

THE HYPERBOLIC FUNCTIONS

$$\text{HYPERBOLIC SINE} : \sinh x = \frac{1}{2}(e^x - e^{-x})$$

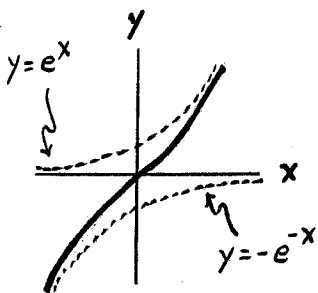
$$\text{HYPERBOLIC COSINE} : \cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\text{HYPERBOLIC TANGENT} : \tanh x = \frac{\sinh x}{\cosh x}$$

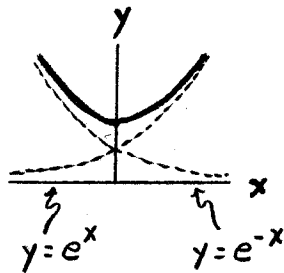
$$\text{HYPERBOLIC COTANGENT} : \coth x = \frac{\cosh x}{\sinh x}$$

$$\text{HYPERBOLIC SECANT} : \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\text{HYPERBOLIC COSECANT} : \operatorname{csch} x = \frac{1}{\sinh x}$$

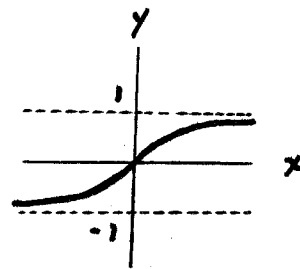


$$y = \sinh x$$



$$y = \cosh x$$

(CATENARY)



$$y = \tanh x$$

HYPERBOLIC FUNCTIONS SATISFY MANY IDENTITIES ANALOGOUS TO STANDARD TRIGONOMETRIC IDENTITIES, E.G.,

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

ETC.

SAMPLE PROOF :

$$\begin{aligned}
 \cosh^2 x - \sinh^2 x &= \left(\frac{1}{2}(e^x + e^{-x})\right)^2 - \left(\frac{1}{2}(e^x - e^{-x})\right)^2 \\
 &= \frac{1}{4}((e^x)^2 + 2e^x e^{-x} + (e^{-x})^2) \\
 &\quad - \frac{1}{4}((e^x)^2 - 2e^x e^{-x} + (e^{-x})^2) \\
 &= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) \\
 &= \frac{1}{4}(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}) \\
 &= 1
 \end{aligned}$$

DERIVATIVES :

$$\begin{aligned}
 (\sinh x)' &= \left(\frac{1}{2}(e^x - e^{-x})\right)' = \frac{1}{2}((e^x)' - (e^{-x})') \\
 &= \frac{1}{2}(e^x - (-e^{-x})) = \frac{1}{2}(e^x + e^{-x}) \\
 &= \cosh x
 \end{aligned}$$

SIMILAR CALCULATIONS GIVE THE FOLLOWING :

$$\begin{aligned}
 (\sinh x)' &= \cosh x \\
 (\cosh x)' &= \sinh x \\
 (\tanh x)' &= \operatorname{sech}^2 x \\
 (\coth x)' &= -\operatorname{csch}^2 x \\
 (\operatorname{sech} x)' &= -\operatorname{sech} x \tanh x \\
 (\operatorname{csch} x)' &= -\operatorname{csch} x \coth x
 \end{aligned}$$



NOTICE THE TWO
UNEXPECTED SIGNS

NOW COMBINE THESE WITH THE CHAIN RULE.

EXAMPLES :

$$1. (\cosh(x^3))' = \sinh(x^3)(x^3)' = 3x^2 \sinh(x^3)$$

$$2. (\ln(\tanh x))' = \frac{1}{\tanh x} (\tanh x)' = \operatorname{coth} x \operatorname{sech}^2 x$$

INTEGRALS :

$$\int \cosh x \, dx = \sinh x + C$$

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \operatorname{sech}^2 x \, dx = \tanh x + C$$

$$\int \operatorname{csech}^2 x \, dx = -\operatorname{coth} x + C$$

$$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csech} x \operatorname{coth} x \, dx = -\operatorname{csech} x + C$$

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NOTICE THE TWO
UNEXPECTED SIGNS

EXAMPLES :

$$1. \int \sinh^5 x \cosh x \, dx = \int u^5 \, du = \frac{1}{6} u^6 + C$$

$$u = \sinh x$$

$$du = \cosh x \, dx$$

$$= \frac{1}{6} \sinh^6 x + C$$

$$2. \int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx = \int \frac{1}{\cosh x} \sinh x \, dx$$

$$u = \cosh x \\ du = \sinh x \, dx$$

$$= \int \frac{1}{u} \, du = \ln |u| + C$$

$$= \ln |\cosh x| + C$$

$$= \ln (\cosh x) + C$$

3. FIND THE LENGTH OF THE CATENARY $y = \cosh x$ FROM $x = -1$ TO $x = 1$.

$$L = \int_{-1}^1 \sqrt{1 + (\cosh x)'}^2} \, dx = \int_{-1}^1 \sqrt{1 + \sinh^2 x} \, dx$$

$$= \int_{-1}^1 \sqrt{\cosh^2 x} \, dx = \int_{-1}^1 \cosh x \, dx = \sinh x \Big|_{-1}^1$$

$$= \sinh 1 - \sinh(-1)$$

$$= \frac{1}{2}(e^1 - e^{-1}) - \frac{1}{2}(e^{-1} - e^1) = \frac{1}{2}(e - \frac{1}{e} - \frac{1}{e} + e)$$

$$= e - \frac{1}{e}$$

INVERSES FOR $y = \sinh x$, $y = \cosh x$ (ON $x \geq 0$), AND
 $y = \tanh x$:

FOR EXAMPLE,

$$y = \sinh x$$

$$y = \frac{1}{2} (e^x - e^{-x})$$

$$2y = e^x - e^{-x}$$

$$(2y)e^x = (e^x)^2 - 1$$

$$(e^x)^2 - (2y)e^x - 1 = 0$$

COMPARE : $z^2 - (2y)z - 1 = 0$ (QUADRATIC EQUATION FOR z)

SOLVE FOR e^x WITH QUADRATIC FORMULA :

$$\begin{aligned} e^x &= \frac{-(-2y) \pm \sqrt{(-2y)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{2y \pm \sqrt{4y^2 + 4}}{2} = \frac{2y \pm 2\sqrt{y^2 + 1}}{2} \\ &= y \pm \sqrt{y^2 + 1} \end{aligned}$$

SINCE $e^x > 0$ WE MUST CHOOSE THE + SIGN.

$$e^x = y + \sqrt{y^2 + 1}$$

$$x = \ln (y + \sqrt{y^2 + 1})$$

THUS,

$$y = \sinh^{-1} x = \ln (x + \sqrt{x^2 + 1}) , \quad -\infty < x < \infty$$

SIMILAR CALCULATIONS YIELD THE REST :

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \quad , \quad -\infty < x < \infty$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \quad , \quad x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad , \quad -1 < x < 1$$

DERIVATIVES OF THE INVERSE FUNCTIONS :

$$\begin{aligned} (\sinh^{-1} x)' &= (\ln(x + \sqrt{x^2 + 1}))' \\ &= \frac{1}{x + \sqrt{x^2 + 1}} (x + (x^2 + 1)^{\frac{1}{2}})' \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \left(1 + \frac{1}{2} (x^2 + 1)^{-\frac{1}{2}} (2x)\right) \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \left(\frac{\sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} + \frac{x}{\sqrt{x^2 + 1}}\right) \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}\right) \\ &= \frac{1}{\sqrt{x^2 + 1}} \end{aligned}$$

SIMILARLY ,

$$(\sinh^{-1} x)' = \frac{1}{\sqrt{x^2 + 1}}$$

$$(\cosh^{-1} x)' = \frac{1}{\sqrt{x^2 - 1}}$$

$$(\tanh^{-1} x)' = \frac{1}{1 - x^2}$$

NOW COMBINE THESE WITH THE CHAIN RULE.

EXAMPLES :

$$\begin{aligned}
 1. \quad (\cosh^{-1}(\frac{2}{x}))' &= \frac{1}{\sqrt{(\frac{2}{x})^2 - 1}} (\frac{2}{x})' = \frac{1}{\sqrt{\frac{4}{x^2} - 1}} (-\frac{2}{x^2}) \\
 &= \frac{1}{\sqrt{\frac{4-x^2}{x^2}}} (-\frac{2}{x^2}) = \frac{1}{\frac{\sqrt{4-x^2}}{x}} (-\frac{2}{x^2}) \\
 &= \frac{x}{\sqrt{4-x^2}} (-\frac{2}{x^2}) = -\frac{2}{x\sqrt{4-x^2}}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad y &= (1 + x \tanh^{-1} x)^{10} \\
 y' &= 10 (1 + x \tanh^{-1} x)^9 (1 + x \tanh^{-1} x)' \\
 &= 10 (1 + x \tanh^{-1} x)^9 (0 + x (\tanh^{-1} x)' + \tanh^{-1} x) \\
 &= 10 (1 + x \tanh^{-1} x)^9 (\frac{x}{1-x^2} + \tanh^{-1} x)
 \end{aligned}$$

INTEGRALS :

$$\int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + C = \ln(x + \sqrt{x^2+1}) + C$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + C = \ln(x + \sqrt{x^2-1}) + C$$

EXAMPLE :

$$\int \frac{dx}{\sqrt{4x^2-9}} = \int \frac{1}{\sqrt{9\left(\frac{4x^2}{9}-1\right)}} dx$$

$$= \int \frac{1}{3\sqrt{\left(\frac{2x}{3}\right)^2-1}} dx$$

$$= \frac{1}{3} \int \frac{1}{\sqrt{\left(\frac{2x}{3}\right)^2-1}} dx$$

$$u = \frac{2x}{3}$$

$$du = \frac{2}{3} dx$$

$$= \frac{1}{3} \frac{3}{2} \int \frac{1}{\sqrt{u^2-1}} du$$

$$= \frac{1}{2} \cosh^{-1} u + C = \frac{1}{2} \cosh^{-1} \left(\frac{2x}{3}\right) + C$$

$$= \frac{1}{2} \ln \left(\frac{2x}{3} + \sqrt{\left(\frac{2x}{3}\right)^2-1} \right) + C$$

$$= \frac{1}{2} \ln \left(\frac{2x}{3} + \sqrt{\frac{4x^2-9}{9}} \right) + C$$

$$= \frac{1}{2} \ln \left(\frac{2x + \sqrt{4x^2-9}}{3} \right) + C$$

$$= \frac{1}{2} \ln (2x + \sqrt{4x^2-9}) - \frac{1}{2} \ln 3 + C$$

$$= \frac{1}{2} \ln (2x + \sqrt{4x^2-9}) + C$$

(ABSORB $-\frac{1}{2} \ln 3$ INTO THE
ARBITRARY CONSTANT)