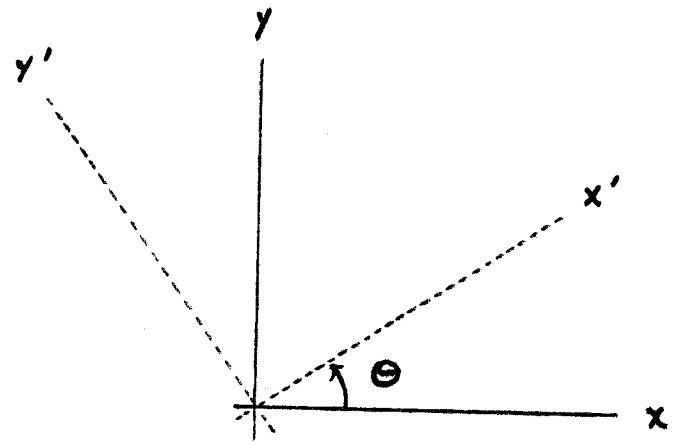
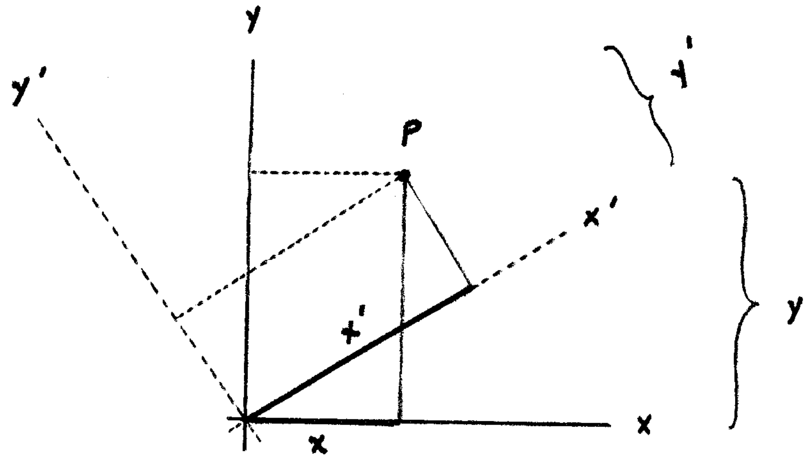
HYPERBOLIC FORM OF THE LORENTZ TRANSFORMATIONS AND THE "ADDITION OF VELOCITIES"

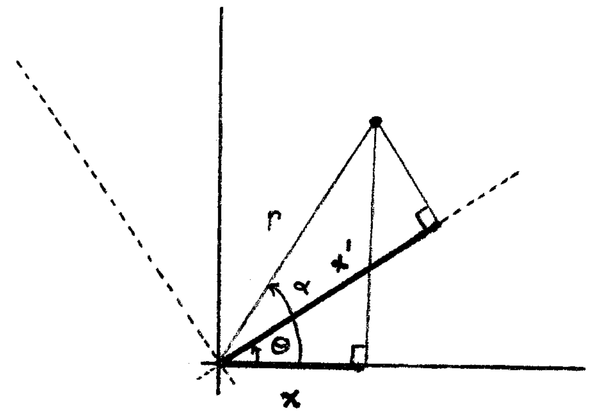
WE BEGIN WITH SOMETHING EASIER, I.E., THE ROTATION OF A COORDINATE SYSTEM IN THE PLANE.



ANY POINT P IN THE PLANE NOW HAS TWO SETS OF COORDINATES (x, y) AND (x', y').



WE WANT TO FIND THE RELATIONSHIP BETWEEN (x, y) AND (x', y'), I.E., THE "COORDINATE TRANSFORMATION".



FOR EXAMPLE,

$$\cos\alpha = \frac{x}{r} \Rightarrow x = r\cos\alpha$$

$$\cos(\alpha-\theta) = \frac{x'}{r} \Rightarrow x' = r\cos(\alpha-\theta)$$

$$\sin\alpha = \frac{y}{r} \Rightarrow y = r\sin\alpha$$

$$\sin(\alpha-\theta) = \frac{y'}{r} \Rightarrow y' = r\sin(\alpha-\theta)$$

SO

$$\begin{aligned} x' = r\cos(\alpha-\theta) &= r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) \\ &= (r\cos\alpha)\cos\theta + (r\sin\alpha)\sin\theta \\ &= x\cos\theta + y\sin\theta \end{aligned}$$

AND SIMILARLY

$$y' = -x\sin\theta + y\cos\theta$$

<u>COORDINATE</u> <u>TRANSFORMATION</u> :

$$(1) \qquad \begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \end{cases}$$

NOTICE THAT WE COULD ALSO DESCRIBE THE RELATIVE POSITION OF THE TWO COORDINATE SYSTEMS BY THE SLOPE m OF THE x'-AXIS RATHER THAN THE ANGLE θ (ASSUME $0 \leq \theta < \frac{\pi}{2}$ FOR CONVENIENCE).

$$\tan\theta = m$$

$$\sin\theta = m\cos\theta$$

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow$$

$$\cos^2\theta + m^2\cos^2\theta = 1$$

$$\cos^2\theta\,(1+m^2) = 1$$

$$\cos^2\theta = \frac{1}{1+m^2}$$

$$\cos\theta = \frac{1}{\sqrt{1+m^2}}$$

$$\sin\theta = \frac{m}{\sqrt{1+m^2}}$$

SO

$$(2) \quad \begin{cases} x' = \dfrac{1}{\sqrt{1+m^2}}\,x + \dfrac{m}{\sqrt{1+m^2}}\,y \\[2ex] y' = -\dfrac{m}{\sqrt{1+m^2}}\,x + \dfrac{1}{\sqrt{1+m^2}}\,y \end{cases}$$

THE DESCRIPTION IN TERMS OF THE ANGLE θ IS PRETTIER. IT ALSO HAS ANOTHER ADVANTAGE :

SUPPOSE WE ROTATE TWICE, FROM (x,y) TO (x',y') THROUGH ANGLE θ

$$\begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \end{cases}$$

AND THEN FROM (x',y') TO (x'',y'') THROUGH ANGLE ϕ

$$\begin{cases} x'' = x'\cos\phi + y'\sin\phi \\ y'' = -x'\sin\phi + y'\cos\phi \end{cases}$$

THEN THE RELATIONSHIP BETWEEN (x'',y'') AND (x,y) IS

$$\begin{cases} x'' = x\cos(\theta+\phi) + y\sin(\theta+\phi) \\ y'' = -x\sin(\theta+\phi) + y\cos(\theta+\phi) \end{cases}$$

THE ANGLES JUST "ADD". HOWEVER, SLOPES ARE NOT SO NICELY BEHAVED :

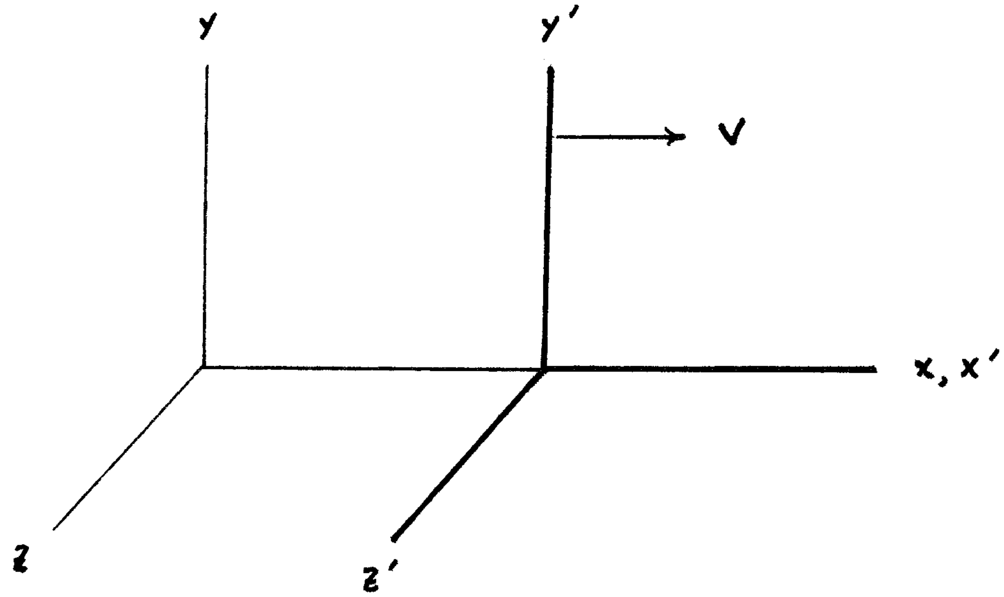
$$m = \tan\theta \qquad \bar{m} = \tan\phi$$

$$\Longrightarrow \qquad \tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

$$= \frac{m + \bar{m}}{1 - m\bar{m}}$$

AND THIS IS WHAT WE WOULD HAVE TO SUBSTITUTE FOR m IN EQUATION (2) TO GET (x'',y'') IN TERMS OF (x,y).

NOW LET'S TURN TO THE LORENTZ TRANSFORMATIONS RELATING TWO INERTIAL FRAMES OF REFERENCE (x, y, z, t) AND (x', y', z', t').



SINCE $y' = y$ AND $z' = z$ I WILL NOT BOTHER TO WRITE THESE. ALSO, SINCE SUCH OBSERVERS AGREE ON THE SPEED OF LIGHT WE CAN CHOOSE UNITS IN WHICH $c = 1$ SO THE LORENTZ TRANSFORMATION FROM (x, t) TO (x', t') CAN BE WRITTEN

$$(3) \qquad \begin{cases} x' = \dfrac{1}{\sqrt{1-v^2}}\, x - \dfrac{v}{\sqrt{1-v^2}}\, t \\[2ex] t' = -\dfrac{v}{\sqrt{1-v^2}}\, x + \dfrac{1}{\sqrt{1-v^2}}\, t \end{cases}$$

EXCEPT FOR A FEW MINUS SIGNS THIS LOOKS REMARKABLY LIKE OUR ROTATION (2) DESCRIBED IN TERMS OF SLOPE.

WE WILL SHOW NOW THAT BY DESCRIBING THE RELATIVE MOTION OF THE TWO FRAMES NOT BY THE VELOCITY, BUT RATHER BY THE "RAPIDITY" φ DEFINED BY

$$v = \tanh \varphi \qquad (\text{I.E., } \varphi = \tanh^{-1} v)$$

WE OBTAIN A SIMPLER DESCRIPTION OF THE LORENTZ TRANSFORMATION (3) AS A "HYPERBOLIC ROTATION":

$$\tanh \varphi = v$$

$$\sinh \varphi = v \cosh \varphi$$

$$\cosh^2 \varphi - \sinh^2 \varphi = 1 \quad \Rightarrow$$

$$\cosh^2 \varphi - v^2 \cosh^2 \varphi = 1$$

$$\cosh^2 \varphi \, (1 - v^2) = 1$$

$$\cosh^2 \varphi = \frac{1}{1 - v^2}$$

$$\cosh \varphi = \frac{1}{\sqrt{1 - v^2}}$$

$$\sinh \varphi = \frac{v}{\sqrt{1 - v^2}}$$

THUS,

$$(4) \quad \begin{cases} x' = x \cosh \varphi - t \sinh \varphi \\ t' = -x \sinh \varphi + t \cosh \varphi \end{cases}$$

THIS IS THE SO-CALLED HYPERBOLIC FORM OF THE LORENTZ TRANSFORMATIONS. IT'S VERY CONVENIENT FOR EXPOSING THE UNDERLYING GEOMETRY OF SPECIAL RELATIVITY (FIRST NOTICED BY HERMANN MINKOWSKI).

NOTICE THAT THE ESSENTIAL FEATURE OF AN ORDINARY ROTATION OF THE PLANE IS THAT IT LEAVES THE (SQUARED) EUCLIDEAN LENGTH OF A VECTOR "INVARIANT":

$$
\begin{aligned}
(x')^2 + (y')^2 &= (x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2 \\
&= x^2\cos^2\theta + 2xy\cos\theta\sin\theta + y^2\sin^2\theta + \\
&\qquad x^2\sin^2\theta - 2xy\cos\theta\sin\theta + y^2\cos^2\theta \\
&= x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta) \\
&= x^2 + y^2
\end{aligned}
$$

THE ESSENTIAL FEATURE OF A LORENTZ TRANSFORMATION IS THAT IT LEAVES THE SPACETIME INTERVAL INVARIANT:

$$
\begin{aligned}
(x')^2 - (t')^2 &= (x\cosh\varphi - t\sinh\varphi)^2 - (-x\sinh\varphi + t\cosh\varphi)^2 \\
&= x^2\cosh^2\varphi - 2xt\cosh\varphi\sinh\varphi + t^2\sinh^2\varphi \\
&\qquad - x^2\sinh^2\varphi + 2xt\cosh\varphi\sinh\varphi - t^2\cosh^2\varphi \\
&= x^2(\cosh^2\varphi - \sinh^2\varphi) - t^2(\cosh^2\varphi - \sinh^2\varphi) \\
&= x^2 - t^2.
\end{aligned}
$$

FINALLY, LET US PERFORM TWO SUCCESSIVE LORENTZ TRANSFORMATIONS.

$$\begin{cases} x' = x\cosh\varphi - t\sinh\varphi \\ t' = -x\sinh\varphi + t\cosh\varphi \end{cases} \quad \text{FOLLOWED BY} \quad \begin{cases} x'' = x'\cosh\xi - t'\sinh\xi \\ t'' = -x'\sinh\xi + t'\cosh\xi \end{cases}$$

THEN

$$\begin{aligned} x'' &= x'\cosh\xi - t'\sinh\xi \\ &= (x\cosh\varphi - t\sinh\varphi)\cosh\xi - (-x\sinh\varphi + t\cosh\varphi)\sinh\xi \\ &= x(\cosh\varphi\cosh\xi + \sinh\varphi\sinh\xi) \\ &\qquad - t(\sinh\varphi\cosh\xi + \cosh\varphi\sinh\xi) \\ &= x\cosh(\varphi+\xi) - t\sinh(\varphi+\xi) \end{aligned}$$

AND SIMILARLY FOR t'' SO

$$\begin{cases} x'' = x\cosh(\varphi+\xi) - t\sinh(\varphi+\xi) \\ t'' = -x\sinh(\varphi+\xi) + t\cosh(\varphi+\xi) \end{cases}$$

THE RAPIDITIES JUST "ADD" (LIKE ANGLES). VELOCITIES ARE A DIFFERENT STORY:

$$V = \tanh\varphi \qquad \bar{V} = \tanh\xi$$

$$\Rightarrow \qquad \tanh(\varphi+\xi) = \frac{\tanh\varphi + \tanh\xi}{1 + \tanh\varphi\tanh\xi} = \frac{V+\bar{V}}{1+V\bar{V}}$$

WHICH IS THE <u>RELATIVISTIC</u> <u>ADDITION</u> <u>OF</u> <u>VELOCITIES</u> FORMULA.