR EXAMPLE, $g(R(X,Y)Z, W) = -g(Z, R(X,Y)W)$ GIVES

$$\begin{aligned} R_{ijk\ell} &= g(R(\partial_i,\partial_j)\partial_k, \partial_\ell) \\ &= -g(\partial_k, R(\partial_i,\partial_j)\partial_\ell) \\ &= -g(R(\partial_i,\partial_j)\partial_\ell, \partial_k) \\ R_{ijk\ell} &= -R_{ij\ell k} \end{aligned}$$

SIMILARLY,

$$R_{ijk\ell} + R_{kij\ell} + R_{jki\ell} = 0$$

FOR AN n-DIMENSIONAL MANIFOLD THERE ARE

$$n^4$$

SUCH COMPONENTS $R_{ijk\ell}$ FOR R IN ANY CHART, E.G., 256 FOR $n = 4$.

TAKING INTO ACCOUNT ALL OF THE AVAILABLE SYMMETRIES REDUCES THE NUMBER OF INDEPENDENT COMPONENTS TO

$$\frac{n(n-1)}{2},$$

E.G., 6 FOR $n = 4$.

SOON WE WILL ISOLATE OBJECTS THAT ARE OF A MORE MANAGEABLE SIZE AND CONTAIN AT LEAST SOME OF THE INFORMATION IN R:

"SECTIONAL CURVATURE"

"RICCI CURVATURE"

"SCALAR CURVATURE"

FIRST, HOWEVER, WE CONSIDER THE ONE "SIMPLE" CASE:

$$n = 2 \qquad n^4 = 16 \qquad \frac{n(n-1)}{2} = 1$$

LET (M, g) BE A 2-DIMENSIONAL RIEMANNIAN MANIFOLD (I.E., RIEMANNIAN SURFACE).

USING ALL OF THE AVAILABLE SYMMETRIES OF $R_{ijk\ell}$ WE FIND THAT THE ONLY (POTENTIALLY) NONZERO COMPONENTS ARE

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112} .$$

LET (U, φ) BE A CHART WITH COORDINATE FUNCTIONS x^1, x^2 AND WRITE ∂_i FOR $\frac{\partial}{\partial x^i}$, $i = 1, 2$. THE METRIC COMPONENTS ARE

$$g_{ij} = g(\partial_i, \partial_j) \quad , \quad i, j = 1, 2$$

AND WE WRITE

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$$(g^{ij}) = (g_{ij})^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}$$

$$\det(g) = g_{11} g_{22} - g_{12} g_{21}$$

COMPUTE THE CHRISTOFFEL SYMBOLS :

E.G.,

$$\Gamma^1_{11} = \tfrac{1}{2} g^{1m} \left\{ \frac{\partial g_{1m}}{\partial x^1} + \frac{\partial g_{1m}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^m} \right\}$$

$$= \tfrac{1}{2} g^{1m} \left\{ 2 \frac{\partial g_{1m}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^m} \right\}$$

$$= \tfrac{1}{2} g^{11} \left\{ 2 \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1} \right\} + \tfrac{1}{2} g^{12} \left\{ 2 \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2} \right\}$$

$$= \tfrac{1}{2} \frac{1}{\det(g)} g_{22} \frac{\partial g_{11}}{\partial x^1} - \tfrac{1}{2} \frac{1}{\det(g)} g_{12} \left\{ 2 \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2} \right\}$$

$$= \frac{1}{\det(g)} \left\{ \tfrac{1}{2} g_{22} \frac{\partial g_{11}}{\partial x^1} - g_{12} \frac{\partial g_{12}}{\partial x^1} + \tfrac{1}{2} g_{12} \frac{\partial g_{11}}{\partial x^2} \right\}$$

$$\Gamma^1_{11} = \frac{1}{\det(g)} \begin{vmatrix} \tfrac{1}{2} \frac{\partial g_{11}}{\partial x^1} & g_{12} \\ \frac{\partial g_{12}}{\partial x^1} - \tfrac{1}{2} \frac{\partial g_{11}}{\partial x^2} & g_{22} \end{vmatrix}$$

<u>EXERCISE</u> : DERIVE THE REST :

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{\det(g)} \begin{vmatrix} \tfrac{1}{2} \frac{\partial g_{11}}{\partial x^2} & g_{12} \\ \tfrac{1}{2} \frac{\partial g_{12}}{\partial x^1} & g_{22} \end{vmatrix}$$

$$\Gamma^1_{22} = \frac{1}{\det(g)} \begin{vmatrix} \frac{\partial g_{12}}{\partial x^2} - \tfrac{1}{2} \frac{\partial g_{22}}{\partial x^1} & g_{12} \\ \tfrac{1}{2} \frac{\partial g_{22}}{\partial x^2} & g_{22} \end{vmatrix}$$

$$\Gamma^{2}_{11} = \frac{1}{\det(g)} \begin{vmatrix} g_{11} & \frac{1}{2}\frac{\partial g_{11}}{\partial x^1} \\ g_{12} & \frac{\partial g_{12}}{\partial x^1} - \frac{1}{2}\frac{\partial g_{11}}{\partial x^2} \end{vmatrix}$$

$$\Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{\det(g)} \begin{vmatrix} g_{11} & \frac{1}{2}\frac{\partial g_{11}}{\partial x^2} \\ g_{21} & \frac{1}{2}\frac{\partial g_{22}}{\partial x^1} \end{vmatrix}$$

$$\Gamma^{2}_{22} = \frac{1}{\det(g)} \begin{vmatrix} g_{11} & \frac{\partial g_{12}}{\partial x^2} - \frac{1}{2}\frac{\partial g_{22}}{\partial x^1} \\ g_{21} & \frac{1}{2}\frac{\partial g_{22}}{\partial x^2} \end{vmatrix}$$

<u>NOTE</u> : FOR <u>ORTHOGONAL</u> COORDINATES, $g_{12} = g_{21} = 0$ AND $\det(g) = g_{11}\, g_{22}$ SO THESE REDUCE TO

$$\Gamma^{1}_{11} = \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial x^1} \qquad \Gamma^{1}_{12} = \Gamma^{1}_{21} = \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial x^2}$$

$$\Gamma^{1}_{22} = -\frac{1}{2g_{11}}\frac{\partial g_{22}}{\partial x^1} \qquad \Gamma^{2}_{11} = -\frac{1}{2g_{22}}\frac{\partial g_{11}}{\partial x^2}$$

$$\Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial x^1} \qquad \Gamma^{2}_{22} = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial x^2}$$

FROM THIS POINT ON WE WILL DO ALL OF OUR CALCULATIONS IN SUCH ORTHOGONAL COORDINATES. EVERY RIEMANNIAN 2-MANIFOLD HAS SUCH COORDINATES IN A NEIGHBORHOOD OF

EACH POINT. IN FACT, IT IS A DEEP THEOREM THAT LOCAL ISOTHERMAL COORDINATES EXIST NEAR ANY POINT ($g_{12} = g_{21} = 0$ AND $g_{22} = g_{11}$ SO THE METRIC ASSUMES THE FORM $g_{11}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)$). MOST OF THE EXAMPLES WE DEAL WITH COME NATURALLY EQUIPPED WITH ORTHOGONAL COORDINATES.

NOW WRITE OUT $\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$.

$$\nabla_{\partial_1} \partial_1 = \Gamma^k_{11} \partial_k = \Gamma^1_{11} \partial_1 + \Gamma^2_{11} \partial_2$$

$$\nabla_{\partial_1} \partial_1 = \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial x^1} \partial_1 - \frac{1}{2g_{22}} \frac{\partial g_{11}}{\partial x^2} \partial_2$$

AND SIMILARLY

$$\nabla_{\partial_1} \partial_2 = \nabla_{\partial_2} \partial_1 = \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial x^2} \partial_1 + \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial x^1} \partial_2$$

$$\nabla_{\partial_2} \partial_2 = -\frac{1}{2g_{11}} \frac{\partial g_{22}}{\partial x^1} \partial_1 + \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial x^2} \partial_2$$

SINCE ALL OF THE BRACKETS $[\partial_i, \partial_j]$ ARE ZERO,

$$R(\partial_1, \partial_2)\partial_1 = \nabla_{\partial_2} \nabla_{\partial_1} \partial_1 - \nabla_{\partial_1} \nabla_{\partial_2} \partial_1$$

$$g(R(\partial_1, \partial_2)\partial_1, \partial_2) = g(\nabla_{\partial_2} \nabla_{\partial_1} \partial_1, \partial_2) - g(\nabla_{\partial_1} \nabla_{\partial_2} \partial_1, \partial_2)$$

NOW OBSERVE THAT

$$\frac{\partial}{\partial x^1} g(\nabla_{\partial_2}\partial_1, \partial_2) = g(\nabla_{\partial_2}\partial_1, \nabla_{\partial_1}\partial_2) + g(\nabla_{\partial_1}\nabla_{\partial_2}\partial_1, \partial_2)$$

(DEFINING PROPERTY OF LEVI-CIVITA)

SO

$$g(\nabla_{\partial_1}\nabla_{\partial_2}\partial_1, \partial_2) = \frac{\partial}{\partial x^1} g(\nabla_{\partial_2}\partial_1, \partial_2) - g(\nabla_{\partial_2}\partial_1, \nabla_{\partial_1}\partial_2)$$

AND, SIMILARLY,

$$g(\nabla_{\partial_2}\nabla_{\partial_1}\partial_1, \partial_2) = \frac{\partial}{\partial x^2} g(\nabla_{\partial_1}\partial_1, \partial_2) - g(\nabla_{\partial_1}\partial_1, \nabla_{\partial_2}\partial_2).$$

THUS,

$$\begin{aligned} g(R(\partial_1,\partial_2)\partial_1, \partial_2) = {} & \frac{\partial}{\partial x^2} g(\nabla_{\partial_1}\partial_1, \partial_2) - g(\nabla_{\partial_1}\partial_1, \nabla_{\partial_2}\partial_2) \\ & - \frac{\partial}{\partial x^1} g(\nabla_{\partial_2}\partial_1, \partial_2) + g(\nabla_{\partial_2}\partial_1, \nabla_{\partial_1}\partial_2) \\ = {} & \frac{\partial}{\partial x^2}\left(-\frac{1}{2}\frac{\partial g_{11}}{\partial x^2}\right) - \left(-\frac{1}{4g_{11}}\frac{\partial g_{11}}{\partial x^1}\frac{\partial g_{22}}{\partial x^1}\right. \\ & \qquad \left. - \frac{1}{4g_{22}}\frac{\partial g_{11}}{\partial x^2}\frac{\partial g_{22}}{\partial x^2}\right) \\ & - \frac{\partial}{\partial x^1}\left(\frac{1}{2}\frac{\partial g_{22}}{\partial x^1}\right) + \left(\frac{1}{4g_{11}}\left(\frac{\partial g_{11}}{\partial x^2}\right)^2\right. \\ & \qquad \left. + \frac{1}{4g_{22}}\left(\frac{\partial g_{22}}{\partial x^1}\right)^2\right) \end{aligned}$$

$$g(R(\partial_1,\partial_2)\partial_1,\partial_2) = -\frac{1}{2}\frac{\partial^2 g_{11}}{(\partial x^2)^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{(\partial x^1)^2} + \frac{1}{4g_{11}}\left(\frac{\partial g_{11}}{\partial x^1}\frac{\partial g_{22}}{\partial x^1} + \left(\frac{\partial g_{11}}{\partial x^2}\right)^2\right)$$
$$+ \frac{1}{4g_{22}}\left(\frac{\partial g_{11}}{\partial x^2}\frac{\partial g_{22}}{\partial x^2} + \left(\frac{\partial g_{22}}{\partial x^1}\right)^2\right)$$

EXERCISE : BY DIRECTLY COMPUTING THE PARTIAL DERIVATIVES SHOW THAT R_{1212} IS GIVEN BY

$$R_{1212} = g(R(\partial_1,\partial_2)\partial_1,\partial_2) =$$
$$-\frac{1}{2}\sqrt{g_{11}g_{22}}\left[\frac{\partial}{\partial x^1}\left(\frac{\frac{\partial g_{22}}{\partial x^1}}{\sqrt{g_{11}g_{22}}}\right) + \frac{\partial}{\partial x^2}\left(\frac{\frac{\partial g_{11}}{\partial x^2}}{\sqrt{g_{11}g_{22}}}\right)\right]$$

DEFINITION : IF (M,g) IS A RIEMANNIAN 2-MANIFOLD WITH RIEMANN CURVATURE TENSOR R, THEN THE GAUSSIAN CURVATURE OF (M,g) IS THE REAL-VALUED FUNCTION

$$\mathcal{K}_g : M \to \mathbb{R}$$

ON M DEFINED BY

$$\mathcal{K}_g = \frac{R_{1212}}{\det(g)}$$

IN ORTHOGONAL COORDINATES IT IS GIVEN BY

$$K_g = -\frac{1}{2\sqrt{g_{11}g_{22}}}\left[\frac{\partial}{\partial x^1}\left(\frac{\frac{\partial g_{22}}{\partial x^1}}{\sqrt{g_{11}g_{22}}}\right) + \frac{\partial}{\partial x^2}\left(\frac{\frac{\partial g_{11}}{\partial x^2}}{\sqrt{g_{11}g_{22}}}\right)\right]$$

TAKE A FEW MINUTES TO COMPUTE THIS FOR SOME FAMILIAR EXAMPLES.

<u>EXERCISE</u> : COMPUTE THE GAUSSIAN CURVATURE K_g FOR EACH OF THE FOLLOWING.

1. THE HYPERBOLIC PLANE H^2.

   ANSWER : $K_g \equiv -1$

2. THE SPHERE OF RADIUS r WITH ITS USUAL RIEMANNIAN METRIC.

   ANSWER : $K_g \equiv \frac{1}{r^2}$

3. THE EUCLIDEAN PLANE $\mathbb{R}^2$ WITH ITS USUAL RIEMANNIAN METRIC.

   ANSWER : $K_g \equiv 0$

IN CLASSICAL DIFFERENTIAL GEOMETRY MANY OF THE MOST IMPORTANT EXAMPLES ARE 2-DIMENSIONAL SUBMANIFOLDS OF $\mathbb{R}^3$ WITH THE RIEMANNIAN METRIC INDUCED FROM $\mathbb{R}^3$. THE FOLLOWING EXERCISES WILL DESCRIBE SOME OF THESE.

<u>EXERCISE</u> (<u>SURFACES OF REVOLUTION</u>): LET M BE A 2-DIMENSIONAL SMOOTH SUBMANIFOLD OF $\mathbb{R}^3$ THAT CAN BE PARAMETRIZED BY A MAP ξ FROM THE uv-PLANE INTO $\mathbb{R}^3$ OF THE FORM

$$\xi(u,v) = (f(u)\cos v,\ f(u)\sin v,\ g(u))$$

WHERE $\alpha(u) = \xi(u,0) = (f(u), 0, g(u))$ IS A SMOOTH CURVE IN THE XZ-PLANE WITH $f(u) \neq 0 \ \forall u$ AND $\alpha'(u) = (f'(u), 0, g'(u))$ NONZERO $\forall u$. THUS, M IS OBTAINED BY REVOLVING $\alpha(u)$ ABOUT THE Z-AXIS.

1. SHOW THAT u AND v ARE LOCAL COORDINATES ON A NEIGHBORHOOD OF EACH POINT IN M (<u>HINT</u>: INVERSE FUNCTION THEOREM)

2. SKETCH EACH OF THE FOLLOWING EXAMPLES.

   (a) <u>TORUS</u> : $\alpha(u) = (a + b\cos u,\ 0,\ b\sin u)$
   WHERE $0 < b < a$

   (b) <u>CATENOID</u> : $\alpha(u) = (a\cosh(\frac{u}{a}),\ 0,\ u)$, $a > 0$

   (c) <u>PSEUDOSPHERE</u> : $\alpha(u) = (a\sin u,\ 0,\ a(\cos u + \ln(\tan u)))$,
   $a > 0$, $0 < u < \frac{\pi}{2}$

   <u>NOTE</u> : THIS CURVE IS CALLED A <u>TRACTRIX</u>.

3. LET g BE THE RIEMANNIAN METRIC ON M INDUCED FROM $\mathbb{R}^3$. SHOW THAT

$$g = ((f'(u))^2 + (g'(u))^2)\, du \otimes du + (f(u))^2 dv \otimes dv$$

4. WRITE OUT THE METRIC FOR EACH OF THE EXAMPLES IN PROBLEM #2 AND COMPUTE ITS GAUSSIAN CURVATURE.

ANSWERS:

TORUS: $\mathcal{K} = \dfrac{\cos u}{b(a + b\cos u)}$

CATENOID: $\mathcal{K} = -\dfrac{1}{a^2}\operatorname{sech}^4\left(\dfrac{u}{a}\right)$

PSEUDOSPHERE: $\mathcal{K} = -\dfrac{1}{a^2}$

<u>NOTE</u>: THE PSEUDOSPHERE HAS CONSTANT NEGATIVE GAUSSIAN CURVATURE AND IS A SUBMANIFOLD OF $\mathbb{R}^3$. HILBERT PROVED A REMARKABLE THEOREM TO THE EFFECT THAT NO <u>COMPLETE</u> SURFACE OF CONSTANT NEGATIVE GAUSSIAN CURVATURE CAN BE EMBEDDED IN $\mathbb{R}^3$ ("COMPLETE" MEANS EVERY CAUCHY SEQUENCE IN IT CONVERGES TO SOMETHING IN IT).

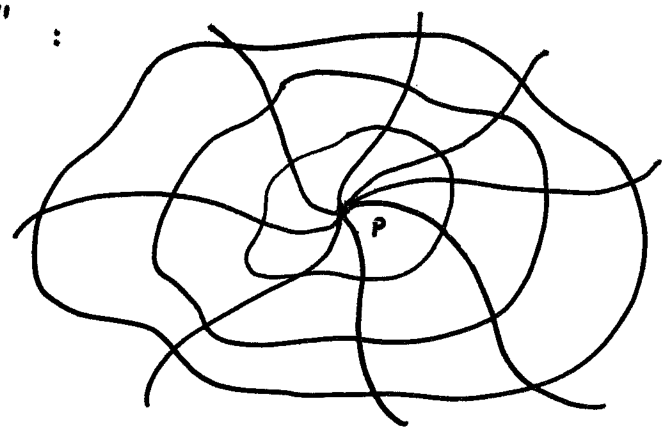
5. SHOW THAT THE PSEUDOSPHERE IS NOT COMPLETE (WITHOUT APPEALING TO HILBERT'S THEOREM).

WE HAVE ALREADY SEEN THAT A LOCAL ISOMETRY PRESERVES THE LEVI-CIVITA CONNECTION SO, IN THE CASE OF RIEMANNIAN SURFACES, IT WILL PRESERVE GAUSSIAN CURVATURE. IN PARTICULAR,

A HYPERBOLIC 2-MANIFOLD (LOCALLY ISOMETRIC TO H^2) HAS CONSTANT GAUSSIAN CURVATURE -1.

THIS CAN BE REVERSED ALSO. IN THE INTEREST OF TIME I WILL JUST SKETCH THE ARGUMENT:

FOR ANY RIEMANNIAN SURFACE M ONE CAN USE THE EXPONENTIAL MAP AT ANY POINT TO PRODUCE A "GEODESIC POLAR COORDINATE SYSTEM":



ANY POINT IS IDENTIFIED BY THE ARC LENGTH r ALONG THE GEODESIC CONNECTING IT TO p AND THE ANGLE θ IN $T_p(M)$ SPECIFYING THE DIRECTION OF THE GEODESIC THAT GETS THERE FROM p.

IN TERMS OF THESE COORDINATES THE METRIC TAKES THE FORM

$$g = dr \otimes dr + G(r,\theta)\, d\theta \otimes d\theta$$

AND THE EQUATION WE DERIVED FOR THE GAUSSIAN CURVATURE BECOMES

$$\mathcal{K}_g = - \frac{\frac{\partial^2 \sqrt{G}}{\partial r^2}}{\sqrt{G}}$$

NOW SUPPOSE (M, g) HAS CONSTANT GAUSSIAN CURVATURE -1. THEN

$$\frac{\partial^2 \sqrt{G}}{\partial r^2} = \sqrt{G}$$

ONE SHOWS THAT

$$\sqrt{G}\,(0,\theta) = 0 \quad , \quad \frac{\partial \sqrt{G}}{\partial r}(0,\theta) = 1$$

AND CONCLUDES THAT $\sqrt{G}\,(r,\theta) = \varphi(r)$ IS THE UNIQUE SOLUTION TO

$$\varphi''(r) - \varphi(r) = 0$$
$$\varphi(0) = 0$$
$$\varphi'(0) = 1$$

BUT THE SOLUTION TO THIS IS CLEAR : $\varphi(r) = \sinh r$

$$G(r,\theta) = \sinh^2 r$$

$$g = dr \otimes dr + \sinh^2 r \, d\theta \otimes d\theta$$

LOCALLY, THEY ARE ALL THE SAME.