INVARIANT SUBSPACES

A THEOREM FROM LINEAR ALGEBRA :

V : A VECTOR SPACE (OVER $\mathbb{R}$ OR $\mathbb{C}$) WITH $\dim V > 2$ (PERHAPS ∞)

$A : V \to V$ A LINEAR TRANSFORMATION

THEN $\exists$ LINEAR SUBSPACE W OF V, $W \neq \{0\}$ AND $W \neq V$ S.T.

$$AW \subseteq W$$

A QUESTION FROM FUNCTIONAL ANALYSIS :

X = A BANACH SPACE (OVER $\mathbb{C}$ AND INFINITE-DIMENSIONAL)

$\mathcal{B}(X)$ = THE ALGEBRA OF BOUNDED LINEAR OPERATORS ON X

DOES EVERY ELEMENT OF $\mathcal{B}(X)$ HAVE A NONTRIVIAL CLOSED INVARIANT SUBSPACE ?

THE THEOREM WE WILL PROVE :

$\mathcal{A}$ = A SUBALGEBRA OF $\mathcal{B}(X)$

$\mathcal{A}'$ = THE COMMUTANT OF $\mathcal{A}$

$\mathcal{A}''$ = THE COMMUTANT OF $\mathcal{A}'$ $\supseteq \mathcal{A}$

IF $\exists$ NONZERO COMPACT OPERATOR $K \in \mathcal{A}'$, THEN $\exists$ CLOSED SUBSPACE M OF X, $M \neq \{0\}$ AND $M \neq X$, S.T.

$$AM \subseteq M$$

FOR EVERY $A \in \mathcal{A}''$.

COROLLARIES : 1. $K \in \mathcal{B}(X)$ COMPACT

$\mathcal{A} = [K]$ = ALGEBRA GENERATED BY K

$K \in \mathcal{A}' \Rightarrow \exists$ NONTRIVIAL INVARIANT SUBSPACE FOR K (ARONSZAJN AND SMITH, 1954)

2. $A \in \mathcal{B}(X)$ S.T. $p(A)$ COMPACT FOR SOME POLYNOMIAL p

$\mathcal{A} = [A]$: ALGEBRA GENERATED BY A

$p(A) \in \mathcal{A}' \Rightarrow \exists$ NONTRIVIAL INVARIANT SUBSPACE FOR A (BERNSTEIN AND ROBINSON, 1966)

PROOF :

SUPPOSE FIRST THAT K HAS AN EIGENVALUE λ. LET

$$M_\lambda = \{ x \in X : Kx = \lambda x \}$$

CLOSED LINEAR SUBSPACE.

$M_\lambda \neq \{0\}$ BY DEFINITION

$M_\lambda \neq X$: SUPPOSE NOT.

$$\lambda = 0 \Rightarrow K \equiv 0 \quad \nleftarrow$$

$$\lambda \neq 0 \Rightarrow Kx = \lambda x \ \forall x \in X \Rightarrow K \text{ NOT COMPACT} \quad \nleftarrow$$

$A \in \mathcal{A}'' \Rightarrow A$ COMMUTES WITH K ($K \in \mathcal{A}'$) SO $\forall x \in M_\lambda$

$$K(Ax) = A(Kx) = A(\lambda x) = \lambda Ax$$

$$Ax \in M_\lambda$$

$$AM_\lambda \subseteq M_\lambda$$

NOW SUPPOSE K HAS NO EIGENVALUES.

RECALL : $\sigma(K) = \{\lambda : K - \lambda I \text{ SINGULAR}\}$ CONSISTS OF 0 AND A COUNTABLE SET OF NONZERO EIGENVALUES.

THUS,

$$\sigma(K) = \{0\}$$

SO

$$\begin{aligned} r(K) &= \text{SPECTRAL RADIUS OF } K \\ &= \sup\{|\lambda| : \lambda \in \sigma(K)\} \\ &= 0 \end{aligned}$$

$$\boxed{\lim_{j \to \infty} \|K^j\|^{\frac{1}{j}} = 0} \qquad (\text{SPECTRAL RADIUS FORMULA})$$

NOW HERE'S THE IDEA BEHIND THE PROOF :

SELECT SOME NONZERO $y_0 \in X$ AND CONSIDER

$$M_{y_0} = \overline{\{Ay_0 : A \in \mathcal{A}''\}}$$

$\mathcal{A}''$ A LINEAR SUBSPACE OF $\mathcal{B}(X)$ $\Rightarrow$ M_{y_0} A CLOSED SUBSPACE OF X

$\mathcal{A}''$ A SUBALGEBRA OF $\mathcal{B}(X)$ AND M_{y_0} CLOSED $\Rightarrow$ M_{y_0} INVARIANT UNDER EVERY ELEMENT OF $\mathcal{A}''$

$I \in \mathcal{A}'' \Rightarrow M_{y_0} \neq \{0\}$

WE WOULD BE DONE IF $M_{y_0} \neq X$, BUT THIS NEED NOT BE TRUE. NEED TO CHOOSE y_0 CAREFULLY. HERE'S HOW TO DO IT :

K $\neq 0$ SO WE CAN NORMALIZE AND ASSUME

$$\|K\| = 1$$

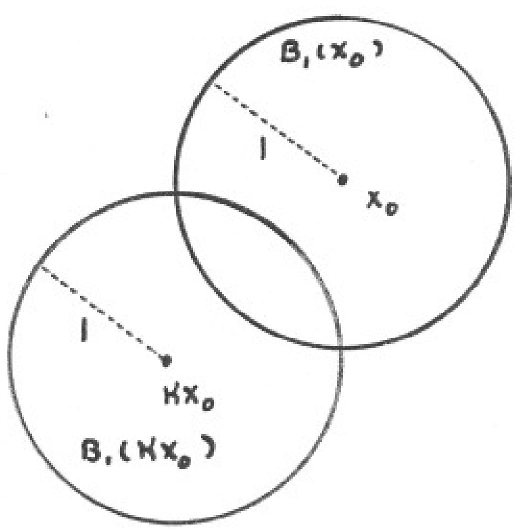
SELECT $x_0 \in X$ WITH

$$\|Kx_0\| > 1$$

AND LET

$$B_1(x_0) = \{x \in X : \|x - x_0\| \leq 1\}$$

$$B_1(Kx_0) = \{x \in X : \|x - Kx_0\| \leq 1\}$$



K IS A COMPACT OPERATOR SO

$$\overline{K(B_1(x_0))}$$

IS A COMPACT SUBSET OF X.

NOTE THAT

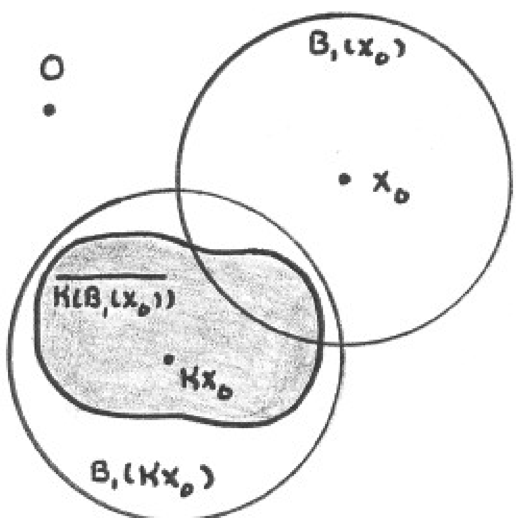
(1) $0 \notin B_1(x_0)$: $\|x_0 - 0\| = \|x_0\| = \|K\|\|x_0\| \geq \|Kx_0\| > 1$

(2) $\overline{K(B_1(x_0))} \subseteq B_1(Kx_0)$: FOR $x \in B_1(x_0)$,

$$\|Kx - Kx_0\| = \|K(x - x_0)\|$$
$$\leq \|K\|\|x - x_0\| \leq 1$$

SO $K(B_1(x_0)) \subseteq B_1(Kx_0)$ AND THEREFORE $\overline{K(B_1(x_0))} \subseteq B_1(Kx_0)$.

(3) $0 \notin \overline{K(B_1(x_0))}$: $\overline{K(B_1(x_0))} \subseteq B_1(Kx_0)$, BUT $\|Kx_0 - 0\| = \|Kx_0\| > 1$.



TO PRODUCE THE $y_0 \in X$ WE'RE LOOKING FOR CONSIDER THE FAMILY OF OPEN SETS

$$\{ A^{-1}(\operatorname{int} B_1(x_0)) : A \in \mathcal{A}'' \}$$

<u>CLAIM</u> : THESE SETS <u>DO</u> <u>NOT</u> COVER $\overline{K(B_1(x_0))}$.

<u>NOTE</u> : THIS WILL FINISH THE PROOF OF THE THEOREM BECAUSE IF y_0 IS AN ELEMENT OF $\overline{K(B_1(x_0))}$ THAT IS NOT COVERED, THEN

$$Ay_0 \notin \operatorname{int} B_1(x_0)$$

$\forall A \in \mathcal{A}''$ SO $\operatorname{int} B_1(x_0)$ IS AN OPEN SET CONTAINING x_0 AND MISSING $\{ Ay_0 : A \in \mathcal{A}'' \}$, I.E.,

$$x_0 \notin M_{y_0} = \overline{\{ Ay_0 : A \in \mathcal{A}'' \}}$$

SO

$$M_{y_0} \neq X.$$

<u>PROOF</u> <u>OF</u> <u>THE</u> <u>CLAIM</u> : SUPPOSE TO THE CONTRARY THAT

$$\overline{K(B_1(x_0))} \subseteq \bigcup_{A \in \mathcal{A}''} A^{-1}(\operatorname{int} B_1(x_0))$$

(WE INTEND TO CONTRADICT THE SPECTRAL RADIUS FORMULA).

COMPACTNESS OF $\overline{K(B_1(x_0))}$ $\Rightarrow$ $\exists\, A_1, \ldots, A_n \in \mathcal{A}''$ S.T.

$$\overline{K(B_1(x_0))} \subseteq A_1^{-1}(\operatorname{int} B_1(x_0)) \cup \cdots \cup A_n^{-1}(\operatorname{int} B_1(x_0))$$

$Kx_0 \in \overline{K(B_1(x_0))}$ $\Rightarrow$ $\exists\, i_1$ S.T. $x_1 = A_{i_1}(Kx_0) \in \operatorname{int} B_1(x_0)$

$K(A_{i_1}(Kx_0)) \in \overline{K(B_1(x_0))}$ $\Rightarrow$ $\exists\, i_2$ S.T. $x_2 = (A_{i_2}K)(A_{i_1}K)x_0 \in \operatorname{int} B_1(x_0)$

$$\vdots$$

CONTINUE INDUCTIVELY TO OBTAIN A SEQUENCE $A_{i_1}, A_{i_2}, \ldots$ OF ELEMENTS OF $\{A_1, \ldots, A_n\}$ S.T. $\forall\, j = 1, 2, \ldots$

$$x_j = (A_{i_j}K)\cdots(A_{i_2}K)(A_{i_1}K)x_0 \in \operatorname{int} B_1(x_0)$$

NOW CHOOSE $d > 0$ S.T.

$$x \in B_1(x_0) \Rightarrow \|x\| \geq d \qquad (\text{SEE } (1), \text{ PAGE } 5)$$

AND LET

$$m = \text{MAX}\,\{\|A_1\|, \ldots, \|A_n\|\}.$$

THEN $m > 0$.

NOW, FOR EVERY $j = 1, 2, \ldots$

$$
\begin{aligned}
d \le \|x_j\| &= \|(A_{i_j}K)\cdots(A_{i_2}K)(A_{i_1}K)x_0\| \\
&= \|A_{i_j}\cdots A_{i_2}A_{i_1}K^j x_0\| \quad (K \in \mathcal{Q}' \text{ AND } A_i \in \mathcal{Q}'') \\
&\le m^j \|K^j\| \|x_0\|
\end{aligned}
$$

$\|x_0\| \neq 0$ AND $m > 0$ SO THIS GIVES

$$\frac{1}{m^j}\left(\frac{d}{\|x_0\|}\right) \le \|K^j\|$$

TAKING j^{TH} ROOTS GIVES

$$\frac{1}{m}\left(\frac{d}{\|x_0\|}\right)^{\frac{1}{j}} \le \|K^j\|^{\frac{1}{j}}$$

BUT $\|x_0\| \geq d \Rightarrow \frac{d}{\|x_0\|} \le 1 \Rightarrow \left(\frac{d}{\|x_0\|}\right)^{\frac{1}{j}} \geq \frac{d}{\|x_0\|} \Rightarrow$

$$0 < \frac{d}{m\|x_0\|} \le \|K^j\|^{\frac{1}{j}}$$

$\forall\ j = 1, 2, \ldots$ SO

$$\lim_{j\to\infty} \|K^j\|^{\frac{1}{j}} \geq \frac{d}{m\|x_0\|} > 0$$

AND THIS CONTRADICTS THE SPECTRAL RADIUS FORMULA (PAGE 3). $\square$