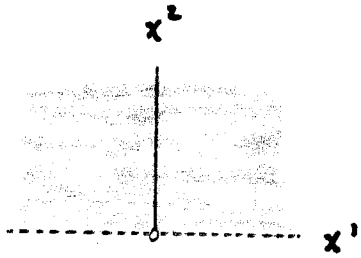
TIME NOW FOR SOME CONCRETE CALCULATIONS. RATHER THAN THE USUAL SURFACES IN $\mathbb{R}^3$, OR SUBMANIFOLDS OF $\mathbb{R}^n$, WITH THEIR INDUCED RIEMANNIAN METRICS, I WOULD LIKE TO SPEND SOME TIME WITH "2-DIMENSIONAL HYPERBOLIC SPACE" FOR WHICH THERE ARE SEVERAL (ISOMETRIC) MODELS.

1. <u>POINCARÉ</u> <u>UPPER</u> <u>HALF</u>-<u>PLANE</u> :

$$\mathbb{R}^2_+ = \{ (x^1,x^2) \in \mathbb{R}^2 : x^2 > 0 \}$$
$$= \{ w \in \mathbb{C} : \operatorname{Im}(w) > 0 \}$$



$$g = g_{ij}(x^1,x^2)\, dx^i \otimes dx^j$$

$$g_{11}(x^1,x^2) = g_{22}(x^1,x^2) = \frac{1}{(x^2)^2}$$

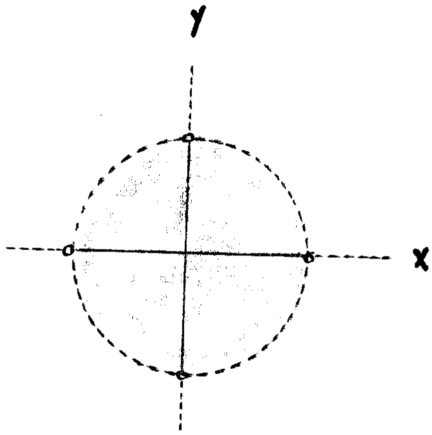
$$g_{12}(x^1,x^2) = g_{21}(x^1,x^2) = 0$$

SO

$$g = \frac{1}{(x^2)^2}\left(dx^1 \otimes dx^1 + dx^2 \otimes dx^2 \right)$$

2. <u>POINCARÉ</u> <u>DISC</u> :

$$B^2_1(0) = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 < 1 \}$$
$$= \{ z \in \mathbb{C} : |z|^2 < 1 \}$$



$$g = \frac{4}{(1-x^2-y^2)^2}(dx \otimes dx + dy \otimes dy)$$

EXERCISE : THE FRACTIONAL LINEAR TRANSFORMATION

$$F : \{ z \in \mathbb{C} : |z|^2 < 1 \} \to \{ w \in \mathbb{C} : \mathrm{Im}(w) > 0 \}$$

$$w = F(z) = \frac{i+z}{1+iz}$$

IS A DIFFEOMORPHISM OF THE UNIT DISC ONTO THE UPPER HALF-PLANE. MOREOVER, THE PULLBACK UNDER F OF THE METRIC ON THE POINCARÉ UPPER HALF-PLANE IS THE METRIC ON THE POINCARÉ DISC, I.E., WITH THESE TWO RIEMANNIAN METRICS, F IS AN ISOMETRY.

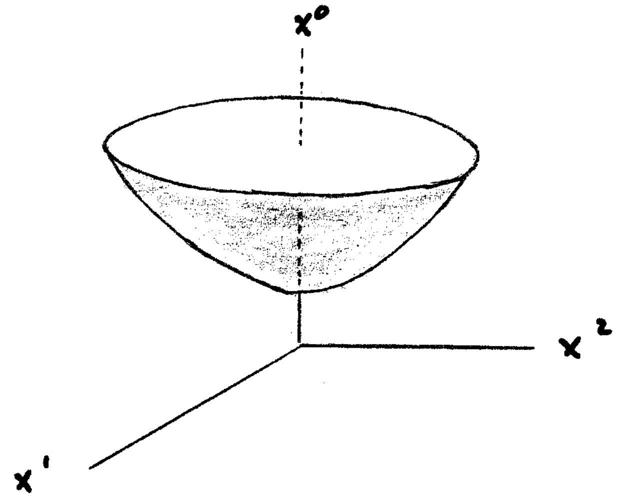
3. MINKOWSKI MODEL

I WILL NOT LINGER ON THIS ONE, BUT INCLUDE IT JUST TO SUGGEST WHERE THE TERMINOLOGY " HYPERBOLIC GEOMETRY " COMES FROM. ON $\mathbb{R}^3 = \{ (x^0, x^1, x^2) : x^0, x^1, x^2 \in \mathbb{R} \}$ ONE DEFINES A SEMI-RIEMANNIAN ("MINKOWSKI") METRIC BY

$$g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2$$

THEN CONSIDER THE SMOOTH SUBMANIFOLD M_+ OF $\mathbb{R}^3$ GIVEN BY

$$M_+ = \{ (x^0, x^1, x^2) \in \mathbb{R}^3 : -(x^0)^2 + (x^1)^2 + (x^2)^2 = -1 ,\ x^0 > 0 \}$$



THE RESTRICTION OF THE MINKOWSKI SEMI-RIEMANNIAN METRIC ON $\mathbb{R}^3$ TO M_+ IS, IN FACT, A RIEMANNIAN METRIC AND THE RESULTING RIEMANNIAN MANIFOLD IS ISOMETRIC TO BOTH OF THE PRECEDING MODELS OF 2-DIMENSIONAL HYPEBOLIC SPACE.

<u>OPTIONAL</u> <u>EXERCISE</u> : PROVE ALL OF THIS.

HERE'S THE PLAN : WE'LL FIRST DO A DIRECT CALCULATION OF ALL OF THE GEODESICS IN THE POINCARÉ UPPER HALF-PLANE. THEN WE'LL " TRANSFER " ALL OF THEM TO THE POINCARÉ DISC BY THE ISOMETRY F. WE'LL LOOK AT SOME PRETTY PICTURES, THEN WE'LL

USE ALL OF THIS TO MOTIVATE THE GENERAL NOTION OF A "HYPERBOLIC 2-MANIFOLD" AND BRIEFLY DESCRIBE THE FAMOUS "UNIFORMIZATION THEOREM" WHICH IS SOMETHING OF A 2-DIMENSIONAL, BABY VERSION OF THE THURSTON PROGRAM FOR 3-MANIFOLDS.

$$\mathbb{R}^2_+ = \{ (x^1, x^2) \in \mathbb{R}^2 : x^2 > 0 \}$$

$$g = (x^2)^{-2} (dx^1 \otimes dx^1 + dx^2 \otimes dx^2), \text{ I.E.,}$$

$$g_{11} = (x^2)^{-2} \qquad g_{12} = g_{21} = 0 \qquad g_{22} = (x^2)^{-2}$$

FOR THE GEODESIC EQUATIONS WE NEED THE CHRISTOFFEL SYMBOLS.

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right)$$

$$i, j, k = 1, 2$$

NOW,

$$(g_{ij}) = \begin{pmatrix} (x^2)^{-2} & 0 \\ 0 & (x^2)^{-2} \end{pmatrix}$$

SO

$$(g^{ij}) = \begin{pmatrix} (x^2)^2 & 0 \\ 0 & (x^2)^2 \end{pmatrix}$$

$$g^{11} = (x^2)^2 \qquad g^{12} = g^{21} = 0 \qquad g^{22} = (x^2)^2$$

NOW WE JUST COMPUTE ALL OF THE Γ^k_{ij} , E.G.,

$$\Gamma^1_{12} = \frac{1}{2} g^{1m} \left(\frac{\partial g_{2m}}{\partial x^1} + \frac{\partial g_{1m}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^m} \right)$$

$$= \frac{1}{2} g^{11} \left(\frac{\partial g_{21}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^1} \right)$$

$$= \frac{1}{2} (x^2)^2 \left(0 - 2(x^2)^{-3} - 0 \right)$$

$$= -(x^2)^{-1}$$

$$= \Gamma^1_{21} \qquad \text{(LEVI-CIVITA IS "SYMMETRIC")}$$

<u>EXERCISE</u> : DO THE REST AND SHOW

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = 0$$

$$\Gamma^2_{11} = (x^2)^{-1}$$

$$\Gamma^2_{22} = -(x^2)^{-1}$$

NOW THE GEODESIC EQUATIONS

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \quad , \quad k = 1, 2$$

BECOME

$k = 1$:

$$\frac{d^2x^1}{dt^2} + \Gamma^1_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0$$

$$\frac{d^2x^1}{dt^2} + \Gamma^1_{12} \frac{dx^1}{dt} \frac{dx^2}{dt} + \Gamma^1_{21} \frac{dx^2}{dt} \frac{dx^1}{dt} = 0$$

$$\frac{d^2x^1}{dt^2} + 2\Gamma^1_{12}\frac{dx^1}{dt}\frac{dx^2}{dt} = 0$$

$$(1) \qquad \boxed{\frac{d^2x^1}{dt^2} - \frac{2}{x^2}\frac{dx^1}{dt}\frac{dx^2}{dt} = 0}$$

EXERCISE : DERIVE, IN THE SAME WAY, THE $k=2$ EQUATION

$$(2) \qquad \boxed{\frac{d^2x^2}{dt^2} + \frac{1}{x^2}\left(\left(\frac{dx^1}{dt}\right)^2 - \left(\frac{dx^2}{dt}\right)^2\right) = 0}$$

NOW ALL WE NEED TO DO IS SOLVE THEM.

TO FIND ALL OF THE GEODESICS THROUGH SOME FIXED POINT

$$p = (x^1_0, x^2_0)$$

OF $\mathbb{R}^2_+$ WE NEED ONLY FIND ONE IN EACH "DIRECTION" $v_p \in T_p(\mathbb{R}^2_+)$ AT p.

FOR $v_p = 0 \in T_p(\mathbb{R}^2_+)$ THIS IS CLEARLY THE CONSTANT CURVE

$$x^1(t) = x^1_0$$
$$x^2(t) = x^2_0$$

FOR $t \in \mathbb{R}$.

HENCEFORTH WE IGNORE THIS ONE AND THEREFORE WE CAN ASSUME THAT NOT BOTH $\frac{dx^1}{dt}$ AND $\frac{dx^2}{dt}$ ARE IDENTICALLY ZERO.

MOREOVER, IF $\frac{dx^2}{dt} \equiv 0$, THEN $\frac{d^2x^2}{dt^2} \equiv 0$ SO (2) IMPLIES $\frac{dx^1}{dt} \equiv 0$. THUS, WE MAY ASSUME $\frac{dx^2}{dt}$ IS NOT IDENTICALLY ZERO.

GEOMETRICALLY, THERE ARE NO NONTRIVIAL "HORIZONTAL" GEODESICS SO THE ONES THROUGH p IN HORIZONTAL DIRECTIONS $v_p = a \frac{\partial}{\partial x^1}\Big|_p$ CANNOT BE STRAIGHT LINES.

WHAT ABOUT "VERTICAL" GEODESICS? LET'S SUPPOSE $\frac{dx^1}{dt} \equiv 0$. THEN $\frac{dx^2}{dt}(p) \neq 0$ (THIS WOULD GIVE THE CONSTANT GEODESIC AT p WHICH WE HAVE ALREADY SET ASIDE). THUS, $\frac{dx^2}{dt} \neq 0$ ON SOME INTERVAL ABOUT $t = 0$. THE CORRESPONDING GEODESIC α CAN BE PARAMETRIZED BY ARC LENGTH s, I.E., WE CAN IMPOSE THE "UNIT SPEED" CONDITION

$$g(\alpha', \alpha') = 1$$

$$g\left((x^1)' \frac{\partial}{\partial x^1} + (x^2)' \frac{\partial}{\partial x^2} \,,\, (x^1)' \frac{\partial}{\partial x^1} + (x^2)' \frac{\partial}{\partial x^2} \right) = 1$$

$$\frac{1}{(x^2)^2} \left(((x^1)')^2 + ((x^2)')^2 \right) = 1$$

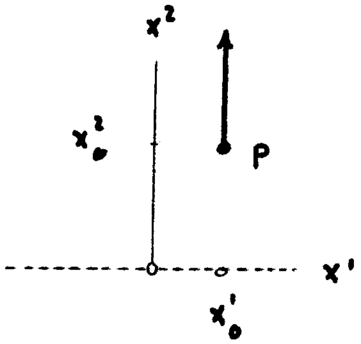
WHICH, WITH $(x^1)' = 0$, GIVES

$$\frac{(x^2)'}{(x^2)^2} = 1 .$$

THUS, EITHER

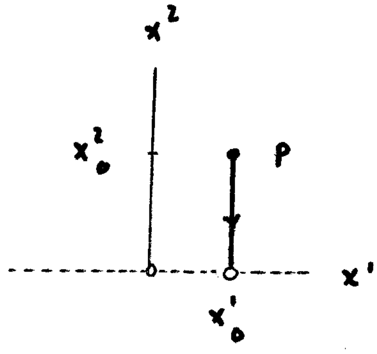
$$\frac{dx^2}{dt} = x^2 \qquad \text{OR} \qquad \frac{dx^2}{dt} = -x^2$$

NOTING THAT, ON $\mathbb{R}^2_+$, x^2 IS ALWAYS POSITIVE WE SEE THAT WE HAVE THE FOLLOWING TWO POSSIBILITIES, CORRESPONDING TO $v_p \in T_p(\mathbb{R}^2_+)$ WITH $g_p(v_p, v_p) = 1$ AND v_p POINTED "UP" OR "DOWN":



$$\frac{dx^2}{ds} = x^2$$

$$x^2(s) = x^2_0 e^{s}$$



$$\frac{dx^2}{ds} = -x^2$$

$$x^2(s) = x^2_0 e^{-s}$$

NOTICE THAT THIS ONE NEVER "REACHES" THE MISSING $x^2 = 0$ LEVEL.

<u>NOTE</u>: WE HAVEN'T USED GEODESIC EQUATION (2) TO DERIVE THESE SO YOU NEED TO PLUG IN AND CHECK THAT THESE REALLY ARE GEODESICS (THEY ARE!). ALSO NOTE THAT, DESPITE APPEARENCES TO THE CONTRARY, THE "DOWNWARD" GEODESIC THROUGH p HAS <u>INFINITE</u> <u>LENGTH</u> (s IS THE ARC LENGTH AND $x^2 \to 0 \Leftrightarrow s \to \infty$).

WITH THESE DIRECTIONS AT P HANDLED WE NOW ASSUME THAT <u>NEITHER</u> $\frac{dx^1}{dt}$ <u>NOR</u> $\frac{dx^2}{dt}$ <u>IS</u> <u>IDENTICALLY</u> <u>ZERO</u>.

ON ANY INTERVAL ON WHICH $\frac{dx^1}{dt}$ AND $\frac{dx^2}{dt}$ ARE NONZERO WE CAN DIVIDE (1) BY $\frac{dx^1}{dt}$ AND PROCEED AS FOLLOWS:

$$\frac{(x^1)''}{(x^1)'} - \frac{2(x^2)'}{x^2} = 0 \tag{3}$$

NOW NOTICE THAT

$$\left(\frac{(x^1)'}{(x^2)^2}\right)' = \frac{(x^1)''(x^2)^2 - (x^1)'(2x^2(x^2)')}{(x^2)^4}$$

$$= \frac{(x^1)'' - \frac{2(x^1)'(x^2)'}{x^2}}{(x^2)^2}$$

$$= \frac{\frac{(x^1)''}{(x^1)'} - \frac{2(x^2)'}{x^2}}{\frac{(x^2)^2}{(x^1)'}}$$

SO, ALONG A GEODESIC (WHERE (3) IS SATISFIED)

$$\left(\frac{(x^1)'}{(x^2)^2}\right)' = 0$$

$\frac{(x^1)'}{(x^2)^2}$ = A NONZERO CONSTANT WHICH, FOR CONVENIENCE, WE WRITE

$$\frac{(x^1)'}{(x^2)^2} = \frac{1}{R}$$

NOTE THAT

$$R > 0 \iff (x^1)' > 0$$

SO $R > 0$ CORRESPONDS TO GEODESICS " GOING TO THE RIGHT " WHILE $R < 0$ GIVES GEODESICS " GOING TO THE LEFT ".
THE TWO CASES ARE ANALOGOUS SO WE WILL CONSIDER ONLY

$$R > 0 .$$

ONCE AGAIN WE IMPOSE THE "UNIT SPEED" CONDITION

$$\frac{1}{(x^2)^2}\left(((x^1)')^2 + ((x^2)')^2 \right) = 1$$

(SO THAT $t = s$).

THEN $(x^1)' = \dfrac{(x^2)^2}{R}$ IMPLIES

$$\frac{(x^2)^2}{R^2} + \left(\frac{(x^2)'}{x^2} \right)^2 = 1$$

$$\left(\frac{dx^2}{ds} \right)^2 = (x^2)^2 \left(1 - \frac{(x^2)^2}{R^2} \right)$$

IN PARTICULAR,

$$1 - \frac{(x^2)^2}{R^2} \geq 0$$

$$(x^2)^2 \leq R^2$$

$$0 < x^2 \leq R$$

NOW, ANY x^2 SATISFYING $0 < x^2 \leq R$ CAN BE WRITTEN AS

$$x^2 = R \sin\theta$$

FOR SOME θ WITH

$$0 < \theta < \pi .$$

FROM THIS

$$(x^1)' = \frac{(x^2)^2}{R}$$

$$\frac{dx^1}{ds} = \frac{R^2 \sin^2\theta}{R} = R \sin^2\theta .$$

INTEGRATE TO GET x^1:

$$x^1 = \int \frac{dx^1}{ds} ds = \int R \sin^2\theta \, ds$$

$$= \int R \sin^2\theta \frac{ds}{d\theta} d\theta$$

NOW,

$$\frac{ds}{d\theta} = \frac{ds}{dx^2} \frac{dx^2}{d\theta} \quad \frac{1}{\pm x^2 \sqrt{1 - (x^2)^2/R^2}} (R \cos\theta)$$

$$= - \frac{1}{R \sin\theta \cos\theta} (R \cos\theta)$$

$$= - \frac{1}{\sin\theta}$$

THUS,

$$x^1 = \int R \sin^2\theta \left(-\frac{1}{\sin\theta}\right) d\theta$$

$$= -R \int \sin\theta \, d\theta$$

$$= R\cos\theta + C$$

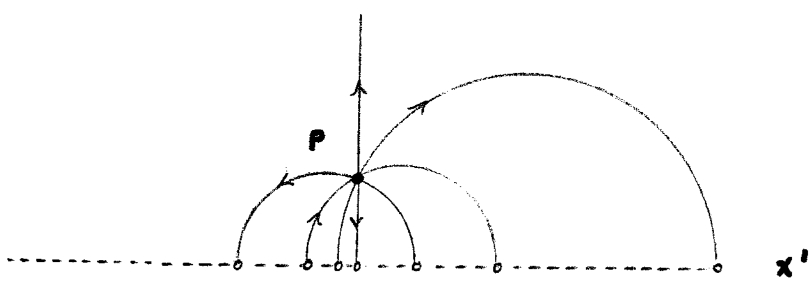
SO

$$\begin{cases} x^1 = R\cos\theta + C \\ x^2 = R\sin\theta \end{cases}$$

$$0 < \theta < \pi$$

EACH OF THESE IS A SEMI-CIRCLE OF RADIUS R CENTERED AT THE POINT $(C, 0)$ ON THE (MISSING) x^1-AXIS AND TRAVERSED CLOCKWISE (BECAUSE $\frac{ds}{d\theta} < 0$)

SO, HERE ARE THE GEODESICS THROUGH p :



EXERCISE : SHOW THAT THEY ALL HAVE INFINITE LENGTH.

<u>OPTIONAL</u> <u>EXERCISE</u> : IF "STRAIGHT LINE" MEANS "GEODESIC OF THE POINCARÉ UPPER HALF PLANE" AND "PARALLEL" MEANS "NONINTERSECTING", THEN THE GEOMETRY OF $\mathbb{R}^2_+$ DOES <u>NOT</u> SATIS EUCLID'S PARALLEL POSTULATE (FOR EACH STRAIGHT LINE ℓ AND EACH POINT p NOT ON ℓ THERE IS EXACTLY ONE STRAIGHT LINE THROUGH p AND PARALLEL TO ℓ). SHOW THAT IT <u>DOES</u> SATISFY THE REST AND SO IS A MODEL OF "LOBACHEVSKY GEOMETRY".

NOW WE WOULD LIKE TO SEE WHAT ALL OF THIS LOOKS LIKE IN THE POINCARÉ DISC.

WE KNOW THAT THE DISC IS ISOMETRIC TO THE UPPER HALF PLANE, BUT WE NEED TO KNOW THAT ISOMETRIES CARRY GEODESICS OF THE LEVI-CIVITA CONNECTION ONTO GEODESICS OF THE LEVI-CIVITA CONNECTION. THIS IS STRAIGHTFORWARD, BUT NOT COMPLETELY TRIVIAL SO I WILL LET YOU WRITE IT OUT. IN FACT, YOU WILL DEDUCE THIS FROM THE FOLLOWING MORE GENERAL FACT.

<u>EXERCISE</u> : LET M AND N BE SMOOTH MANIFOLDS WITH RIEMANNIAN METRICS ${}^M g$ AND ${}^N g$ AND LET $F : M \to N$ BE AN ISOMETRY ($F^*({}^N g) = {}^M g$). SHOW THAT F PRESERVES LEVI-CIVITA CONNECTIONS IN THE FOLLOWING SENSE :

LET V AND W BE SMOOTH VECTOR FIELDS ON M AND $F_*(V)$ AND $F_*(W)$ THEIR IMAGES UNDER THE DIFFEOMORPHISM F (E.G., $(F_*(V))(F(p)) = F_{*p}(V(p))$). THEN

$$F_*\left({}^M\nabla_V W\right) = {}^N\nabla_{F_*(V)} F_*(W).$$

<u>HINTS</u>: PROVE THIS LOCALLY ON A NEIGHBORHOOD OF $F(p)$ AS FOLLOWS: LET ψ BE A CHART ON SOME NEIGHBORHOOD OF $F(p)$ IN N, WITH COORDINATE FUNCTIONS $y^1, \dots, y^n$. THEN $\phi = \psi \circ F$ IS A CHART ON SOME NEIGHBORHOOD OF p IN M, WITH COORDINATE FUNCTIONS $x^1, \dots, x^n$. NOW PROVE THE FOLLOWING:

1. $F_*\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}$

2. IF $V = V^i \frac{\partial}{\partial x^i}$ AND $F_*(V) = \tilde{V}^i \frac{\partial}{\partial y^i}$, THEN $V^i = \tilde{V}^i \circ F$. SIMILARLY FOR W.

3. ${}^N g_{ij}(F(q)) = {}^M g_{ij}(q)$ FOR EVERY q IN THE COORDINATE NEIGHBORHOOD OF p.

4. ${}^N\Gamma^k_{ij}(F(q)) = {}^M\Gamma^k_{ij}(q)$ FOR EVERY q IN THE COORDINATE NEIGHBORHOOD OF p

NOW WRITE OUT ${}^N\nabla_{F_*(V)} F_*(W)$ LOCALLY. FINALLY, DEDUCE THAT α IS A GEODESIC OF ${}^M\nabla$ IF AND ONLY IF $F \circ \alpha$ IS A GEODESIC OF ${}^N\nabla$.

NOTE: THIS ARGUMENT ACTUALLY SHOWS THAT THE LEVI-CIVITA CONNECTION IS PRESERVED BY LOCAL ISOMETRIES.

THUS, TO GET THE GEODESICS OF THE POINCARÉ DISC WE NEED ONLY FIND THE IMAGES OF THE GEODESICS OF THE POINCARÉ UPPER HALF-PLANE UNDER AN ISOMETRY.

WE KNOW THAT THE FRACTIONAL LINEAR TRANSFORMATION

$$F : \{ z \in \mathbb{C} : |z|^2 < 1 \} \to \{ w \in \mathbb{C} : \operatorname{Im}(w) > 0 \}$$

$$w = F(z) = \frac{z + i}{1 + iz}$$

IS AN ISOMETRY FROM THE DISC TO THE HALF-PLANE. SOLVING FOR z GIVES THE INVERSE

$$z = F^{-1}(w) = \frac{w - i}{1 - iw} .$$

THIS, OF COURSE, IS ALSO FRACTIONAL LINEAR AND IS AN ISOMETRY OF THE HALF-PLANE ONTO THE DISC.

ONE COULD COMPUTE DIRECTLY THE IMAGE UNDER F^{-1} OF ALL OF THE GEODESICS OF THE HALF-PLANE, BUT GENERAL PROPERTIES OF FRACTIONAL LINEAR TRANSFORMATIONS SUFFICE TO IDENTIFY THEM.

RECALL THAT ANY FRACTIONAL LINEAR TRANSFORMATION ON $\mathbb{C}$

1. CARRIES {CIRCLES, STRAIGHT LINES} ONTO {CIRCLES, STRAIGHT LINES}

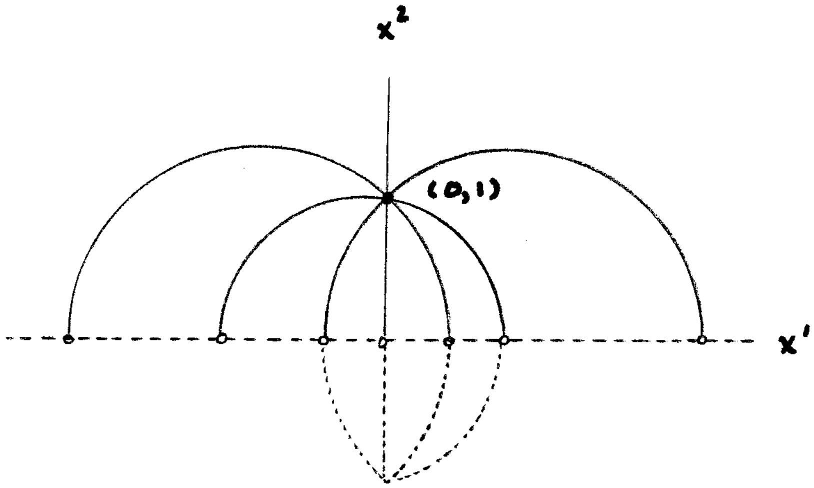
2. IS CONFORMAL

THE F.L.T. $z = \dfrac{w - i}{1 - iw}$ CARRIES $\operatorname{Im}(w) = 0$ ONTO $|z|^2 = 1$

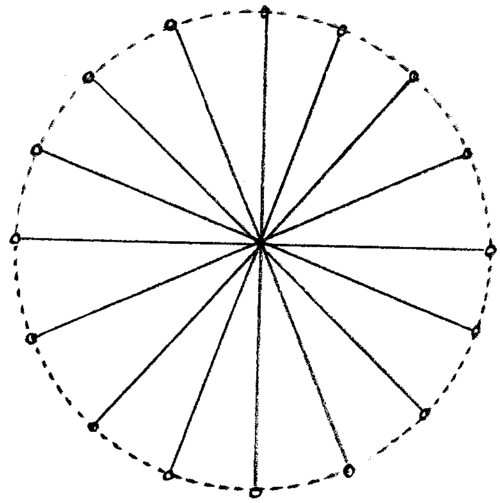
AND TAKES $w = i$ TO $z = 0$.

THUS, ANY GEODESIC OF THE HALF-PLANE THROUGH $(0,1)$ IS MAPPED TO A GEODESIC OF THE DISC THROUGH $(0,0)$.

CHECK THAT ANY GEODESIC OF THE HALF-PLANE THROUGH $(0,1)$ ALSO PASSES THROUGH $(0,-1)$ $(w = -i)$ SO ITS IMAGE IS A STRAIGHT LINE ("CIRCLE THROUGH $z = \infty$").

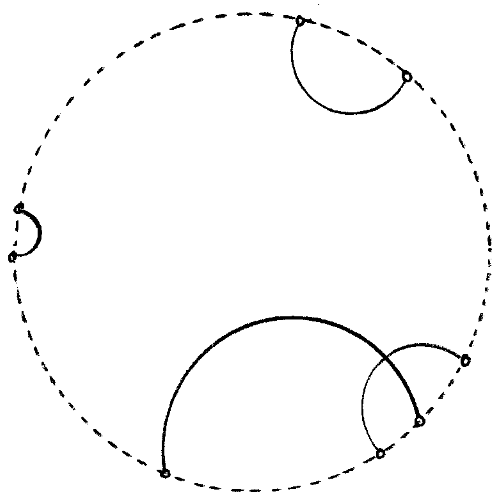


THE GEODESICS OF THE DISC THROUGH $(0,0)$ ARE THEREFORE RADIAL STRAIGHT LINE SEGMENTS (THEY MUST INTERSECT $|z|^2 = 1$ ORTHOGONALLY).

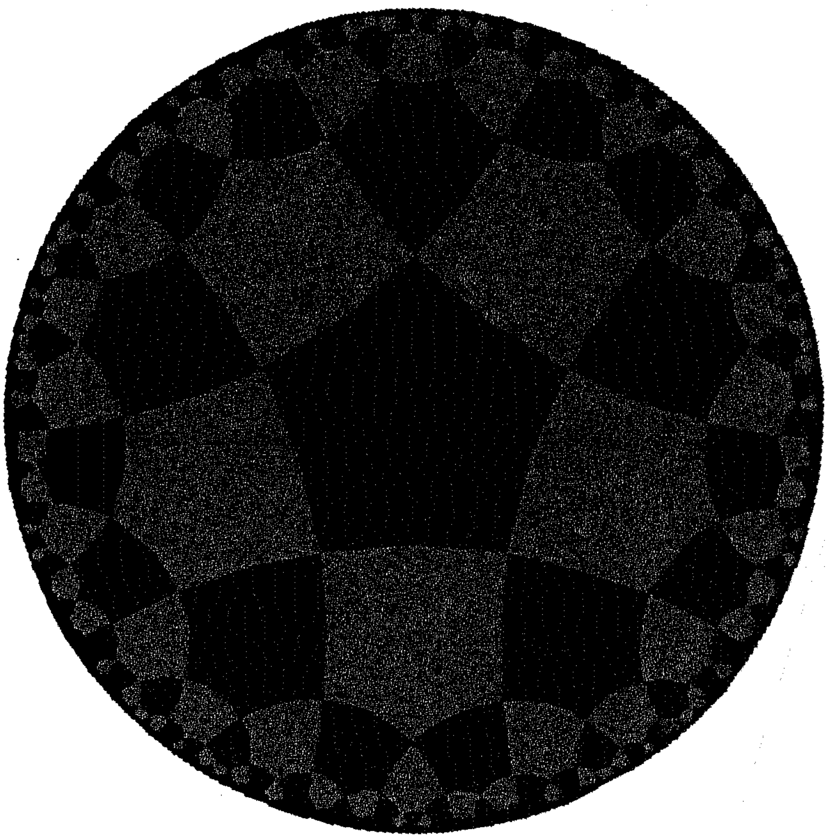


THEIR LENGTHS ARE ALL INFINITE, OF COURSE.

A GEODESIC OF THE HALF-PLANE THAT DOES NOT PASS THROUGH $(0,1)$ MUST MAP TO AN ACTUAL, HONEST-TO-GOD CIRCLE IN THE DISC (BECAUSE IT ALSO DOES NOT PASS THROUGH $(0,-1)$)) WHICH MUST INTERSECT THE BOUNDARY $|z|^2=1$ ORTHOGONALLY.



THESE ALSO HAVE INFINITE LENGTH.



GEODESIC TESSELLATION OF POINCARÉ DISC

ESCHER HAD A SOMEWHAT MORE EXOTIC VIEW OF THE POINCARÉ DISC :